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On the solution of two-dimensional local fractional LWR model of fractal vehicular traffic flow

Pranay Goswami[a], Bhawna Pokhriyal[a,*], Kranti Kumar[a]

[a]*School of Liberal Studies, Ambedkar University Delhi, India.*

Abstract

This study proposed a two-dimensional local fractional Lighthill-Whitham-Richards (LWR) model of fractal vehicular traffic flow. The non-differentiable traffic parameters that arise in traffic flow are addressed by this model. The local fractional Laplace variational iteration method (LFLVIM) is employed to analyse the proposed model and making it serves well for examining the dynamic shifts in non-differentiable traffic density function. The existence and uniqueness of the solution of 2D local fractional LWR model have also proven. Several examples are also covered to further illustrate the effectiveness of applying LFLVIM to the proposed model. Additionally, the numerical simulations for every instance have been demonstrated. This study indicates that the presented 2D fractal model well captures the phenomena of traffic flow, and the iterative approach that has been employed to analyse the model is successful and can be applied to obtain the non-differentiable solution to 2D local fractional LWR model.

Keywords: Local fractional calculus, local fractional Laplace variational iteration method, two-dimensional LWR model, fractal Laplace transform, traffic flow model

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1. Introduction

The purpose of traffic models is to serve mechanisms for traffic control, with the aim of optimising transportation and achieving financial and environmental advantages, such as decreased traffic congestion and pollution. Depending on the observational scale, traffic models fall into three basic categories: microscopic, macroscopic and kinetic models. Microscopic models measure each vehicle's dynamics. They simulate single vehicle-driver units and explain the intricacies of traffic flow and the interactions that occur within it. The position and velocity of individual vehicles are represented by the dynamic variables of the models, which are microscopic attributes. Using an analogy with the flow of continuous media, such as fluids, macroscopic models simulate traffic flow using a first or higher order continuum and take aggregated characteristics like vehicle density into account. Kinetic models fall between the preceding two classifications because they can be generated from microscopic models and macroscopic models can be obtained from kinetic descriptions.

First-order models depending on scalar hyperbolic equations and second-order models composed up of

*Corresponding author

Email addresses: pranaygoswami83@gmail.com, pranay@aud.ac.in (Pranay Goswami), pokhriyalbhawna17@gmail.com (Bhawna Pokhriyal), kranti311u@gmail.com (Kranti Kumar)

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systems of hyperbolic equations are the two common categories used to categorise the (inviscid) macroscopic models. The model developed by Lighthill and Whitham [31] and Richards [35] (LWR) is the pioneering study of the first case. While the Aw and Rascle [3] and Zhang [43] model (ARZ) provides a particular illustration of the second scenario. In order to evaluate various models against data, different levels of granularity in the underlying process description have been used. It has been suggested in recent research that the macroscopic models offer a framework that is appropriate for incorporating online traffic data, especially data from fundamental diagrams [11, 14]. Although continuum models have been mathematically investigated and are now commonly utilised in traffic engineering, very little effort has been made to validate them using traffic data [2, 8]. There are several well-established and validated continuum traffic flow models for lane based ordered traffic. The first macroscopic traffic flow model, the LWR model, primarily consists of three components: a hydrodynamic relation, a conservation equation, and an equilibrium speed-density relationship as described below:

$$\frac{\partial \xi}{\partial \tau} + \frac{\partial Q}{\partial \omega} = 0, \tag{1.1}$$

with $Q = \xi\upsilon$ and $\upsilon = \upsilon(\xi)$, where ξ represents traffic density in time τ and space ω, Q denotes traffic flow and υ is traffic speed, a function of density.

Some researchers addressed the process of lane switching in macroscopic frameworks. Among all the studies in this direction, the first was done by Gazis et al. [15]. It was later enhanced by the work of Munjal and Pipes [34], they made the assumption that lane changes would take place in order to achieve uniform density along the road segment. Daganzo [13] developed a multi-lane traffic behavioural theory. The lane-changing rate was measured by Laval and Daganzo [27], who then used the data in hybrid models, which treated lane shifters as movable obstructions to traffic with limited acceleration. Taking into consideration the lateral spacing of vehicles, Gupta and Dhiman [16] used the car-following model with full velocity variation to construct a non lane based higher order vehicular traffic model. Aw and Rascle's [3] second-order AR model on a three-dimensional flow concentrating surface for multiclass traffic was extended by Mohan and Ramadurai [33]. The literature is well aware of the drawbacks of first order models, such as the infeasibility of solutions with unbounded acceleration and the incapability to replicate intricate traffic phenomena like stop-and-go waves. To address the shortcomings of the LWR model and be able to explain intricate traffic phenomena, first order models have been extended to multiclass [33], multi lane [13, 34, 15], or multi-dimensional [20] models. Several initial proposals have been made to create two-dimensional continuum frameworks. Herty et al. [20] suggested a unique macroscopic model of first-order in two dimensions based on empirical traffic data. Both lateral and longitudinal traffic variables are continuous in this model. Balzotti and Goettlich [7] extended upon the single class model to create a multiclass model in two-dimensions. Higher order traffic models in two dimensions that describe disorder and chaos in traffic were proposed by Vikram et al. [36]. The literature review leads to the conclusion that study of the first and second order vehicular macroscopic traffic frameworks in two-dimensions is still in its infancy. However, none of the aforementioned studies produced a complete two-dimensional model that took into account the effects of lateral density gradients and road boundaries. First-order (LWR) models are unable to explain traffic instabilities. However, they are among the most effective at explaining traffic disruption and traffic chaos behind obstacles like signals, lane dips, on-ramps and accidents. Additionally, compared to second-order models, first-order models are simpler to calibrate, contain less number of parameters, and are analytically tractable. Agrawal et al. [1] extend the standard equation of continuity to two dimensions and incorporate boundary repulsion and diffusion-driven lateral dynamics to present a comprehensive first-order 2D directed lane-free macroscopic traffic model, expressed as

$$\frac{\partial \xi}{\partial \tau} + \frac{\partial Q_\omega}{\partial \omega} + \frac{\partial Q_\mu}{\partial \mu} = 0, \ \forall \tau > 0, -\infty < \omega < \infty, -b \leqslant \mu \leqslant b, \tag{1.2}$$

where $\xi(\omega, \mu, \tau)$ denotes traffic density and Q_ω and Q_μ represent vehicle flux in longitudinal (ω) direction and lateral (μ) direction.

The LWR model is still widely used to simulate traffic flow because of its efficiency and power in explaining the qualitative behaviour of road traffic. However, the traditional conservation law is invalidated and the classic standard LWR traffic model is rendered ineffective when physical characteristics like traffic density or speed in the classical traffic model are a non-differentiable function of space and time viewed on Cantorian sets, a fractal set. The dynamical LWR model was therefore modified fractally by Wang et al. [37] to deal with this circumstance in the context of local fractional calculus [40, 38, 44] and indulging local fractional derivatives (LFDs) inside the local fractional conservation laws. This fractal version of LWR model is expressed below as

$$\frac{\partial^{\varepsilon}\xi}{\partial\tau^{\varepsilon}}+\frac{\partial^{\varepsilon}Q}{\partial\omega^{\varepsilon}}=0,\ 0<\varepsilon\leqslant 1, \tag{1.3}$$

where ξ is non-differentiable density function, Q is vehicle flux, function of density, ξ and ε denotes local fractional derivatives' order. Researchers have recently used a number of approaches to address the local fractional LWR model implementing the local fractal conservation law. The fractal structure of nature has been the subject of several investigations in a variety of scientific and technical fields, starting with Mandelbrot [9]. Yang [41, 39] refined and extended upon the local fractional derivative, which had been thoroughly examined in the works of Babakhani and Gejji [4]. Local fractional calculus has been used to tackle fractal problems in physics, fluid dynamics, applied mathematics, signal processing, quantum mechanics, and other fields [32, 6, 22, 23]. It is among the best and most efficient measures available for handling fractal and continuous functions which are non-differentiable. For example, Hao et al. [17] described local fractional diffusion and Helmholtz equations, Yang et al. [40] suggested local fractional Navier-Stokes equations, Zhao et al. [44] presented the fractal Maxwell's equations, and Yang et al. [38] investigated local fractional nonhomogeneous heat equations.

