

## n-pure submodules of modules

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**Abstract:** Let  $R$  be a commutative ring,  $M$  an  $R$ -module, and  $n \geq 1$  an integer. In this paper, we introduce the concept of  $n$ -pure submodules of  $M$  as a generalization of pure submodules and obtain some related results. We say that a submodule  $N$  of  $M$  is a  $n$ -pure submodule of  $M$  if  $I_1 I_2 \dots I_n N = I_1 N \cap I_2 N \cap \dots \cap I_n N \cap (I_1 I_2 \dots I_n)M$  for all proper ideals  $I_1, I_2, \dots, I_n$  of  $R$ .

**Keywords:** Pure submodules,  $n$ -pure submodule, Multiplication module, Fully  $n$ -pure module.

**2010 Mathematics Subject Classification:** 13C13; 13C99.

**Receive:** 09 July 2024, **Accepted:** 07 November 2024

### 1 Introduction

Throughout this paper,  $R$  will denote a commutative ring with identity and  $\mathbb{Z}$  will denote the ring of integers. Further,  $n$  will denote a positive integer.

Let  $M$  be an  $R$ -module.  $M$  is said to be a *multiplication module* if for every submodule  $N$  of  $M$ , there exists an ideal  $I$  of  $R$  such that  $N = IM$  [5]. It is easy to see that  $M$  is a multiplication module if and only if  $N = (N :_R M)M$  for each submodule  $N$  of  $M$ .

Cohn [6] defined a submodule  $N$  of  $M$  a *pure submodule* if the sequence  $0 \rightarrow N \otimes E \rightarrow M \otimes E$  is exact for every  $R$ -module  $E$ . Anderson and Fuller [2] called the submodule  $N$  a *pure submodule* of  $M$  if  $IN = N \cap IM$  for every ideal  $I$  of  $R$ . Ribenboim [10] called  $N$  to be *pure* in  $M$  if  $rM \cap N = rN$  for each  $r \in R$ . Although the first condition implies the second [9, p.158], and the second obviously implies the third, these definitions are not equivalent in general, see [9, p.158] for an example. The three definitions of purity given above are equivalent if  $M$  is flat. In particular, if  $M$  is a faithful multiplication module [1].

In this paper, our definition of purity will be that of Anderson and Fuller [2].

Let  $n$  be a positive integer. The main purpose of this paper is to introduce the concept of  $n$ -pure submodules of an  $R$ -module  $M$  as a generalization of pure submodules and investigate some results concerning this notion.

### 2 2-pure submodules

**Definition 2.1.** We say that a submodule  $N$  of an  $R$ -module  $M$  is a 2-pure submodule of  $M$  if  $IJN = IN \cap JN \cap IJM$  for all proper ideals  $I, J$  of  $R$ . Also, we say that an ideal  $I$  of  $R$  is a 2-pure ideal of  $R$  if  $I$  is a 2-pure submodule of  $R$ .

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**Remark 2.2.** Clearly every pure submodule of an  $R$ -module  $M$  is a 2-pure submodule of  $M$ . But we see in the Example 2.3 that the converse is not true in general.

**Example 2.3.** The submodule  $\bar{2}\mathbb{Z}_4$  of the  $\mathbb{Z}_4$ -module  $\mathbb{Z}_4$  is a 2-pure submodule but it is not a pure submodule.

**Example 2.4.** (a) The submodule  $2\mathbb{Z}$  of the  $\mathbb{Z}$ -module  $\mathbb{Z}$  is not a 2-pure submodule.

(b) Take  $R = K[x, y]$ , where  $K$  is a field, and take  $M = R^2/(xy, y^2)(x + y)R$ . Then one can see that the submodule  $N = (x, y)R$  is not a 2-pure submodule of  $M$ .

(c) Since  $4\mathbb{Z} = (2\mathbb{Z})(2\mathbb{Z})\mathbb{Z} \neq (2\mathbb{Z})\mathbb{Z} \cap (2\mathbb{Z})\mathbb{Z} \cap (4\mathbb{Z})\mathbb{Q} = 2\mathbb{Z}$ , the submodule  $\mathbb{Z}$  of the  $\mathbb{Z}$ -module  $\mathbb{Q}$  is not 2-pure.

