

Some fixed point results on C-class functions for generalized cyclic contractive mappings in b -metric spaces

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Abstract: The main objective of this paper is to utilize C -class sum generalized cyclic contractive rational type mappings and investigate the existence of fixed point with an auxiliary and control function in b -metric spaces. As an applications of our results, we have deduce some new and interesting corollaries. An example and some remarks also have been given to notify the importance of our main results.

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1 Introduction

From last one decade, several researchers have focused on different contractive conditions in metric spaces endowed with a partial order and obtained many fixed point results in such spaces. Ran and Reurings [20] gave the first result on fixed point with partial order as a generalization of Banach contraction principle. Nieto and Lopez [18] remove the continuity condition on mappings and extended their (Ran and Reurings) result by utilizing the notion of non-decreasing mappings. Suzuki [25] proved a generalization of Banach contraction principle by using a contraction condition depending on a non-increasing function. Recently, Han and Hieu [12] introduced the notion of generalized cyclic contractive mapping in b -metric spaces. Gupta et al. [11] proved a new fixed point theorem satisfying generalized Geraghty type contraction for pair of self mappings in partially ordered metric spaces. For more details on fixed point results with different kind of contractions, we refer the readers to see [10, 23, 26, 22, 17, 24, 19] and the references there in.

In 1993, Czerwik [7] introduced the concept of b -metric space. A b -metric is a metric if we take $s = 1$. Every metric is a b -metric, but in general a b -metric need not to be a metric. Following example (due to Aghajani [1])

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justify the statement.

Example 1.1. [1] Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p \geq 1$ is a real number. Then ρ is a b -metric with $s = 2^{p-1}$. However, (X, ρ) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b -metric on \mathbb{R} with $s = 2$, but it is not a metric on \mathbb{R} .

In 2009, Boriceanu [6] gave the following definitions and propositions on b -metric space and proved some fixed point theorems.

Definition 1.2. [6] Let (X, d) be a b -metric space. Then a sequence $\{x_n\}$ in X is called:

1. b -convergent if and only if there exists $x \in X$ such that $d(x_n, x) \rightarrow 0$, as $n \rightarrow \infty$. In this case, we write $\lim_{n \rightarrow \infty} x_n = x$.
2. b -Cauchy if and only if $d(x_n, x_m) \rightarrow 0$, as $n, m \rightarrow \infty$.

Proposition 1.3. [6] In a b -metric space (X, d) , the following assertions hold:

1. A b -convergent sequence has a unique limit.
2. Each b -convergent sequence is b -Cauchy.

Definition 1.4. [6] A b -metric space (X, d) is called b -complete if every b -Cauchy sequence in X is b -convergent.

Remark 1.5. The first important difference between a metric and a b -metric is that the b -metric need not be a continuous function in its two variables.

Aghajani et al. [1] proved the following lemma about the convergence in b -metric spaces.

Lemma 1.6. [1] Let (X, d) be a b -metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b -convergent to x, y , respectively. Then we have

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow \infty} d(x_n, z) \leq \limsup_{n \rightarrow \infty} d(x_n, z) \leq sd(x, z).$$

Han and Hieu [12] gave the following lemma, which is the equivalent condition for the Cauchy property of $\{x_n\}$ in b -metric spaces.

Lemma 1.7. [12] Let (X, d, s) be a b -metric space, and $\{x_n\}$ be a sequence in (X, d, s) . Then the following statements are equivalent.

1. $\{x_n\}$ is a Cauchy Sequence in (X, d, s) ,
2. $\{x_{2n}\}$ is a Cauchy Sequence in (X, d, s) and $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Definition 1.8. [13] A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

- (i) ψ is non-decreasing and continuous,
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.9. We let Ψ denote the class of the altering distance functions.

Definition 1.10. A function $\psi : R \rightarrow R$ is called an infinite altering distance function, if the following properties are satisfied:

- (i) ψ is non-decreasing and continuous,
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.11. We let Ψ_{inf} denote the class of the infinite altering distance functions.

Definition 1.12. [2] We say $\phi : [0, +\infty) \rightarrow [0, +\infty)$ ultra distance function if it is continuous and $\phi(0) \geq 0$, and $\phi(t) > 0, t > 0$.