Likewise, to deal with the problem that arises when physical attributes such as density or speed in the vehicle traffic flow model are taken as non-differentiable functions in space and time, we extend Agrawal et al.'s two-dimensional LWR model to a fractal version with LFDs under the local fractional conservation laws and the fractal two-dimensional LWR model is given by

$$\frac{\partial^{\varepsilon}}{\partial\tau^{\varepsilon}}\xi+\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}Q_{\omega}+\frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}Q_{\mu}=0,\ \forall\tau>0,-\infty<\omega<\infty,-b\leqslant\mu\leqslant b,0<\varepsilon\leqslant 1, \tag{1.4}$$

where $\xi(\omega,\mu,\tau)$ is non-differentiable traffic density function, and ε denotes local fractional derivatives' order. A generalisation taken on fractal sets for differentiation and integration of functions is known as local fractional calculus, or fractal calculus [26, 25, 42]. The phenomena that engage non-differentiability of functions is not well described by the classical derivatives. Moreover, fractal domains like the cantor set lack differentiability. Local fractional derivatives enter the picture to solve the problem that arises with non-differentiable functions. There are various approaches to define fractal derivatives such as fractal derivative by means of fractal geometry [29, 18, 5, 12, 24], fractal derivative via Hausdorff measure [10, 30]. The fractal derivatives are defined by using fractal geometry in this study.

The literature employs a number of numerical techniques to solve the traffic flow models. Lebacque [28] showed that a Godunov-type method may be used to solve the LWR model for any feasible Riemann issue. McCormack's approach has been effectively applied by Helbing and Treiber [19] to solve non-equilibrium models.

This research presented a study to analyse a local fractional two-dimensional LWR model of traffic flow and approximate the analytical solution. The study employed a local fractional Laplace variational iteration method (LFLVIM) to derive the approximate solution of the proposed local fractional two-dimensional LWR traffic model with local fractional derivatives.

2. Preliminaries

This section deals with the concepts related to local fractional calculus and local fractional Laplace transform.

Definition 2.1. [39] A function $\sigma(\nu)$, $\nu \in (e,f)$ is called local fractional continuous (LFC) at $\nu = \nu_o$ if for $\beta, \delta > 0$, $|\nu - \nu_o| < \delta$ implies

$$|\sigma(\nu) - \sigma(\nu_o)| < \beta^{\varepsilon}, 0 < \varepsilon \leqslant 1. \tag{2.1}$$

Furthermore, if this is true $\forall \nu \in (e,f)$ then $\sigma(\nu)$ is LFC on (e,f) and represented by $\sigma(\nu) \in C_{\varepsilon}(e,f)$.

Definition 2.2. [39] Let $\sigma(\nu) \in C_{\varepsilon}(e,f)$, then local fractional derivative of $\sigma(\nu)$ at $\nu = \nu_o$ is defined by

$$D_{\nu}^{\varepsilon}\sigma(\nu_o) = \sigma^{(\varepsilon)}(\nu_0) = \frac{d^{\varepsilon}\sigma(\nu_o)}{d\nu^{\varepsilon}} = \lim_{\nu \to \nu_o} \frac{\Delta^{\varepsilon}(\sigma(\nu) - \sigma(\nu_o))}{(\nu - \nu_o)^{\varepsilon}}, \tag{2.2}$$

where

$$\Delta^{\varepsilon}(\sigma(\nu) - \sigma(\nu_o)) \cong \Gamma(1+\varepsilon)(\sigma(\nu) - \sigma(\nu_o)), \tag{2.3}$$

and ε, $(0 < \varepsilon < 1)$ is fractional order.

Definition 2.3. [39] Let partition of the closed interval $[r,s]$ is (f_h, f_{h+1}), $h = 0, 1, \cdots, P-1$ with $f_P = m$, $\Delta f_h = f_{h+1} - f_h$ and $\Delta f = \max\{\Delta f_0, \Delta f_1, ...\}$. Then for the function $\sigma(\nu)$, local fractional integral in $[r,s]$ is defined as

$${}_rJ_s^{\varepsilon}\sigma(\nu) = \frac{1}{\Gamma(1+\varepsilon)}\int_r^s \sigma(f)(df)^{\varepsilon} = \frac{1}{\Gamma(1+\varepsilon)}\lim_{\Delta f \to 0}\sum_{h=0}^{P-1}\sigma(f_h)(\Delta f_h)^{\varepsilon}. \tag{2.4}$$

Definition 2.4. [21] Suppose that the norm $\|\cdot\|_{\varepsilon}$ is defined on a generalized Banach space $B = C_{\varepsilon}$. If a map $\vartheta : B \to B$ satisfies $\|\vartheta\upsilon^{\varepsilon} - \upsilon^{\varepsilon}\|_{\varepsilon} = 0$, $\upsilon^{\varepsilon} \in B$ then υ^{ε} is called a fixed point of ϑ. Further, for $\eta^{\varepsilon} \in B$, if ϑ follows

$$\|\vartheta\upsilon^{\varepsilon} - \vartheta\eta^{\varepsilon}\|_{\varepsilon} \leqslant k^{\varepsilon}\|\upsilon^{\varepsilon} - \eta^{\varepsilon}\|_{\varepsilon}, 0 < k^{\varepsilon} \leqslant 1, \tag{2.5}$$

then the mapping ϑ is referred as a contraction.

Theorem 2.5. *[21] Suppose $(B, \|.\|_{\varepsilon})$ is a complete generalized Banach space and $\vartheta : B \to B$ be a mapping. If $\exists\, \beta \geqslant 1$ such that σ^{β} is a contraction, then it guarantees the existence and uniqueness of a fix point for the map σ.*

Definition 2.6. [39] The Mittage Leffler function in fractal space is defined as

$$E_{\varepsilon}(\nu^{\varepsilon}) = \sum_{h=0}^{\infty}\frac{\nu^{h\varepsilon}}{\Gamma(1+h\varepsilon)}, 0 < \varepsilon \leqslant 1. \tag{2.6}$$

Definition 2.7. [39] Local fractional derivative of the following functions is defined in a fractal space as

$$D_{\nu}^{\varepsilon}\nu^{K\varepsilon} = \frac{\Gamma(1+K\varepsilon)}{\Gamma(1+(K-1)\varepsilon)}\nu^{(K-1)\varepsilon}, \tag{2.7}$$

$$D_{\nu}^{\varepsilon}E_{\varepsilon}(K\nu^{\varepsilon}) = KE_{\varepsilon}(K\nu^{\varepsilon}), \tag{2.8}$$

Definition 2.8. [41, 38] (Local Fractional Laplace Transform) Local fractional Laplace transform (LFLT) of a continuously non-differentiable function $\sigma : \mathfrak{R} \to C$ is defined as

$$M''\{\sigma(\nu)\} = \sigma_s^{M,\varepsilon}(s) = \frac{1}{\Gamma(1+\varepsilon)}\int_0^{\infty}\sigma(\nu)E_{\varepsilon}(-\nu^{\varepsilon}s^{\varepsilon})(d\nu)^{\varepsilon}, 0 < \varepsilon \leqslant 1, \tag{2.9}$$

where M'' represents local fractional Laplace operator.
Moreover, local fractional inverse Laplace transform (LFILT) is given by

$$M''^{-1}\{\sigma_s^{M,\varepsilon}(s)\} = \sigma(\nu) = \frac{1}{(2\pi)^{\varepsilon}}\int_{\eta - i\infty}^{\eta + i\infty}\sigma_s^{M,\varepsilon}(s)E_{\varepsilon}(\nu^{\varepsilon}s^{\varepsilon})(ds)^{\varepsilon}, 0 < \varepsilon \leqslant 1, \tag{2.10}$$

where $M_{''}^{-1}$ denotes the fractal inverse Laplace operator with $E_\varepsilon\left((2\pi)^\varepsilon i^\varepsilon\right) = 1$, and $\operatorname{Re}(s) = \eta > 0$. Additionally, the convergence sufficient condition is given as

$$\frac{1}{\Gamma(1+\varepsilon)} \int_0^\infty |\sigma(\nu)| (d\nu)^\varepsilon < k < \infty. \tag{2.11}$$