A non-zero submodule  $N$  of an  $R$ -module  $M$  is said to be a *weakly strongly 2-absorbing second submodule* of  $M$  if whenever  $a, b \in R$ ,  $K$  is a submodule of  $M$ ,  $abM \not\subseteq K$ , and  $abN \subseteq K$ , then  $aN \subseteq K$  or  $bN \subseteq K$  or  $ab \in \text{Ann}_R(N)$  [4].

**Proposition 2.5.** Let  $N$  be a submodule of an  $R$ -module  $M$ . Then we have the following.

- (a) If  $IJN = IN \cap JN$  for all proper ideals  $I, J$  of  $R$ , then  $N$  is a 2-pure submodule of  $M$ .
- (b) If for each ideal  $I$  of  $R$ ,  $IN$  is a pure submodule of  $M$ , then  $N$  is a 2-pure submodule of  $M$ .
- (c) If  $N$  is a weakly strongly 2-absorbing second submodule of  $M$ , then for any  $ab \in R \setminus \text{Ann}_R(N)$ , we have  $abN = aN \cap bN \cap abM$ .

*Proof.* (a) Let  $I$  and  $J$  be proper ideals of  $R$ . Then by assumption,  $IJN = IN \cap JN$ . Thus  $IN \cap JN = IJN \subseteq IJM$ . This implies that  $IN \cap JN \cap IJM = IN \cap JN$ . Therefore,  $IN \cap JN \cap IJM = IJN$  as required.

(b) Let  $I$  and  $J$  be proper ideals of  $R$ . Then by assumption,  $I(JN) = JN \cap IM \supseteq JN \cap IN$ . This implies that  $N$  is a 2-pure submodule of  $M$  since the reverse inclusion is clear.

(c) Let  $N$  be a weakly strongly 2-absorbing second submodule of  $M$  and  $ab \in R \setminus \text{Ann}_R(N)$ . Clearly,  $abN \subseteq aN \cap bN \cap abM$ . Now let  $L$  be a completely irreducible submodule of  $M$  such that  $abN \subseteq L$ . If  $abM \subseteq L$ , then we are done. If  $abM \not\subseteq L$ , then  $aN \subseteq L$  or  $bN \subseteq L$  because  $N$  is a weakly strongly 2-absorbing second submodule of  $M$  and the proof is completed.  $\square$

**Proposition 2.6.** Let  $M$  be an  $R$ -module and  $N \subseteq K$  be two submodules of  $M$ . Then the 2-absorbing purity relation satisfies the following.

- (a) *Transitive:* If  $N$  is a 2-pure submodule of  $K$  and  $K$  is a 2-pure submodule of  $M$ , then  $N$  is a 2-pure submodule of  $M$ .
- (b) *Hereditary:* If  $K$  is a 2-pure submodule of  $M$ , then  $K/N$  is a 2-pure submodule of  $M/N$ .
- (c) If  $K/N$  is a 2-pure submodule of  $M/N$  and  $N$  is a pure submodule of  $M$ , then  $K$  is a 2-pure submodule of  $M$ .

*Proof.* (a) and (b) are clear.

(c) Let  $I$  and  $J$  be two proper ideals of  $R$ . Since  $K/N$  is a 2-pure submodule of  $M/N$ , we have

$$IJK + N = (IK + N) \cap (JK + N) \cap (IJM + N) \supseteq (IK \cap JK \cap IJM) + N.$$

Now let  $x \in IK \cap JK \cap IJM$ . Then  $x + N \in IJK + N$ . Let  $n_1 \in N$ . Then  $x + n_1 = y + n_2$  for some  $y \in IJK$  and  $n_2 \in N$ . This implies that  $x - y \in N$ . Therefore,  $x - y \in N \cap IJM = IJN$ . It follows that  $x - y \in IJK$  and so  $x \in IJK$ . Thus  $JK \cap IK \cap IJM \subseteq IJK$ . This implies that  $K$  is a 2-pure submodule of  $M$  since the reverse inclusion is clear.  $\square$

**Proposition 2.7.** *Let  $R$  be a PID,  $N$  a submodule of an  $R$ -module  $M$ , and  $p_i$  ( $i \in \mathbb{N}$ ) be a prime element in  $R$ . Then  $p_1^{s_1} \dots p_t^{s_t} N = \bigcap_{i=1}^t p_i^{s_i} N$ .*