Remark 1.13. We let Φ_u denote the class of the ultra altering distance functions.

Han and Hieu [12] introduced the notions of generalized cyclic contractive mapping and proved a unique fixed point theorem.

Definition 1.14. [12] Let (X, d, s) be a b-metric space, A and B be non-empty subsets of $X, Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Then T is called a generalized cyclic contractive mapping if:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exist $\psi \in \Psi, \varphi \in \Phi_u$ and a constant $L \geq 0$, such that

$$\psi(s^4 d(Tx, Ty)) \leq \phi(\psi(M(x, y))) + LN(x, y)$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2s}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) \right\}$$

and

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(T^2x, T^2y)\}.$$

Theorem 1.15. [12] Let (X, d, s) be a b-metric space, A and B be non-empty closed subsets of $X, Y = A \cup B$ and $T : Y \rightarrow Y$ be a generalized cyclic contractive mapping. Then T has a unique fixed point in $A \cap B$.

In 2014-15, Ansari [2, 3] introduced the notion of C -class function as a major generalization of Banach contraction principle. Author introduced the following definitions and examples to justify the importance of his result.

Definition 1.16. [2] A family of mappings $F : [0, \infty)^2 \rightarrow \mathbb{R}$ is called C -class function if it is continuous and satisfies following axioms:

1. $F(r, t) \leq r$;
2. $F(r, t) = r$ implies that either $r = 0$ or $t = 0$; for all $r, t \in [0, \infty)$.

For brevity, we denote C -class functions as \mathcal{C} .

Remark 1.17. Clearly, for some F we have $F(0, 0) = 0$.

Example 1.18. [2] The following functions $F : [0, \infty)^2 \rightarrow \mathbb{R}$ are elements of \mathcal{C} , for all $r, t \in [0, \infty)$:

1. $F(r, t) = r - t, F(r, t) = r \Rightarrow t = 0$;
2. $F(r, t) = mr, 0 < m < 1, F(r, t) = r \Rightarrow r = 0$;
3. $F(r, t) = \frac{r}{(1+t)^h}; h \in (0, \infty), F(r, t) = r \Rightarrow r = 0$ or $t = 0$;
4. $F(r, t) = \log(t + a^r)/(1 + t), a > 1, F(r, t) = r \Rightarrow r = 0$ or $t = 0$;

5. $F(r, t) = \ln(1 + a^r)/2$, $a > e$, $F(r, t) = r \Rightarrow r = 0$;
6. $F(r, t) = (r + l)^{(1/(1+t)^p)} - l$, $l > 1$, $p \in (0, \infty)$, $F(r, t) = r \Rightarrow t = 0$;
7. $F(r, t) = r \log_{t+a} a$, $a > 1$, $F(r, t) = r \Rightarrow r = 0$ or $t = 0$;
8. $F(r, t) = r - (\frac{1+r}{2+r})(\frac{t}{1+t})$, $F(r, t) = r \Rightarrow t = 0$;
9. $F(r, t) = r\beta(r)$, $\beta : [0, \infty) \rightarrow [0, 1)$, $F(r, t) = r \Rightarrow r = 0$;
10. $F(r, t) = r - \frac{t}{k+t}$, $F(r, t) = r \Rightarrow t = 0$;
11. $F(r, t) = r - \varphi(r)$, $F(r, t) = r \Rightarrow r = 0$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
12. $F(r, t) = rh(r, t)$, $F(r, t) = r \Rightarrow r = 0$, where $h : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $h(r, t) < 1$ for all $t, s > 0$;
13. $F(r, t) = r - (\frac{2+t}{1+t})t$, $F(r, t) = r \Rightarrow t = 0$;
14. $F(r, t) = \sqrt[n]{\ln(1 + r^n)}$, $F(r, t) = r \Rightarrow r = 0$;
15. $F(r, t) = \phi(r)$, $F(r, t) = r \Rightarrow r = 0$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a upper semi-continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,
16. $F(r, t) = \frac{r}{(1+r)^s}$; $s \in (0, \infty)$, $F(r, t) = r$ implies $r = 0$.