Proposition 2.9. *[41, 38] Suppose* $p\varepsilon$ *is a order of fractal derivative of the function* $\sigma(\nu)$, *where* $0 < \varepsilon \leqslant 1$ *and* p *is any positive integer, then*

$$M_{''}\{\sigma^{p\varepsilon}(\nu)\} = s^{p\varepsilon} M_{''}\{\sigma(\nu)\} - s^{(p-1)\varepsilon}\sigma(0) - s^{(p-2)\varepsilon}\sigma^{(\varepsilon)}(0) - \cdots - \sigma^{((p-1)\varepsilon)}(0). \tag{2.12}$$

Properties of LFLT

$$M_\varepsilon\{ik(\nu) + jl(\nu)\} = iM_\varepsilon\{k(\nu)\} + jM_\varepsilon\{l(\nu)\},\ i, j \in C, \tag{2.13}$$

for $r \in C$,

$$M_\varepsilon\left\{\frac{\nu^{r\varepsilon}}{\Gamma(1+r\varepsilon)}\right\} = \frac{1}{s^{\varepsilon(r+1)}}, \tag{2.14}$$

$$M_\varepsilon\left\{\frac{\nu^{h\varepsilon}}{\Gamma(1+h\varepsilon)} E_\varepsilon\left(r^\varepsilon \nu^\varepsilon\right)\right\} = \frac{1}{(s-r)^{\varepsilon(h+1)}}, \tag{2.15}$$

$$M_\varepsilon\left\{E_\varepsilon\left(-r\nu^\varepsilon\right) + \frac{r\nu^\varepsilon}{\Gamma(1+\varepsilon)} - 1\right\} = \frac{r^2}{(s^\varepsilon + r)\, s^{2\varepsilon}}, \tag{2.16}$$

3. Model Specification

We have given a local fractional two-dimensional LWR model with local fractional derivatives, as follow

$$\frac{\partial^\varepsilon}{\partial\tau^\varepsilon}\xi + \frac{\partial^\varepsilon}{\partial\omega^\varepsilon}Q_\omega + \frac{\partial^\varepsilon}{\partial\mu^\varepsilon}Q_\mu = 0,\ \forall\tau > 0,\ -\infty < \omega < \infty,\ -b \leqslant \mu \leqslant b,\ 0 < \varepsilon \leqslant 1, \tag{3.1}$$

where $\xi(\omega, \mu, \tau)$ is non-differentiable traffic density function, Q_ω and Q_μ represent vehicle flux density in longitudinal direction (ω) and in lateral (μ) direction, respectively, and ε local fractional derivatives' order. In contrary to pedestrian traffic, traffic flow in two dimensions exhibits substantial anisotropy, with longitudinal dynamics that differ significantly from lateral dynamics. Therefore, the hydrodynamic relationship based flux density function is expressed separately for the longitudinal and lateral flow-density, as follows

$$Q_\omega = \xi\upsilon_\omega, \tag{3.2}$$

$$Q_\mu = \xi\upsilon_\mu - D\frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\xi, \tag{3.3}$$

where υ_ω and υ_μ are the velocities in longitudinal and lateral direction, respectively, and D is the diffusion coefficient. Substitute (3.2) and (3.3) in (3.1), we have

$$\frac{\partial^\varepsilon}{\partial\tau^\varepsilon}\xi + \frac{\partial^\varepsilon}{\partial\omega^\varepsilon}(\xi\upsilon_\omega) + \frac{\partial^\varepsilon}{\partial\mu^\varepsilon}(\xi\upsilon_\mu) - D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi = 0,\ \forall\tau > 0,\ -\infty < \omega < \infty,\ -b \leqslant \mu \leqslant b,\ 0 < \varepsilon \leqslant 1, \tag{3.4}$$

A relationship between speed and density must be initially determined in longitudinal (x) direction. In regards to the speed-density relationship, several functional relationships have been put forth. The longitudinal movement in this study is modelled using Greenshields' approach. The patterns in actual traffic flows can be understood by using Greenshields' model. Speed, a function of density, is a decreasing function as speed is free flow (desired) when density is zero and zero when density is maximum, that is, at jam density. The relationship for longitudinal speed and density is expressed as

$$\upsilon_{\omega} = \upsilon_{f}\left(1 - \frac{\xi}{\xi_{max}}\right), \tag{3.5}$$

where υ_f is free flow speed in longitudinal direction and ξ_{max} is jam density.
In contrast, the road width physically constraints the lateral dimension. We introduced a lateral speed given as

$$\upsilon_{\mu} = \upsilon_{\mu b}\varphi\left(\mu\right)\left(1 - \frac{\xi}{\xi_{max}}\right), \tag{3.6}$$

where $\upsilon_{\mu b}$ indicates the lateral desired speed directly near one of the boundaries, b is the road's half width, μ is the lateral distance from centre of the road and $\varphi\left(\mu\right)$ is a function of lateral distance. The term $\left(1 - \frac{\xi}{\xi_{max}}\right)$ is used to signify consistency in the model. Vehicles move under the initial condition assuming a constant maximum density if this term is absent from the equation. Later, the boundary repulsion forces push the density beyond its maximum within the road. Because there is no room to move away from the boundary if the maximum density is achieved [1].
Thus, (3.4) together with (3.5) and (3.6) represents a local fractional two-dimensional LWR model of traffic flow.

4. Existence and Uniqueness of Solution of 2D local fractional LWR model

The presented 2D local fractional LWR model (3.4) with longitudinal and lateral speed (3.5) and (3.6), respectively, can be written as

$$L_{\varepsilon}\xi\left(\omega,\mu,\tau\right) = \Lambda\left(\xi\left(\omega,\mu,\tau\right)\right), \tag{4.1}$$

where $L_{\varepsilon} = \frac{\partial^{\varepsilon}}{\partial\tau^{\varepsilon}}$ and

$$\Lambda\left(\xi\left(\omega,\mu,\tau\right)\right) = -\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\left(\xi\upsilon_{\omega}\right) - \frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}\left(\xi\upsilon_{\mu}\right) + D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi. \tag{4.2}$$

Theorem 4.1. *Let the function* $\Lambda\left(\xi\left(\omega,\mu,\tau\right)\right) = -\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\left(\xi\upsilon_{\omega}\right) - \frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}\left(\xi\upsilon_{\mu}\right) + D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi$ *is LFC and satisfies the condition for Lipschitz continuity, that is*

$$\left|\Lambda\left(\xi_1\left(\omega,\mu,\tau\right)\right) - \Lambda\left(\xi_2\left(\omega,\mu,\tau\right)\right)\right| \leqslant \lambda^{\varepsilon}\left|\xi_1\left(\omega,\mu,\tau\right) - \xi_2\left(\omega,\mu,\tau\right)\right|,\ 0<\lambda<1,\ 0<\varepsilon\leqslant 1, \tag{4.3}$$

then the system

$$L_{\varepsilon}\xi\left(\omega,\mu,\tau\right) = \Lambda\left(\xi\left(\omega,\mu,\tau\right)\right),\ \left(\omega,\mu\right)\in[e,f]\times[e,f],\ \tau>0, \tag{4.4}$$

along with the initial condition

$$\xi\left(\omega,\mu,0\right) = h\left(\omega,\mu\right) \tag{4.5}$$

has a unique solution in $C_{\varepsilon}\left([e,f]\times[e,f]\right)$.