*Proof.* Let  $p$  and  $q$  be two prime elements in  $R$  and  $k, s \in \mathbb{N}$ . Clearly,  $p^k q^s N \subseteq p^t N \cap q^s N$ . Now let  $x \in p^k N \cap q^s N$ . Then  $x = p^k n_1 = q^s n_2$  for some  $n_1, n_2 \in N$ . Since  $R$  is a PID,  $p^k R + q^s R = R$ . Thus there exist  $a, b \in R$  such that  $1 = ap^k + bq^s$ . Hence  $x = 1x = ap^k x + bq^s x = ap^k q^s n_2 + bq^s p^k n_1$ . Thus  $x \in p^k q^s N$ . Therefore,  $p^k q^s N = p^t N \cap q^s N$ . Now the result follows by induction on  $t$ .  $\square$

**Corollary 2.8.** *Let  $M$  be a  $\mathbb{Z}$ -module,  $m, n$  be square-free integers such that  $(m, n) = 1$ . Then for each submodule  $N$  of  $M$  we have  $(n\mathbb{Z})(m\mathbb{Z})N = (n\mathbb{Z})N \cap (m\mathbb{Z})N$ .*

*Proof.* This follows from Proposition 2.7.  $\square$

**Proposition 2.9.** *A submodule  $N$  of an  $R$ -module  $M$  is a 2-pure submodule if and only if  $N_P$  is a 2-pure submodule of  $M_P$  for every maximal ideal  $P$  of  $R$ .*

*Proof.* This follows from the fact that for each ideal  $I$  and  $J$  of  $R$ , by [11, 9.11],

$$I_P J_P N_P = (IJN)_P = (IN \cap JN \cap IJM)_P = I_P N_P \cap J_P N_P \cap I_P J_P M_P.$$

$\square$

**Proposition 2.10.** *Let  $M$  be an  $R$ -module. Then we have the following.*

- (a) *If  $\{N_\lambda\}_{\lambda \in \Lambda}$  is a chain of 2-pure submodules of  $M$ , then  $\bigcup_{\lambda \in \Lambda} N_\lambda$  is a 2-pure submodules of  $M$ .*
- (b) *If  $\{N_\lambda\}_{\lambda \in \Lambda}$  is a chain of submodules of  $M$  and  $K$  is a 2-pure submodule of  $N_\lambda$  for each  $\lambda \in \Lambda$ , then  $K$  is a 2-pure submodule of  $\bigcup_{\lambda \in \Lambda} N_\lambda$ .*

*Proof.* (a) Let  $I$  and  $J$  be two proper ideals of  $R$ . Clearly,

$$IJ(\bigcup_{\lambda \in \Lambda} N_\lambda) \subseteq I(\bigcup_{\lambda \in \Lambda} N_\lambda) \cap J(\bigcup_{\lambda \in \Lambda} N_\lambda) \cap IJM.$$

Now let  $x \in I(\bigcup_{\lambda \in \Lambda} N_\lambda) \cap J(\bigcup_{\lambda \in \Lambda} N_\lambda) \cap IJM$ . Then  $x = \sum_{i=1}^n a_i n_i = \sum_{j=1}^m b_j m_j$ , where  $a_i \in I$ ,  $b_j \in J$  and  $n_i, m_j \in \bigcup_{\lambda \in \Lambda} N_\lambda$ . Since  $\{N_\lambda\}_{\lambda \in \Lambda}$  is a chain, there exists  $\alpha \in \Lambda$  such that  $x \in IN_\alpha$  and  $x \in JN_\alpha$ . Thus  $x \in IN_\alpha \cap JN_\alpha \cap IJM$ . This implies that  $x \in IJN_\alpha \subseteq IJ(\bigcup_{\lambda \in \Lambda} N_\lambda)$  as needed.