Lemma 1.19. [3] Let $\{x_n\}$ be a sequence in a b -metric type space (X, d) such that

$$d(x_n, x_{n+1}) \leq \lambda d(x_n, x_{n-1})$$

for some λ , $0 < \lambda < \frac{1}{s}$, and for $n = 1, 2, \dots$. Then $\{x_n\}$ is a b -Cauchy sequence in (X, d) .

In 2018, Saleem et al. [21] introduced the concept of the inverse C -class function and established several fixed point theorems. Subsequently, other researchers applied this concept to prove analogous versions of well-known fixed point theorems [4, 9, 5, 14, 15, 16]. Recently, Ansari, Gupta and Mani [8] gave a sufficient condition for the existence and uniqueness of common fixed points pair of self-maps satisfying a generalized \int_{φ}^{ψ} -weakly contractive condition involving C -class functions in complete metric spaces.

In this paper, we first introduce a C -class generalized cyclic contractive mappings and then prove a fixed point result in the set up of b -metric spaces with the help of monotone triplet (ψ, φ, F) . In next section, as an applications, we have deduce some new and interesting corollaries.

2 Main Results

The following theorem is the main result of this paper.

Definition 2.1. A triplet (ψ, φ, F) , where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$, is say to be monotone, if for any $x, y \in [0, \infty)$

$$x \leq y \implies F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y)).$$

Example 2.2. Let $F(s, t) = s - t$, $\phi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x > 1 \end{cases},$$

then (ψ, ϕ, F) is monotone.

Example 2.3. Let $F(s, t) = s - t$, $\phi(x) = x^2$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x > 1 \end{cases},$$

then triplet (ψ, ϕ, F) is not monotone.

Definition 2.4. Let (X, d, s) be a b -metric space, A and B be non-empty subsets of X , $Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Then T is called a C -class sum generalized cyclic contractive mapping if:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exist $F \in C$, $\psi \in \Psi_{\text{inf}}$, $\varphi \in \Phi_u$ such that (ψ, ϕ, F) is monotone and a constant $L \geq 0, \varepsilon > 0$ such that

$$\psi(s^{1+\varepsilon} d(Tx, Ty)) \leq F(\psi(M(x, y)), \varphi(M(x, y))) + LN(x, y) \quad (2.1)$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where

$$M(x, y) = \frac{1}{a + b + c + 2e + 2f + r + s + h} \left\{ \begin{array}{l} ad(x, y) + bd(x, Tx) + cd(y, Ty) \\ + e \frac{d(x, Ty) + d(y, Tx)}{d(T^2x, x) + d(T^2x, Ty)} \\ + f \frac{d(T^2x, x) + d(T^2x, Ty)}{+rd(T^2x, Tx)} \\ + sd(T^2x, y) + hd(T^2x, Ty) \end{array} \right\} \quad (2.2)$$

$a, b, c, e, f, r, s, h \geq 0$, with $a + b + c + e + f + r + s + h > 0$ and

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(T^2x, T^2y)\}. \quad (2.3)$$

Theorem 2.5. Let (X, d, s) be a b -metric space, A and B be non-empty closed subsets of X , $Y = A \cup B$ and $T : Y \rightarrow Y$ be a C -class sum generalized cyclic contractive mapping. Then T has a unique fixed point in $A \cap B$.

Proof. Let $x_0 \in A$. We construct the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. Since $x_0 \in A$, $x_1 = Tx_0 \in T(A) \subset B$. So, $x_2 = Tx_1 \in T(B) \subset A$. Continuing this process for all $n \geq 0$, we have

$$x_{2n} \in A \text{ and } x_{2n+1} \in B \quad (2.4)$$

If there exists $k \geq 0$ such that $x_{k+1} = x_k$, then $Tx_k = x_k$, that is, x_k is a fixed point of T . Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. From (2.4), we have $(x_{2n-1}, x_{2n}) \in B \times A$. Since T is a C -class generalized cyclic contractive mapping, we have

$$\begin{aligned} \psi(s^{1+\varepsilon}(d(x_{2n}, x_{2n+1}))) &= \psi(s^{1+\varepsilon} d(Tx_{2n-1}, Tx_{2n})) \\ &\leq F(\psi(M(x_{2n-1}, x_{2n})), \varphi(M(x_{2n-1}, x_{2n}))) \\ &\quad + LN(x_{2n-1}, x_{2n}) \\ &\leq \psi(M(x_{2n-1}, x_{2n})) + LN(x_{2n-1}, x_{2n}), \end{aligned} \quad (2.5)$$