Proof. Defining a map $\vartheta : C_{\varepsilon} \to C_{\varepsilon}$ as

$$\vartheta\left(\xi\left(\omega,\mu,\tau\right)\right) = \xi_0\left(\omega,\mu\right) + \frac{1}{\Gamma\left(1+\varepsilon\right)}\int_{e}^{f}\Lambda\left(\xi\left(\omega,\mu,\theta\right)\right)(d\theta)^{\varepsilon}. \tag{4.6}$$

We will show that for $m = 1, 2, \cdots$

$$\left\|\vartheta^{m}\left(\xi_1\left(\omega,\mu,\tau\right)\right) - \vartheta^{m}\left(\xi_2\left(\omega,\mu,\tau\right)\right)\right\|_{\varepsilon} \leqslant \frac{\lambda^{m\varepsilon}|f-e|^{m\varepsilon}}{\Gamma^{m}\left(1+\varepsilon\right)}\left\|\xi_1\left(\omega,\mu,\tau\right) - \xi_2\left(\omega,\mu,\tau\right)\right\|_{\varepsilon}. \tag{4.7}$$

For $m = 1$,

$$\begin{aligned}\|\vartheta(\xi_1(\omega,\mu,\tau)) - \vartheta(\xi_2(\omega,\mu,\tau))\|_\varepsilon &= \left|\frac{1}{\Gamma(1+\varepsilon)}\int_e^f (\Lambda(\xi_1(\omega,\mu,\tau)) - \Lambda(\xi_2(\omega,\mu,\tau)))(d\theta)^\varepsilon\right| \\ &\leqslant \left|\frac{1}{\Gamma(1+\varepsilon)}\int_e^f \lambda^\varepsilon|\xi_1(\omega,\mu,\theta) - \xi_2(\omega,\mu,\theta)|(d\theta)^\varepsilon\right| \\ &\leqslant \frac{\lambda^\varepsilon|f-e|^\varepsilon}{\Gamma(1+\varepsilon)}\|\xi_1(\omega,\mu,\tau) - \xi_2(\omega,\mu,\tau)\|_\varepsilon,\end{aligned} \tag{4.8}$$

which implies

$$\|\vartheta(\xi_1(\omega,\mu,\tau)) - \vartheta(\xi_2(\omega,\mu,\tau))\|_\varepsilon \leqslant \frac{\lambda^\varepsilon|f-e|^\varepsilon}{\Gamma(1+\varepsilon)}\|\xi_1(\omega,\mu,\tau) - \xi_2(\omega,\mu,\tau)\|_\varepsilon. \tag{4.9}$$

Now assuming the inequality holds for $m = k$

$$\left\|\vartheta^k(\xi_1(\omega,\mu,\tau)) - \vartheta^k(\xi_2(\omega,\mu,\tau))\right\|_\varepsilon \leqslant \frac{\lambda^{k\varepsilon}|f-e|^{k\varepsilon}}{\Gamma^k(1+\varepsilon)}\|\xi_1(\omega,\mu,\tau) - \xi_2(\omega,\mu,\tau)\|_\varepsilon. \tag{4.10}$$

Consider for $m = k+1$

$$\begin{aligned}&\left\|\vartheta^{k+1}(\xi_1(\omega,\mu,\tau)) - \vartheta^{k+1}(\xi_2(\omega,\mu,\tau))\right\|_\varepsilon \\ &= \left|\frac{1}{\Gamma(1+\varepsilon)}\int_e^f \left(\Lambda\left(\vartheta^k(\xi_1(\omega,\mu,\theta))\right) - \Lambda\left(\vartheta^k(\xi_1(\omega,\mu,\theta))\right)\right)(d\theta)^\varepsilon\right| \\ &\leqslant \left|\frac{1}{\Gamma(1+\varepsilon)}\int_e^f \lambda^\varepsilon\left|\vartheta^k(\xi_1(\omega,\mu,\theta)) - \vartheta^k(\xi_2(\omega,\mu,\theta))\right|(d\theta)^\varepsilon\right| \\ &\leqslant \frac{\lambda^{(k+1)\varepsilon}|f-e|^{(k+1)\varepsilon}}{\Gamma^{(k+1)}(1+\varepsilon)}\|\xi_1(\omega,\mu,\tau) - \xi_2(\omega,\mu,\tau)\|_\varepsilon,\end{aligned} \tag{4.11}$$

which implies

$$\left\|\vartheta^{k+1}(\xi_1(\omega,\mu,\tau)) - \vartheta^{k+1}(\xi_2(\omega,\mu,\tau))\right\|_\varepsilon \leqslant \frac{\lambda^{(k+1)\varepsilon}|f-e|^{(k+1)\varepsilon}}{\Gamma^{(k+1)}(1+\varepsilon)}\|\xi_1(\omega,\mu,\tau) - \xi_2(\omega,\mu,\tau)\|_\varepsilon. \tag{4.12}$$

Therefore, the claim (4.7) is proven and hence the map ϑ is a contraction. This proves the system (4.4) has a unique solution. □

5. Local fractional laplace variational iteration scheme

Consider a local fractional two-dimensional LWR model (3.4) of traffic flow with υ_ω and υ_μ defined by (3.5) and (3.6), respectively, subject to the initial condition $\xi(\omega,\mu,0) = h(\omega,\mu)$.
Then the local fractional correctional functional [39] that is associated with (3.4) is obtained by applying the local fractional variational iteration method (LFVIM) and is given by

$$\xi_{n+1}(\omega,\mu,\tau) = \xi_n(\omega,\mu,\tau) + {}_0J_\tau^\varepsilon\left\{\lambda^\varepsilon(\omega,\mu,\theta)\left[\frac{\partial^\varepsilon\xi_n}{\partial\theta^\varepsilon} + \frac{\partial^\varepsilon(\widetilde{\xi_n\upsilon_{n\omega}})}{\partial\omega^\varepsilon} + \frac{\partial^\varepsilon(\widetilde{\xi_n\upsilon_{n\mu}})}{\partial\mu^\varepsilon} - D\frac{\partial^{2\varepsilon}\tilde{\xi}_n}{\partial\mu^{2\varepsilon}}\right]\right\}, \tag{5.1}$$

which implies

$$\xi_{n+1}(\omega,\mu,\tau) = \xi_n(\omega,\mu,\tau) + {}_0J_\tau^\varepsilon\left\{\lambda^\varepsilon(\omega,\mu,\theta)\begin{bmatrix}\frac{\partial^\varepsilon \xi_n}{\partial\theta^\varepsilon} + \xi_n\frac{\partial^\varepsilon \tilde{\upsilon}_{n\omega}}{\partial\omega^\varepsilon} + \upsilon_{n\omega}\frac{\partial^\varepsilon \tilde{\xi}_n}{\partial\omega^\varepsilon} + \xi_n\frac{\partial^\varepsilon \tilde{\upsilon}_{n\mu}}{\partial\mu^\varepsilon} + \\ \upsilon_{n\mu}\frac{\partial^\varepsilon \tilde{\xi}_n}{\partial\mu^\varepsilon} - D\frac{\partial^{2\varepsilon} \tilde{\xi}_n}{\partial\mu^{2\varepsilon}}\end{bmatrix}\right\}, \tag{5.2}$$

where $\lambda^\varepsilon(\omega,\mu,\theta)$ defines fractal Langrange multiplier and $\tilde{\xi}_n$, $\tilde{\upsilon}_{n\omega}$ and $\tilde{\upsilon}_{n\mu}$ represent restricted fractal variations, that is, $\delta^\varepsilon\xi_n = 0$, $\delta^\varepsilon\tilde{\upsilon}_{n\omega} = 0$ and $\delta^\varepsilon\tilde{\upsilon}_{n\mu} = 0$. The LFLT is applying to (5.2), and hence we obtained

$$M_\varepsilon\{\xi_{n+1}\} = M_\varepsilon\{\xi_n\} + M_\varepsilon\{\lambda^\varepsilon\}M_\varepsilon\left\{\begin{array}{l}\frac{\partial^\varepsilon \xi_n}{\partial\theta^\varepsilon} + \xi_n\frac{\partial^\varepsilon \tilde{\upsilon}_{n\omega}}{\partial\omega^\varepsilon} + \upsilon_{n\omega}\frac{\partial^\varepsilon \tilde{\xi}_n}{\partial\omega^\varepsilon} + \xi_n\frac{\partial^\varepsilon \tilde{\upsilon}_{n\mu}}{\partial\mu^\varepsilon} + \\ \upsilon_{n\mu}\frac{\partial^\varepsilon \tilde{\xi}_n}{\partial\mu^\varepsilon} - D\frac{\partial^{2\varepsilon} \tilde{\xi}_n}{\partial\mu^{2\varepsilon}}\end{array}\right\}. \tag{5.3}$$

Taking fractal variation of (5.3), we have

$$\delta^\varepsilon M_\varepsilon\{\xi_{n+1}\} = \delta^\varepsilon M_\varepsilon\{\xi_n\} + M_\varepsilon\{\lambda^\varepsilon\}\delta^\varepsilon M_\varepsilon\left\{\begin{array}{l}\frac{\partial^\varepsilon \xi_n}{\partial\theta^\varepsilon} + \xi_n\frac{\partial^\varepsilon \tilde{\upsilon}_{n\omega}}{\partial\omega^\varepsilon} + \upsilon_{n\omega}\frac{\partial^\varepsilon \tilde{\xi}_n}{\partial\omega^\varepsilon} + \xi_n\frac{\partial^\varepsilon \tilde{\upsilon}_{n\mu}}{\partial\mu^\varepsilon} + \\ \upsilon_{n\mu}\frac{\partial^\varepsilon \tilde{\xi}_n}{\partial\mu^\varepsilon} - D\frac{\partial^{2\varepsilon} \tilde{\xi}_n}{\partial\mu^{2\varepsilon}}\end{array}\right\}. \tag{5.4}$$