(b) Let  $I$  and  $J$  be two proper ideals of  $R$ . Clearly,

$$IJK \subseteq IK \cap JK \cap IJ(\bigcup_{\lambda \in \Lambda} N_\lambda).$$

Now let  $x \in IK \cap JK \cap IJ(\bigcup_{\lambda \in \Lambda} N_\lambda)$ . Then  $x = \sum_{i=1}^n a_i b_i n_i$ , where  $a_i \in I$ ,  $b_j \in J$  and  $n_i \in \bigcup_{\lambda \in \Lambda} N_\lambda$ . Since  $\{N_\lambda\}_{\lambda \in \Lambda}$  is a chain, there exists  $\beta \in \Lambda$  such that  $x \in IJN_\beta$ . Thus  $x \in IK \cap JK \cap IJN_\beta$ . This in turn implies that  $x \in IJK$  as required.  $\square$

**Theorem 2.11.** *Let  $N$  be a submodule of an  $R$ -module  $M$ . Then there is a submodule  $K$  of  $N$  maximal with respect to  $K \subseteq N$  and  $K$  is a 2-pure submodule of  $M$ .*

*Proof.* Let

$$\Sigma = \{H \leq N \mid H \text{ is a 2-absorbing pure submodule of } M\}.$$

Then  $0 \in \Sigma$  implies that  $\Sigma \neq \emptyset$ . Let  $\{N_\lambda\}_{\lambda \in \Lambda}$  be a totally ordered subset of  $\Sigma$ . Then  $\bigcup_{\lambda \in \Lambda} N_\lambda \leq N$  and by Proposition 2.10 (a),  $\bigcup_{\lambda \in \Lambda} N_\lambda$  is a 2-pure submodule of  $M$ . Thus by using Zorn's Lemma, one can see that  $\Sigma$  has a maximal element,  $K$  say as needed.  $\square$

**Definition 2.12.** *We say that a pure submodule  $N$  of an  $R$ -module  $M$  is a maximal pure submodule of a submodule  $K$  of  $M$ , if  $N \subseteq K$  and there does not exist a pure submodule  $H$  of  $M$  such that  $N \subset H \subset K$ .*

An  $R$ -module  $M$  is called a *fully cancellation module* if for each non-zero ideal  $I$  of  $R$  and for each submodules  $N_1, N_2$  of  $M$  such that  $IN_1 = IN_2$  implies  $N_1 = N_2$  [8].

**Theorem 2.13.** *Every Artinian fully cancellation  $R$ -module  $M$  has only a finite number of maximal pure submodules.*

*Proof.* Suppose that the result is false. Let  $\Sigma$  denote the collection of non-zero submodules  $N$  of  $M$  such that  $N$  has an infinite number of maximal pure submodules. The collection  $\Sigma$  is non-empty because  $M \in \Sigma$  and hence has a minimal member,  $S$  say. Then  $S$  is not a pure submodule. Thus there exists an ideal  $I$  of  $R$  such that  $IS \neq S \cap IM$ . Let  $V$  be a maximal pure submodule of  $M$  contained in  $S$ . If  $IS + V = S$ , then  $(IS + V) \cap IM = S \cap IM$ . Hence by the modular law,  $IS + V \cap IM = S \cap IM$ . Now as  $V$  is a pure submodule of  $M$ ,  $IS + IV = S \cap IM$ . It follows that  $S \cap IM \subseteq IS$ , a contradiction. If  $IS + V = V$ , then  $IS \subseteq V$  and so  $IS = IS \cap IM \subseteq V \cap IM = IV$ . Thus  $IV = IS$ . Since  $M$  is a fully cancellation  $R$ -module,  $V = S$ , a contradiction. Therefore,  $V \subset IS + V \subset S$ . Now by the choice of  $S$ , the module  $IS + V$  has only finitely many maximal pure submodules. Therefore, there is only a finite number of possibilities for the module  $S$  which is a contradiction.  $\square$

**Definition 2.14.** *We say that an  $R$ -module  $M$  is fully 2-pure if every submodule of  $M$  is 2-pure.*

Let  $N$  and  $K$  be two submodules of  $M$ . The *product* of  $N$  and  $K$  is defined by  $(N :_R M)(K :_R M)M$  and denoted by  $NK$  [3].

**Theorem 2.15.** *Let  $M$  be a multiplication  $R$ -module. Then the following statements are equivalent.*

- (a) *For submodules  $N, K, H$  of  $M$ , we have  $NHK = NK \cap NH \cap KH$ .*
- (b)  *$M$  is a fully 2-pure  $R$ -module.*

*Proof.* (a)  $\Rightarrow$  (b). Let  $N$  be a submodule of  $M$  and  $I, J$  be two proper ideals of  $R$ . Then by part (a) and the fact that  $M$  is a multiplication  $R$ -module,

$$\begin{aligned} IN \cap JN \cap IJM &= (IM)(N) \cap (JM)(N) \cap (IJM)(M) = (IN)(JN)(IJM) \subseteq \\ &= (N^2)(IJM) \subseteq N(IJM) \subseteq IJN. \end{aligned}$$

The reverse inclusion is clear.