where,

$$\begin{aligned}
 &M(x_{2n-1}, x_{2n}) \\
 &= \frac{1}{a + b + c + 2e + 2f + r + s + h} \left\{ \begin{aligned} &ad(x_{2n-1}, x_{2n}) + bd(x_{2n-1}, Tx_{2n-1}) \\ &+ cd(x_{2n}, Tx_{2n}) \\ &+ e \frac{d(x_{2n-1}, Tx_{2n}) + d(x_{2n}, Tx_{2n-1})}{s} \\ &+ f \frac{d(T^2x_{2n-1}, x_{2n-1}) + d(T^2x_{2n-1}, Tx_{2n})}{s} \\ &+ rd(T^2x_{2n-1}, Tx_{2n-1}) + \\ &sd(T^2x_{2n-1}, x_{2n}) + hd(T^2x_{2n-1}, Tx_{2n}) \end{aligned} \right\} \\
 &= \frac{1}{a + b + c + 2e + 2f + r + s + h} \left\{ \begin{aligned} &ad(x_{2n-1}, x_{2n}) + bd(x_{2n-1}, x_{2n}) \\ &+ cd(x_{2n}, x_{2n+1}) \\ &+ e \frac{d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})}{s} \\ &+ f \frac{d(x_{2n+1}, x_{2n-1}) + d(x_{2n+1}, x_{2n+1})}{s} \\ &+ rd(x_{2n+1}, x_{2n}) + sd(x_{2n+1}, x_{2n}) \\ &+ hd(x_{2n+1}, x_{2n+1}) \end{aligned} \right\} \\
 &= \frac{1}{a + b + c + 2e + 2f + r + s + h} \left\{ \begin{aligned} &ad(x_{2n-1}, x_{2n}) + bd(x_{2n-1}, x_{2n}) \\ &+ cd(x_{2n}, x_{2n+1}) \\ &+ e s \frac{d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})}{s} \\ &+ f s \frac{d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})}{s} \\ &+ rd(x_{2n+1}, x_{2n}) + sd(x_{2n+1}, x_{2n}) \\ &+ hd(x_{2n+1}, x_{2n+1}) \end{aligned} \right\} \\
 &\leq \frac{1}{a + b + c + 2e + 2f + r + s + h} \left\{ \begin{aligned} &(a + b + e + f)d(x_{2n-1}, x_{2n}) + \\ &(c + e + f + r + s)d(x_{2n}, x_{2n+1}) \end{aligned} \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 N(x_{2n-1}, x_{2n}) &= \min \{d(x_{2n-1}, Tx_{2n-1}), d(x_{2n}, Tx_{2n-1}), d(T^2x_{2n-1}, T^2x_{2n})\} \\
 &= \min \{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n}), d(x_{2n+1}, x_{2n+2})\} = 0.
 \end{aligned}$$

From (2.5)

$$\begin{aligned}
 \psi(s^{1+\varepsilon}(d(x_{2n}, x_{2n+1}))) &\leq \psi \left\{ \frac{1}{a+b+c+2e+2f+r+s+h} \left\{ \begin{aligned} &(a + b + e + f)d(x_{2n-1}, x_{2n}) + \\ &(c + e + f + r + s)d(x_{2n}, x_{2n+1}) \end{aligned} \right\} \right\} \\
 &+ L(0),
 \end{aligned}$$

Using the properties of ψ , we get

$$s^{1+\varepsilon}(d(x_{2n}, x_{2n+1})) \leq \frac{1}{a + b + c + 2e + 2f + r + s + h} \left\{ \begin{aligned} &(a + b + e + f)d(x_{2n-1}, x_{2n}) + \\ &(c + e + f + r + s)d(x_{2n}, x_{2n+1}) \end{aligned} \right\}$$

Consequently, we get

$$d(x_{2n}, x_{2n+1}) \leq \frac{1}{s^{1+\varepsilon}} d(x_{2n-1}, x_{2n}) \tag{2.6}$$