The optimal condition of ξ_{n+1} is given as

$$\delta^\varepsilon M_\varepsilon\{\xi_{n+1}\} = 0. \tag{5.5}$$

Thus, with reference to (5.5), we get

$$\delta^\varepsilon M_\varepsilon\{\xi_n\}(1 + s^\varepsilon M_\varepsilon\{\lambda^\varepsilon\}) = 0. \tag{5.6}$$

Therefore, the LFLT of fractal Lagrange multiplier is provided as

$$M_\varepsilon\{\lambda^\varepsilon\} = -\frac{1}{s^\varepsilon}. \tag{5.7}$$

Hence, the formula for successive iteration is given as

$$M_\varepsilon\{\xi_{n+1}\} = M_\varepsilon\{\xi_n\} - \frac{1}{s^\varepsilon}M_\varepsilon\left\{\frac{\partial^\varepsilon}{\partial\theta^\varepsilon}\xi_n + \frac{\partial^\varepsilon}{\partial\omega^\varepsilon}(\xi_n\upsilon_{n\omega}) + \frac{\partial^\varepsilon}{\partial\mu^\varepsilon}(\xi_n\upsilon_{n\mu}) - D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_n\right\}, \tag{5.8}$$

along with the initial approximation

$$M_\varepsilon\{\xi_0\} = M_\varepsilon\{\xi(\omega,\mu,0)\} = \xi_0(\omega,\mu,s). \tag{5.9}$$

Therefore, the fractal series solution of (3.4) is

$$M_\varepsilon\{\xi(\omega,\mu,\tau)\} = \lim_{n\to\infty} M_\varepsilon\{\xi_n(\omega,\mu,\tau)\}, \tag{5.10}$$

and thus, we have

$$\xi(\omega,\mu,\tau) = \lim_{n\to\infty} M_\varepsilon^{-1}\{M_\varepsilon\{\xi_n(\omega,\mu,\tau)\}\}. \tag{5.11}$$

6. Non-differentiable solution of local fractional 2D LWR model

This section provides various instances illustrating LFLVIM to obtain non-differentiable solutions for the 2D local fractional LWR traffic flow model. Table 1 provides the values for traffic parameters that has used in this study to derive non-differentiable solutions [1].

Table 1: Parametric Values taken in the study

Parameter	Value
Longitudinal free flow speed, υ_f	$15.28ms^{-1}$
Maximum velocity of repulsion at the boundary, $\upsilon_{\mu b}$	$0.5ms^{-1}$
Width of road, b	$10m$
Diffusion Coefficient, D	$1m^2s^{-1}$

Example 6.1. Let us consider a 2D local fractional LWR traffic flow model

$$\frac{\partial^{\varepsilon}\xi}{\partial\tau^{\varepsilon}}+\frac{\partial^{\varepsilon}\xi}{\partial\omega^{\varepsilon}}+\frac{\partial^{\varepsilon}\xi}{\partial\mu^{\varepsilon}}=D\frac{\partial^{2\varepsilon}\xi}{\partial\mu^{2\varepsilon}},\ \forall\tau>0,\ -\infty<\omega<\infty,\ -b\leqslant\mu\leqslant b,\ 0<\varepsilon\leqslant 1, \tag{6.1}$$

along with the initial condition

$$\xi(\omega,\mu,0)=E_{\varepsilon}(\omega^{\varepsilon}) \tag{6.2}$$

In view of (5.8), the successive iteration formula corresponding to (6.1) is

$$\begin{aligned}M_{\varepsilon}\{\xi_{n+1}(\omega,\mu,\tau)\}&=M_{\varepsilon}\{\xi_n(\omega,\mu,\tau)\}-\\&\frac{1}{s^{\varepsilon}}M_{\varepsilon}\left\{\frac{\partial^{\varepsilon}\xi_n(\omega,\mu,\tau)}{\partial\tau^{\varepsilon}}+\frac{\partial^{\varepsilon}\xi_n(\omega,\mu,\tau)}{\partial\omega^{\varepsilon}}+\frac{\partial^{\varepsilon}\xi_n(\omega,\mu,\tau)}{\partial\mu^{\varepsilon}}-D\frac{\partial^{2\varepsilon}\xi_n(\omega,\mu,\tau)}{\partial\mu^{2\varepsilon}}\right\}\end{aligned} \tag{6.3}$$

with the initial approximation

$$M_{\varepsilon}\{\xi_0\}=M_{\varepsilon}\{E_{\varepsilon}(\omega^{\varepsilon})\}=\frac{E_{\varepsilon}(\omega^{\varepsilon})}{s^{\varepsilon}}. \tag{6.4}$$

First approximation to $\xi(\omega,\mu,\tau)$ is

$$\begin{aligned}M_{\varepsilon}\{\xi_1(\omega,\mu,\tau)\}&=M_{\varepsilon}\{\xi_0(\omega,\mu,\tau)\}-\\&\frac{1}{s^{\varepsilon}}M_{\varepsilon}\left\{\frac{\partial^{\varepsilon}\xi_0(\omega,\mu,\tau)}{\partial\tau^{\varepsilon}}+\frac{\partial^{\varepsilon}\xi_0(\omega,\mu,\tau)}{\partial\omega^{\varepsilon}}+\frac{\partial^{\varepsilon}\xi_0(\omega,\mu,\tau)}{\partial\mu^{\varepsilon}}-D\frac{\partial^{2\varepsilon}\xi_0(\omega,\mu,\tau)}{\partial\mu^{2\varepsilon}}\right\}\\&=\xi_0(\omega,\mu,s)-\\&\frac{1}{s^{\varepsilon}}\left\{\begin{array}{l}s^{\varepsilon}\xi_0(\omega,\mu,s)-\xi_0(\omega,\mu,0)+\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\xi_0(\omega,\mu,s)+\frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}\xi_0(\omega,\mu,s)\\-D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_0(\omega,\mu,s)\end{array}\right\}\\&=\frac{E_{\varepsilon}(\omega^{\varepsilon})}{s^{\varepsilon}}-\frac{E_{\varepsilon}(\omega^{\varepsilon})}{s^{2\varepsilon}}.\end{aligned} \tag{6.5}$$

This implies

$$\begin{aligned}\xi_1(\omega,\mu,\tau)&=M_{\varepsilon}^{-1}\left\{\frac{E_{\varepsilon}(\omega^{\varepsilon})}{s^{\varepsilon}}-\frac{E_{\varepsilon}(\omega^{\varepsilon})}{s^{2\varepsilon}}\right\}\\&=E_{\varepsilon}(\omega^{\varepsilon})-\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}E_{\varepsilon}(\omega^{\varepsilon}).\end{aligned} \tag{6.6}$$