(b)  $\Rightarrow$  (a). We have

$$\begin{aligned} NK \cap NH \cap KH &= (N :_R M)(K :_R M)M \cap (N :_R M)H \cap (K :_R M)H = \\ &= (N :_R M)(K :_R M)H = NKH. \end{aligned}$$

$\square$

**Theorem 2.16.** *Let  $M$  be a finitely generated faithful multiplication  $R$ -module and  $N$  be a submodule of  $M$ . Then  $N$  is a 2-pure submodule of  $M$  if and only if  $(N :_R M)$  is a 2-pure ideal of  $R$ .*

*Proof.* Since  $M$  is a multiplication  $R$ -module,  $N = (N :_R M)M$ . Let  $N$  be a 2-pure submodule of  $M$  and let  $I$  and  $J$  be any two proper ideals of  $R$ . Then  $IJN = IN \cap JN \cap IJM$ . Hence,

$$IJ(N :_R M)M = I(N :_R M)M \cap J(N :_R M)M \cap IJM.$$

This implies that  $IJ(N :_R M)M \supseteq (I(N :_R M) \cap J(N :_R M) \cap IJ)M$ . Now by [7, 3.1],  $IJ(N :_R M) \supseteq I(N :_R M) \cap J(N :_R M) \cap IJ$ . This implies that  $(N :_R M)$  is a 2-pure ideal of  $R$  since the reverse inclusion is clear. Conversely, let  $(N :_R M)$  be a 2-pure ideal of  $R$  and let  $I$  and  $J$  be any two proper ideals of  $R$ . Then  $IJ(N :_R M) = I(N :_R M) \cap J(N :_R M) \cap IJ$ . Hence  $IJ(N :_R M)M = (I(N :_R M) \cap J(N :_R M) \cap IJ)M$ . Thus  $IJ(N :_R M)M = I(N :_R M)M \cap J(N :_R M)M \cap IJM$  by [7, 3.1]. Therefore,  $IJN = IN \cap JN \cap IJM$  as desired.  $\square$

### 3 $n$ -pure submodules

**Definition 3.1.** Let  $n$  be a positive integer. We say that a submodule  $N$  of an  $R$ -module  $M$  is a  $n$ -pure submodule of  $M$  if  $I_1 I_2 \dots I_n N = I_1 N \cap I_2 N \cap \dots \cap I_n N \cap (I_1 I_2 \dots I_n)M$  for all proper ideals  $I_1, I_2, \dots, I_n$  of  $R$ . Also, we say that an ideal  $I$  of  $R$  is a  $n$ -pure ideal of  $R$  if  $I$  is a  $n$ -pure submodule of  $R$ .

**Remark 3.2.** Let  $n$  be a positive integer. Clearly every  $(n-1)$ -pure submodule of an  $R$ -module  $M$  is a  $n$ -pure submodule of  $M$ . But we see in the Example 3.3 that the converse is not true in general.

**Example 3.3.** Let  $n$  be a positive integer. The submodule  $2\mathbb{Z}_{2^n}$  of the  $\mathbb{Z}_{2^n}$ -module  $\mathbb{Z}_{2^n}$  is a  $n$ -pure submodule but it is not a  $(n-1)$ -pure submodule.

**Example 3.4.** Let  $n$  be a positive integer.

(a) The submodule  $2\mathbb{Z}$  of the  $\mathbb{Z}$ -module  $\mathbb{Z}$  is not a  $n$ -pure submodule.

(b) Let  $n > 1$  be an integer. Since

$$2^n \mathbb{Z} = \underbrace{(2\mathbb{Z}) \dots (2\mathbb{Z})}_{n \text{ times}} \mathbb{Z} \neq \underbrace{(2\mathbb{Z})\mathbb{Z} \cap (2\mathbb{Z})\mathbb{Z} \cap \dots \cap (2\mathbb{Z})\mathbb{Z}}_{n \text{ times}} \cap (2^n \mathbb{Z})\mathbb{Q} = 2\mathbb{Z},$$

the submodule  $\mathbb{Z}$  of the  $\mathbb{Z}$ -module  $\mathbb{Q}$  is not  $n$ -pure.