Similarly, we get

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s^{1+\varepsilon}} d(x_{2n}, x_{2n+1}) \tag{2.7}$$

Therefore, from (2.6) and (2.7) for all n , we have

$$d(x_{n+1}, x_n) \leq \frac{1}{s^{1+\varepsilon}} d(x_n, x_{n-1})$$

By induction, we get

$$\begin{aligned}
 d(x_{n+1}, x_n) &\leq \left(\frac{1}{s^{1+\varepsilon}}\right) d(x_n, x_{n-1}) \\
 &\leq \left(\frac{1}{s^{1+\varepsilon}}\right)^2 d(x_{n-1}, x_{n-2}) \\
 &\leq \dots \\
 &\leq \left(\frac{1}{s^{1+\varepsilon}}\right)^n d(x_1, x_0).
 \end{aligned} \tag{2.8}$$

Letting $n \rightarrow \infty$ in (2.8), we get

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{2.9}$$

Step II. By Lemma 1.19, $\{x_n\}$ is a b -Cauchy sequence, completeness of X yields that $\{x_n\}$ converges to a point $u \in X$. We shall prove that $u \in A \cap B$. Since $\{x_{2n}\} \subset A$, $\{x_{2n+1}\} \subset B$ and A, B are closed subsets of X , we have $u \in A$ and $u \in B$. Therefore, $u \in A \cap B$.

Step III. We shall now prove that u is a fixed point of T .

Since $(x_{2n}, u) \in A \times B$ and T is a C -class generalized cyclic contractive mapping,

$$\begin{aligned}
 \psi(s^{1+\varepsilon} d(x_{2n+1}, Tu)) &= \psi(s^{1+\varepsilon} d(Tx_{2n}, Tu)) \\
 &\leq F(\psi(M(x_{2n}, u)), \varphi(M(x_{2n}, u))) + LN(x_{2n}, u) \\
 &\leq \psi(M(x_{2n}, u)) + LN(x_{2n}, u)
 \end{aligned} \tag{2.10}$$

where,

$$\begin{aligned}
 M(x_{2n}, u) &= \frac{1}{a+b+c+2e+2f+r+s+h} \left\{ \begin{array}{l} ad(x_{2n}, u) + bd(x_{2n}, Tx_{2n}) \\ +cd(u, Tu) \\ +e \frac{d(x_{2n}, Tu) + d(u, Tx_{2n})}{s} \\ +f \frac{d(T^2x_{2n}, x_{2n}) + d(T^2x_{2n}, Tu)}{s} \\ +rd(T^2x_{2n}, Tx_{2n}) \\ +sd(T^2x_{2n}, u) + hd(T^2x_{2n}, Tu) \end{array} \right\} \\
 &= \frac{1}{a+b+c+2e+2f+r+s+h} \left\{ \begin{array}{l} ad(x_{2n}, u) + bd(x_{2n}, x_{2n+1}) \\ +cd(u, Tu) + \\ +e \frac{d(x_{2n}, Tu) + d(u, x_{2n+1})}{s} \\ +f \frac{d(x_{2n+2}, x_{2n}) + d(x_{2n+2}, Tu)}{s} \\ +rd(x_{2n+2}, x_{2n+1}) + sd(x_{2n+2}, u) \\ +hd(x_{2n+2}, Tu) \end{array} \right\} \\
 &\leq \frac{1}{a+b+c+2e+2f+r+s+h} \left\{ \begin{array}{l} ad(x_{2n}, u) + bd(x_{2n}, x_{2n+1}) \\ +cd(u, Tu) \\ +e \frac{sd(x_{2n}, u) + sd(u, Tu) + d(u, x_{2n+1})}{s} \\ +f \frac{d(x_{2n+2}, x_{2n}) + sd(x_{2n+2}, u) + sd(u, Tu)}{s} \\ +rd(x_{2n+2}, x_{2n+1}) + sd(x_{2n+2}, u) \\ +hd(x_{2n+2}, Tu) \end{array} \right\}
 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
 N(x_{2n}, u) &= \min \{d(x_{2n}, Tx_{2n}), d(u, Tx_{2n}), d(T^2x_{2n}, T^2u)\} \\
 &= \min \{d(x_{2n}, u), d(u, x_{2n+1}), d(x_{2n+2}, T^2u)\} = 0.
 \end{aligned} \tag{2.12}$$