Second approximation to $\xi(\omega,\mu,\tau)$ is

$$\begin{aligned} M_\varepsilon\left\{\xi_2(\omega,\mu,\tau)\right\} &= M_\varepsilon\left\{\xi_1(\omega,\mu,\tau)\right\} - \\ &\frac{1}{s^\varepsilon} M_\varepsilon \left\{ \frac{\partial^\varepsilon \xi_1(\omega,\mu,\tau)}{\partial\tau^\varepsilon} + \frac{\partial^\varepsilon \xi_1(\omega,\mu,\tau)}{\partial\omega^\varepsilon} + \frac{\partial^\varepsilon \xi_1(\omega,\mu,\tau)}{\partial\mu^\varepsilon} - D\frac{\partial^{2\varepsilon} \xi_1(\omega,\mu,\tau)}{\partial\mu^{2\varepsilon}} \right\} \\ &= \xi_1(\omega,\mu,s) - \\ &\frac{1}{s^\varepsilon} \left\{ \begin{array}{l} s^\varepsilon \xi_1(\omega,\mu,s) - \xi_1(\omega,\mu,0) + \frac{\partial^\varepsilon}{\partial\omega^\varepsilon}\xi_1(\omega,\mu,s) + \frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\xi_1(\omega,\mu,s) \\ -D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_1(\omega,\mu,s) \end{array} \right\} \\ &= \frac{1}{s^\varepsilon} E_\varepsilon(\omega^\varepsilon) - \frac{1}{s^\varepsilon}\frac{\partial^\varepsilon}{\partial\omega^\varepsilon}\left(\frac{E_\varepsilon(\omega^\varepsilon)}{s^\varepsilon} - \frac{E_\varepsilon(\omega^\varepsilon)}{s^{2\varepsilon}}\right) - \frac{1}{s^\varepsilon}\frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\left(\frac{E_\varepsilon(\omega^\varepsilon)}{s^\varepsilon} - \frac{E_\varepsilon(\omega^\varepsilon)}{s^{2\varepsilon}}\right) \\ &+ \frac{1}{s^\varepsilon} D \frac{\partial^{2\varepsilon}}{\partial\omega^{2\varepsilon}}\left(\frac{E_\varepsilon(\omega^\varepsilon)}{s^\varepsilon} - \frac{E_\varepsilon(\omega^\varepsilon)}{s^{2\varepsilon}}\right) \\ &= \frac{E_\varepsilon(\omega^\varepsilon)}{s^\varepsilon} - \frac{E_\varepsilon(\omega^\varepsilon)}{s^{2\varepsilon}} + \frac{E_\varepsilon(\omega^\varepsilon)}{s^{3\varepsilon}}. \end{aligned} \tag{6.7}$$

This implies

$$\begin{aligned} \xi_2(\omega,\mu,\tau) &= M_\varepsilon^{-1}\left\{\frac{E_\varepsilon(\omega^\varepsilon)}{s^\varepsilon} - \frac{E_\varepsilon(\omega^\varepsilon)}{s^{2\varepsilon}} + \frac{E_\varepsilon(\omega^\varepsilon)}{s^{3\varepsilon}}\right\} \\ &= E_\varepsilon(\omega^\varepsilon) - \frac{\tau^\varepsilon}{\Gamma(1+\varepsilon)} E_\varepsilon(\omega^\varepsilon) + \frac{\tau^{2\varepsilon}}{\Gamma(1+2\varepsilon)} E_\varepsilon(\omega^\varepsilon). \end{aligned} \tag{6.8}$$

Proceeding in the same manner, we have

$$\begin{aligned} M_\varepsilon\left\{\xi_n(\omega,\mu,\tau)\right\} &= \frac{E_\varepsilon(\omega^\varepsilon)}{s^\varepsilon} - \frac{E_\varepsilon(\omega^\varepsilon)}{s^{2\varepsilon}} + \frac{E_\varepsilon(\omega^\varepsilon)}{s^{3\varepsilon}} + \ldots + (-1)^n \frac{E_\varepsilon(\omega^\varepsilon)}{s^{(n+1)\varepsilon}} \\ &= E_\varepsilon(\omega^\varepsilon)\left(\frac{1}{s^\varepsilon} - \frac{1}{s^{2\varepsilon}} + \frac{1}{s^{3\varepsilon}} - \ldots + (-1)^n \frac{1}{s^{(n+1)\varepsilon}}\right). \end{aligned} \tag{6.9}$$

This implies

$$\begin{aligned} \xi_n(\omega,\mu,\tau) &= M_\varepsilon^{-1}\left\{E_\varepsilon(\omega^\varepsilon)\left(\frac{1}{s^\varepsilon} - \frac{1}{s^{2\varepsilon}} + \frac{1}{s^{3\varepsilon}} - \ldots + (-1)^n \frac{1}{s^{(n+1)\varepsilon}}\right)\right\} \\ &= E_\varepsilon(\omega^\varepsilon)\left(1 - \frac{\tau^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{\tau^{2\varepsilon}}{\Gamma(1+2\varepsilon)} - \ldots + (-1)^n \frac{\tau^{n\varepsilon}}{\Gamma(1+n\varepsilon)}\right). \end{aligned} \tag{6.10}$$

Thus, the solution is given as

$$\begin{aligned} \xi(\omega,\mu,\tau) &= \lim_{n\to\infty} \xi_n(\omega,\mu,\tau) \\ &= E_\varepsilon(\omega^\varepsilon) E_\varepsilon(-\tau^\varepsilon), \end{aligned} \tag{6.11}$$

and the corresponding graphical representation of the solution surface of (6.1) is displayed by Figure 1 with parameter $\varepsilon = \ln 2/\ln 3$. Visualisation that is displayed make it evident that the performed iterations are convergent at the solution's surface.

Example 6.2. Let us consider a 2D local fractional LWR model

$$\frac{\partial^\varepsilon}{\partial\tau^\varepsilon}\xi + \frac{\partial^\varepsilon}{\partial\omega^\varepsilon}(\xi\upsilon_\omega) + \frac{\partial^\varepsilon}{\partial\mu^\varepsilon}(\xi\upsilon_\mu) - D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi = 0,\ \forall\tau > 0, -\infty < \omega < \infty, -b \leqslant \mu \leqslant b, 0 < \varepsilon \leqslant 1, \tag{6.12}$$

with

$$\upsilon_\omega = \upsilon_f\left(1 - \frac{\xi}{\xi_{max}}\right), \tag{6.13}$$

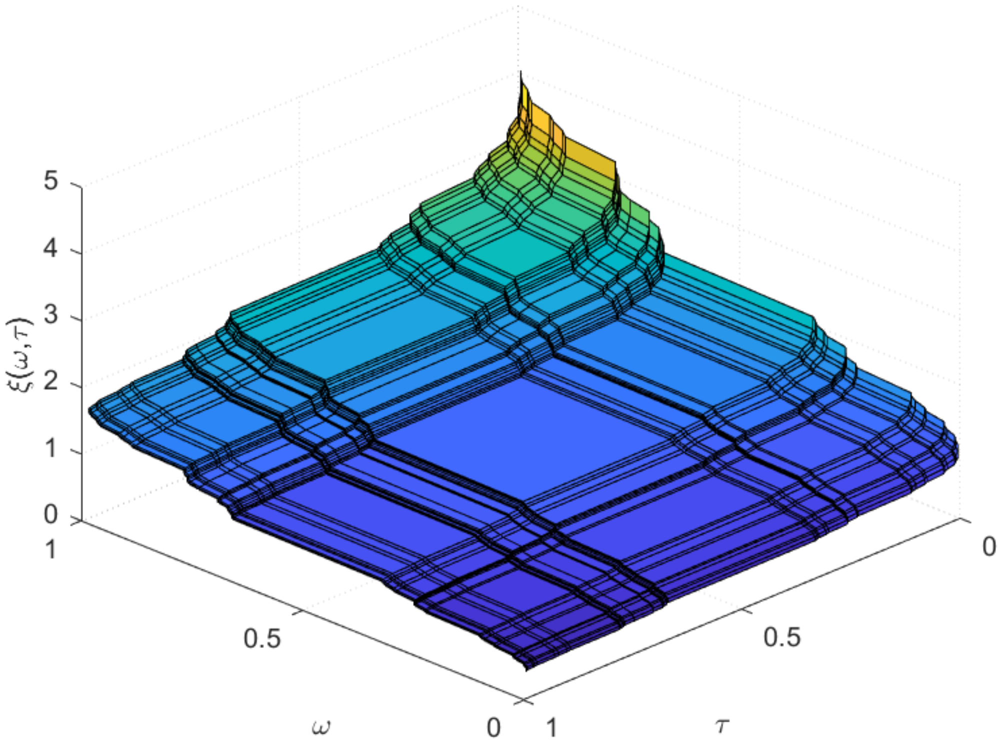

Figure 1: Solution surface of the given system (6.1) with parameter $\varepsilon = \ln 2/\ln 3$

$$\upsilon_{\mu} = \upsilon_{\mu b}\left(1 - \frac{\xi}{\xi_{\max}}\right), \tag{6.14}$$

subject to the initial condition

$$\xi\left(\omega, \mu, 0\right) = \omega^{\varepsilon}\mu^{\varepsilon}. \tag{6.15}$$