**Proposition 3.5.** Let  $N$  be a submodule of an  $R$ -module  $M$  and  $n$  be a positive integer. If  $I_1 \dots I_n N = I_1 N \cap I_2 N \dots \cap I_n N$  for all proper ideals  $I_1, I_2, \dots, I_n$  of  $R$ , then  $N$  is a  $n$ -pure submodule of  $M$ .

*Proof.* Use the technique of Proposition 2.5 (a). □

**Proposition 3.6.** Let  $M$  be an  $R$ -module,  $N \subseteq K$  be two submodules of  $M$ , and  $n$  be a positive integer. Then the  $n$ -absorbing purity relation satisfies the following.

(a) *Transitive:* If  $N$  is a  $n$ -pure submodule of  $K$  and  $K$  is a  $n$ -pure submodule of  $M$ , then  $N$  is a  $n$ -pure submodule of  $M$ .

(b) *Hereditary:* If  $K$  is a  $n$ -pure submodule of  $M$ , then  $K/N$  is a  $n$ -pure submodule of  $M/N$ .

(c) If  $K/N$  is a  $n$ -pure submodule of  $M/N$  and  $N$  is a pure submodule of  $M$ , then  $K$  is a  $n$ -pure submodule of  $M$ .

*Proof.* Use the technique of Proposition 2.11. □

**Proposition 3.7.** Let  $n$  be a positive integer. A submodule  $N$  of an  $R$ -module  $M$  is a  $n$ -pure submodule if and only if  $N_P$  is a  $n$ -pure submodule of  $M_P$  for every maximal ideal  $P$  of  $R$ .

*Proof.* Use the technique of Proposition 2.9. □

**Proposition 3.8.** Let  $M$  be an  $R$ -module and  $n$  be a positive integer. Then we have the following.

(a) If  $\{N_\lambda\}_{\lambda \in \Lambda}$  is a chain of  $n$ -pure submodules of  $M$ , then  $\cup_{\lambda \in \Lambda} N_\lambda$  is a  $n$ -pure submodules of  $M$ .

(b) If  $\{N_\lambda\}_{\lambda \in \Lambda}$  is a chain of submodules of  $M$  and  $K$  is a  $n$ -pure submodule of  $N_\lambda$  for each  $\lambda \in \Lambda$ , then  $K$  is a  $n$ -pure submodule of  $\cup_{\lambda \in \Lambda} N_\lambda$ .

*Proof.* Use the technique of Proposition 2.10. □

**Theorem 3.9.** Let  $N$  be a submodule of an  $R$ -module  $M$  and let  $n$  be a positive integer. Then there is a submodule  $K$  of  $N$  maximal with respect to  $K \subseteq N$  and  $K$  is a  $n$ -pure submodule of  $M$ .

*Proof.* Use the technique of Theorem 2.11. □

**Definition 3.10.** Let  $n$  be a positive integer. We say that an  $R$ -module  $M$  is fully  $n$ -pure if every submodule of  $M$  is  $n$ -pure.

Let  $N$  and  $K$  be two submodules of  $M$ . The *product* of  $N$  and  $K$  is defined by  $(N :_R M)(K :_R M)M$  and denoted by  $NK$  [3].

**Theorem 3.11.** Let  $M$  be a multiplication  $R$ -module and let  $n$  be a positive integer. Then the following statements are equivalent.

- (a) For submodules  $N_1, N_2, \dots, N_n$  of  $M$ , we have  $N_1N_2\dots N_n = N_1N_2 \cap N_1N_3 \cap \dots \cap N_1N_n \cap (N_2N_3\dots N_n)$ .
- (b)  $M$  is a fully  $n$ -pure  $R$ -module.

*Proof.* Use the technique of Theorem 2.15. □

**Theorem 3.12.** Let  $M$  be a finitely generated faithful multiplication  $R$ -module,  $N$  be a submodule of  $M$ , and let  $n$  be a positive integer. Then  $N$  is a  $n$ -pure submodule of  $M$  if and only if  $(N :_R M)$  is a  $n$ -pure ideal of  $R$ .

*Proof.* Use the technique of Theorem 2.16. □

**Acknowledgement(s):** I would like to thank the referee(s) for his comments and suggestions on the manuscript.

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