Taking the upper limit as $n \rightarrow \infty$ in (2.11) and (2.12), we have

$$\begin{aligned}\limsup_{n \rightarrow \infty} M(x_{2n}, u) &= \frac{c + e + f + h}{a + b + c + 2e + 2f + r + s + h} d(u, Tu) \\ \limsup_{n \rightarrow \infty} N(x_{2n}, u) &= 0.\end{aligned}\quad (2.13)$$

Consider,

$$\begin{aligned}\psi(d(u, Tu) - sd(u, x_{2n+1})) &\leq \psi(sd(x_{2n+1}, Tu)) \\ &\leq \psi(s^{1+\varepsilon} d(x_{2n+1}, Tu)).\end{aligned}\quad (2.14)$$

On taking the upper limit as $n \rightarrow \infty$ in (2.14) and (2.10), and using (2.13), we obtain

$$\begin{aligned}\psi(d(u, Tu)) &\leq \psi\left(\frac{c + e + f + h}{a + b + c + 2e + 2f + r + s + h} d(u, Tu)\right) \\ &\leq \psi(d(u, Tu)),\end{aligned}$$

which yields that $d(u, Tu) = 0$, i.e. $u = Tu$.

Step IV. Next we prove that u is a unique fixed point of T . Suppose that v is also a fixed point of T , that is, $Tv = v$. Then, $v \in A \cap B$. Therefore, $(u, v) \in A \times B$. Since T is a C -class generalized cyclic contractive mapping, we have

$$\begin{aligned}\psi(s^{1+\varepsilon} d(u, v)) &= \psi(s^{1+\varepsilon} d(Tu, Tv)) \\ &\leq F(\psi(M(u, v)), \varphi(M(u, v))) + LN(u, v),\end{aligned}\quad (2.15)$$

where,

$$M(u, v) = \frac{a + \frac{2e}{s} + \frac{f}{s} + s + h}{a + b + c + 2e + 2f + r + s + h} d(u, v), \text{ and } N(u, v) = 0.$$

Then (2.15) becomes

$$\begin{aligned}\psi(d(u, v)) &\leq \psi(s^{1+\varepsilon} d(u, v)) \\ &\leq F \left\{ \begin{aligned} &\psi\left(\frac{a + \frac{2e}{s} + \frac{f}{s} + s + h}{a + b + c + 2e + 2f + r + s + h} d(u, v)\right), \\ &\varphi\left(\frac{a + \frac{2e}{s} + \frac{f}{s} + s + h}{a + b + c + 2e + 2f + r + s + h} d(u, v)\right) \end{aligned} \right\} \\ &\leq \psi\left(\frac{a + \frac{2e}{s} + \frac{f}{s} + s + h}{a + b + c + 2e + 2f + r + s + h} d(u, v)\right) \\ &\leq \psi(d(u, v)),\end{aligned}$$

which yields that $d(u, v) = 0$, i.e. $u = v$. So, u is a unique fixed point of T . This proves the result. \square

3 Applications

Next we give several interesting corollaries, as an application of our main result, in the underlying spaces. Some of them are novel in literature.

If we take $F(r, t) = r\beta(r)$ in Theorem 2.5, we get a new result.

Corollary 3.1. Let (X, d, s) be a b-metric space, A and B be non-empty closed subsets of $X, Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Suppose that following assertions hold:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exist $\psi \in \Psi_{\text{inf}}$ and a constant $L \geq 0, \varepsilon > 0$ such that

$$\psi(s^{1+\varepsilon}d(Tx, Ty)) \leq \psi(M(x, y))\beta(\psi(M(x, y))) + LN(x, y)$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where $M(x, y)$ and $N(x, y)$ are given in (2.2) and (2.3) and $\beta : [0, \infty) \rightarrow [0, 1)$ be function. Then T has a unique fixed point in $A \cap B$.

If we take $F(r, t) = r - \gamma(r)$ in Theorem 2.5, we find following result.