In view of (5.8), the successive iteration formula corresponding to (6.12) is

$$M_{\varepsilon}\left\{\xi_{n+1}\left(\omega, \mu, \tau\right)\right\} = M_{\varepsilon}\left\{\xi_{n}\left(\omega, \mu, \tau\right)\right\} - \frac{1}{s^{\varepsilon}}M_{\varepsilon}\left\{\begin{array}{c}\frac{\partial^{\varepsilon}}{\partial \tau^{\varepsilon}}\xi_{n}\left(\omega, \mu, \tau\right) + \frac{\partial^{\varepsilon}}{\partial \omega^{\varepsilon}}\left(\xi_{n}\left(\omega, \mu, \tau\right)\upsilon_{n\omega}\right) \\ + \frac{\partial^{\varepsilon}}{\partial \mu^{\varepsilon}}\left(\xi_{n}\left(\omega, \mu, \tau\right)\upsilon_{n\mu}\right) - D\frac{\partial^{2\varepsilon}}{\partial \mu^{2\varepsilon}}\xi_{n}\left(\omega, \mu, \tau\right)\end{array}\right\}, \tag{6.16}$$

with the initial approximation

$$M_{\varepsilon}\left\{\xi_{0}\left(\omega, \mu, \tau\right)\right\} = M_{\varepsilon}\left\{\omega^{\varepsilon}\mu^{\varepsilon}\right\} = \frac{\omega^{\varepsilon}\mu^{\varepsilon}}{s^{\varepsilon}}. \tag{6.17}$$

First approximation to $\xi\left(\omega, \mu, \tau\right)$ is given as

$$
\begin{aligned}
M_\varepsilon\left\{\xi_1(\omega,\mu,\tau)\right\} &= M_\varepsilon\left\{\xi_0(\omega,\mu,\tau)\right\} - \frac{1}{s^\varepsilon} M_\varepsilon \left\{ \begin{array}{l} \frac{\partial^\varepsilon}{\partial\tau^\varepsilon}\xi_0(\omega,\mu,\tau) + \frac{\partial^\varepsilon}{\partial\omega^\varepsilon}\left(\xi_0(\omega,\mu,\tau)\upsilon_{n\omega}\right) \\ +\frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\left(\xi_0(\omega,\mu,\tau)\upsilon_{n\mu}\right) - D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_0(\omega,\mu,\tau) \end{array} \right\} \\
&= \xi_0(\omega,\mu,s) - \\
&\frac{1}{s^\varepsilon}\left\{ \begin{array}{l} s^\varepsilon\xi_0(\omega,\mu,s) - \xi_0(\omega,\mu,0) + \upsilon_f\frac{\partial^\varepsilon}{\partial\omega^\varepsilon}\left(\xi_0(\omega,\mu,s) - \frac{M_\varepsilon\left\{\xi_0^2(\omega,\mu,\tau)\right\}}{\xi_{max}}\right) \\ +\upsilon_{\mu b}\frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\left(\xi_0(\omega,\mu,s) - \frac{M_\varepsilon\left\{\xi_0^2(\omega,\mu,\tau)\right\}}{\xi_{max}}\right) - D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_0(\omega,\mu,s) \end{array} \right\} \\
&= \frac{\omega^\varepsilon\mu^\varepsilon}{s^\varepsilon} - \frac{\upsilon_f}{s^\varepsilon}\frac{\partial^\varepsilon}{\partial\omega^\varepsilon}\left(\frac{\omega^\varepsilon\mu^\varepsilon}{s^\varepsilon} - \frac{\omega^{2\varepsilon}\mu^{2\varepsilon}}{s^\varepsilon\xi_{max}}\right) - \frac{\upsilon_{\mu b}}{s^\varepsilon}\frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\left(\frac{\omega^\varepsilon\mu^\varepsilon}{s^\varepsilon} - \frac{\omega^{2\varepsilon}\mu^{2\varepsilon}}{s^\varepsilon\xi_{max}}\right) - \\
&D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\frac{\omega^\varepsilon\mu^\varepsilon}{s^\varepsilon} \\
&= \frac{\omega^\varepsilon\mu^\varepsilon}{s^\varepsilon} - \left(\frac{\upsilon_f\mu^\varepsilon}{s^{2\varepsilon}} - \frac{\upsilon_{\mu b}\omega^\varepsilon}{s^{2\varepsilon}}\right)\left(\Gamma(1+\varepsilon) - \frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\omega^\varepsilon\mu^\varepsilon}{\xi_{max}}\right),
\end{aligned} \tag{6.18}
$$

which gives

$$
\begin{aligned}
\xi_1(\omega,\mu,\tau) &= M_\varepsilon^{-1}\left\{\frac{\omega^\varepsilon\mu^\varepsilon}{s^\varepsilon} - \left(\frac{\upsilon_f\mu^\varepsilon}{s^{2\varepsilon}} - \frac{\upsilon_{\mu b}\omega^\varepsilon}{s^{2\varepsilon}}\right)\left(\Gamma(1+\varepsilon) - \frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\omega^\varepsilon\mu^\varepsilon}{\xi_{max}}\right)\right\} \\
&= \omega^\varepsilon\mu^\varepsilon - \left(\upsilon_f\mu^\varepsilon\tau^\varepsilon + \upsilon_{\mu b}\omega^\varepsilon\tau^\varepsilon\right)\left(1 - \frac{\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^\varepsilon\mu^\varepsilon}{\xi_{max}}\right).
\end{aligned} \tag{6.19}
$$