Corollary 3.2. Let (X, d, s) be a b-metric space, A and B be non-empty closed subsets of $X, Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Suppose that following assertions hold:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exist $\psi \in \Psi_{\text{inf}}$ and a constant $L \geq 0, \varepsilon > 0$ such that

$$\psi(s^{1+\varepsilon}d(Tx, Ty)) \leq \psi(M(x, y)) - \gamma(\psi(M(x, y))) + LN(x, y)$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where $M(x, y)$ and $N(x, y)$ are given in (2.2) and (2.3) and $\gamma : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\gamma(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in $A \cap B$.

If we take $F(r, t) = \lambda r, 0 < \lambda < 1$ in Theorem 2.5, then we have following noval result.

Corollary 3.3. Let (X, d, s) be a b-metric space, A and B be non-empty closed subsets of $X, Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Suppose that following assertions hold:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exist $\psi \in \Psi_{\text{inf}}$ and a constant $L \geq 0, \varepsilon > 0$ such that

$$\psi(s^{1+\varepsilon}d(Tx, Ty)) \leq \lambda\psi(M(x, y)) + LN(x, y)$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where $M(x, y)$ and $N(x, y)$ are given in (2.2) and (2.3). Then T has a unique fixed point in $A \cap B$.

If we take $F(m, t) = (a + b + c + 2e + 2f + r + s + h)m, 0 < a + b + c + 2e + 2f + r + s + h < 1$, with $\psi(t) = t$ in Theorem 2.5, then we have following noval result.

Corollary 3.4. Let (X, d, s) be a b-metric space, A and B be non-empty closed subsets of $X, Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Suppose that following assertions hold:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exist constants $L \geq 0, \varepsilon > 0$ such that

$$s^{1+\varepsilon}d(Tx, Ty) \leq \left\{ \begin{array}{l} ad(x, y) + bd(x, Tx) + cd(y, Ty) \\ + e \frac{d(x, Ty) + d(y, Tx)}{s} \\ + f \frac{d(T^2x, x) + d(T^2x, Ty)}{s} \\ + rd(T^2x, Tx) + sd(T^2x, y) \\ + hd(T^2x, Ty) + LN(x, y) \end{array} \right\}$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where $N(x, y)$ is given in (2.3). Then T has a unique fixed point in $A \cap B$.

If we assume that $F(r, t) = \alpha(r)$ in Theorem 2.5, then we obtain a new result.

Corollary 3.5. Let (X, d, s) be a b -metric space, A and B be non-empty closed subsets of X , $Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Suppose that following assertions hold:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exist $\psi \in \Psi_{\text{inf}}$ and a constant $L \geq 0, \varepsilon > 0$ such that

$$\psi(s^{1+\varepsilon}d(Tx, Ty)) \leq \alpha(\psi(M(x, y))) + LN(x, y)$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where $M(x, y)$ and $N(x, y)$ are given in (2.2) and (2.3) and $\alpha : [0, \infty) \rightarrow [0, \infty)$ is a upper semi-continuous function such that $\alpha(0) = 0$, and $\alpha(t) < t$ for all $t > 0$. Then T has a unique fixed point in $A \cap B$.

If we take $F(r, t) = r - t$ in Theorem 2.5, then we obtain the following generalized result.

Corollary 3.6. Let (X, d, s) be a b -metric space, A and B be non-empty closed subsets of X , $Y = A \cup B$ and $T : Y \rightarrow Y$ be a mapping. Suppose that following assertions hold:

- (i) $Y = A \cup B$ is a cyclic representation of Y with respect to T , that is, $T(A) \subset B$ and $T(B) \subset A$.
- (ii) There exists $\psi \in \Psi_{\text{inf}}, \varphi \in \Phi_u$ and a constant $L \geq 0, \varepsilon > 0$ such that

$$\psi(s^{1+\varepsilon}d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) + LN(x, y)$$

for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where $M(x, y)$ and $N(x, y)$ are given in (2.2) and (2.3). Then T has a unique fixed point in $A \cap B$.

4 Conclusion

In this finding, we have defined a new definition and then derived a result Theorem 2.5. We have also placed some innovative and interesting corollaries as an simple application of our main result. Some of the results given here are easily derived with help of our auxiliary function, but in nature these results are proved with additional assumption or condition on mappings or on set. Thus we conclude that our main result is more generalized than other existing results.

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