Second approximation to $\xi(\omega,\mu,\tau)$ is given as

$$
\begin{aligned}
M_\varepsilon\left\{\xi_2(\omega,\mu,\tau)\right\} &= M_\varepsilon\left\{\xi_1(\omega,\mu,\tau)\right\} - \frac{1}{s^\varepsilon} M_\varepsilon \left\{ \begin{array}{l} \frac{\partial^\varepsilon}{\partial\tau^\varepsilon}\xi_1(\omega,\mu,\tau) + \frac{\partial^\varepsilon}{\partial\omega^\varepsilon}\left(\xi_1(\omega,\mu,\tau)\,\upsilon_{n\omega}\right) \\ + \frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\left(\xi_1(\omega,\mu,\tau)\,\upsilon_{n\mu}\right) - D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_1(\omega,\mu,\tau) \end{array} \right\} \\
&= \frac{1}{s^\varepsilon}\xi_1(\omega,\mu,0) - \frac{\upsilon_f}{s^\varepsilon}\frac{\partial^\varepsilon}{\partial\omega^\varepsilon}\left(\xi_1(\omega,\mu,s) - \frac{M_\varepsilon\left\{\xi_1^2(\omega,\mu,\tau)\right\}}{\xi_{max}}\right) \\
&- \frac{\upsilon_{\mu b}}{s^\varepsilon}\frac{\partial^\varepsilon}{\partial\mu^\varepsilon}\left(\xi_1(\omega,\mu,s) - \frac{M_\varepsilon\left\{\xi_1^2(\omega,\mu,\tau)\right\}}{\xi_{max}}\right) - D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_1(\omega,\mu,s) \\
&= \frac{\omega^\varepsilon\mu^\varepsilon}{s^\varepsilon} - \frac{\upsilon_f}{s^\varepsilon}\left[\begin{array}{l}
\Gamma(1+\varepsilon)\frac{\mu^\varepsilon}{s^\varepsilon} + \upsilon_f\Gamma(1+\varepsilon)\frac{\mu^{2\varepsilon}}{s^{2\varepsilon}\xi_{max}} - \upsilon_{\mu b}\Gamma^2(1+\varepsilon)\frac{1}{s^{2\varepsilon}} \\
+\upsilon_{\mu b}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^\varepsilon\mu^\varepsilon}{s^{2\varepsilon}\xi_{max}} - \frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\omega^\varepsilon\mu^{2\varepsilon}}{s^\varepsilon\xi_{max}} - \\
\frac{\Gamma(1+2\varepsilon)}{s^{3\varepsilon}\xi_{max}}\left(\begin{array}{l}
\upsilon_f^2\frac{\Gamma^3(1+2\varepsilon)}{\Gamma^5(1+\varepsilon)}\frac{\omega^\varepsilon\mu^{4\varepsilon}}{\xi_{max}^2} - 2\upsilon_f^2\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\mu^{3\varepsilon}}{\xi_{max}} \\
+\upsilon_{yb}^2\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\omega^\varepsilon - 2\upsilon_{yb}^2\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^\varepsilon}{\xi_{max}} \\
+\upsilon_{yb}^2\frac{\Gamma^2(1+2\varepsilon)\Gamma(1+4\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+3\varepsilon)}\frac{\omega^{3\varepsilon}\mu^{2\varepsilon}}{\xi_{max}^2} \\
+2\upsilon_f\upsilon_{\mu b}\Gamma(1+\varepsilon)\mu^\varepsilon - 4\upsilon_f\upsilon_{\mu b}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\frac{\omega^\varepsilon\mu^{2\varepsilon}}{\xi_{max}} \\
+2\upsilon_f\upsilon_{\mu b}\frac{\Gamma(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^4(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{3\varepsilon}}{\xi_{max}^2}
\end{array}\right) \\
+\frac{2\Gamma(1+2\varepsilon)}{s^{2\varepsilon}\xi_{max}}\left(\begin{array}{l}
\upsilon_f\Gamma(1+\varepsilon)\mu^{2\varepsilon} - \upsilon_f\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\frac{\omega^\varepsilon\mu^{3\varepsilon}}{\xi_{max}} \\
+\upsilon_{\mu b}\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\omega^\varepsilon\mu^\varepsilon - \upsilon_{\mu b}\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\omega^{2\varepsilon}\mu^{2\varepsilon}
\end{array}\right)
\end{array}\right] \\
&- \frac{\upsilon_{\mu b}}{s^\varepsilon}\left[\begin{array}{l}
\frac{\Gamma(1+\varepsilon)}{s^\varepsilon}\omega^\varepsilon - \upsilon_f\frac{\Gamma^2(1+\varepsilon)}{s^{2\varepsilon}} + \upsilon_f\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^\varepsilon\mu^\varepsilon}{s^{2\varepsilon}\xi_{max}} + \\
\upsilon_{\mu b}\Gamma(1+2\varepsilon)\frac{\omega^{2\varepsilon}}{s^{2\varepsilon}\xi_{max}} - \frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^\varepsilon}{s^\varepsilon\xi_{max}} - \frac{\Gamma(1+2\varepsilon)}{s^{3\varepsilon}\xi_{max}} \\
\left(\begin{array}{l}
\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\upsilon_f^2\mu^\varepsilon + \upsilon_f^2\frac{\Gamma^2(1+2\varepsilon)\Gamma(1+4\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+3\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{3\varepsilon}}{\xi_{max}^2} - \\
-2\upsilon_f^2\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^\varepsilon\mu^{2\varepsilon}}{\xi_{max}} + \upsilon_{yb}^2\frac{\Gamma^3(1+2\varepsilon)}{\Gamma^5(1+\varepsilon)}\frac{\omega^{4\varepsilon}\mu^\varepsilon}{\xi_{max}^2} - 2\upsilon_{yb}^2\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\omega^{3\varepsilon}}{\xi_{max}^2} \\
+2\upsilon_f\upsilon_{\mu b}\Gamma(1+\varepsilon)\omega^\varepsilon + 2\upsilon_f\upsilon_{\mu b}\frac{\Gamma(1+2\varepsilon)}{\Gamma^4(1+\varepsilon)}\frac{\omega^{3\varepsilon}\mu^{2\varepsilon}}{\xi_{max}^2} - \\
4\upsilon_f\upsilon_{\mu b}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^\varepsilon}{\xi_{max}}
\end{array}\right) \\
+\frac{2\Gamma(1+2\varepsilon)}{s^{2\varepsilon}\xi_{max}}\left(\begin{array}{l}
\upsilon_f\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\omega^\varepsilon\mu^\varepsilon - \upsilon_f\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{2\varepsilon}}{\xi_{max}} \\
+\upsilon_{\mu b}\Gamma(1+2\varepsilon)\omega^{2\varepsilon} - \upsilon_{\mu b}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\omega^{3\varepsilon}\mu^\varepsilon
\end{array}\right)
\end{array}\right] \\
&- D\upsilon_f\frac{\Gamma^2(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\omega^\varepsilon}{s^{2\varepsilon}\xi_{max}}.
\end{aligned}
\tag{6.20}
$$

Taking the local fractional inverse Laplace transform, we have

$$
\begin{aligned}
\xi_2(\omega,\mu,\tau) = \omega^{\varepsilon}\mu^{\varepsilon} - \upsilon_f & \left[\begin{array}{l}
\mu^{\varepsilon}\tau^{\varepsilon} + \upsilon_f \frac{\mu^{2\varepsilon}\tau^{2\varepsilon}}{\xi_{max}} - \upsilon_{\mu b}\frac{\Gamma^2(1+\varepsilon)}{\Gamma(1+2\varepsilon)}\tau^{2\varepsilon} \\
+\upsilon_{\mu b}\frac{\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}\tau^{2\varepsilon}}{\xi_{max}} - \frac{\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{\varepsilon}\mu^{2\varepsilon}\tau^{\varepsilon}}{\xi_{max}} \\
-\frac{\tau^{2\varepsilon}}{\xi_{max}}\left(\begin{array}{l}
\upsilon_f^2\frac{\Gamma^3(1+2\varepsilon)}{\Gamma^5(1+\varepsilon)}\frac{\omega^{\varepsilon}\mu^{4\varepsilon}}{\xi_{max}^2} - 2\upsilon_f^2\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\mu^{3\varepsilon}}{\xi_{max}} \\
+\upsilon_{yb}^2\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\omega^{\varepsilon} - 2\upsilon_{yb}^2\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{\varepsilon}}{\xi_{max}} \\
+\upsilon_{yb}^2\frac{\Gamma^2(1+2\varepsilon)\Gamma(1+4\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+3\varepsilon)}\frac{\omega^{3\varepsilon}\mu^{2\varepsilon}}{\xi_{max}^2} \\
+2\upsilon_f\upsilon_{\mu b}\Gamma(1+\varepsilon)\mu^{\varepsilon} - 4\upsilon_f\upsilon_{\mu b}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\frac{\omega^{\varepsilon}\mu^{2\varepsilon}}{\xi_{max}} \\
+2\upsilon_f\upsilon_{\mu b}\frac{\Gamma(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^4(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{3\varepsilon}}{\xi_{max}^2}
\end{array}\right) \\
+\frac{\tau^{2\varepsilon}}{\xi_{max}}\left(\begin{array}{l}
\upsilon_f\Gamma(1+\varepsilon)\mu^{2\varepsilon} - \upsilon_f\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\frac{\omega^{\varepsilon}\mu^{3\varepsilon}}{\xi_{max}} \\
+\upsilon_{\mu b}\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\omega^{\varepsilon}\mu^{\varepsilon} - \upsilon_{\mu b}\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\omega^{2\varepsilon}\mu^{2\varepsilon}
\end{array}\right)
\end{array}\right] \\
- \upsilon_{\mu b} & \left[\begin{array}{l}
\omega^{\varepsilon}\tau^{\varepsilon} - \upsilon_f\frac{\Gamma^2(1+\varepsilon)}{\Gamma(1+2\varepsilon)}\tau^{2\varepsilon} + \upsilon_f\frac{\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}\tau^{2\varepsilon}}{\xi_{max}} + \\
\upsilon_{\mu b}\frac{\omega^{2\varepsilon}\tau^{2\varepsilon}}{\xi_{max}} - \frac{\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{\varepsilon}\tau^{2\varepsilon}}{\xi_{max}} - \frac{\tau^{2\varepsilon}}{\xi_{max}} \\
\left(\begin{array}{l}
\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\upsilon_f^2\mu^{\varepsilon} + \upsilon_f^2\frac{\Gamma^2(1+2\varepsilon)\Gamma(1+4\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+3\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{3\varepsilon}}{\xi_{max}^2} - \\
-2\upsilon_f^2\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{\varepsilon}\mu^{2\varepsilon}}{\xi_{max}} + \upsilon_{yb}^2\frac{\Gamma^3(1+2\varepsilon)}{\Gamma^5(1+\varepsilon)}\frac{\omega^{4\varepsilon}\mu^{\varepsilon}}{\xi_{max}^2} - 2\upsilon_{yb}^2\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\frac{\omega^{3\varepsilon}}{\xi_{max}^2} \\
+2\upsilon_f\upsilon_{\mu b}\Gamma(1+\varepsilon)\omega^{\varepsilon} + 2\upsilon_f\upsilon_{\mu b}\frac{\Gamma(1+2\varepsilon)}{\Gamma^4(1+\varepsilon)}\frac{\omega^{3\varepsilon}\mu^{2\varepsilon}}{\xi_{max}^2} - \\
4\upsilon_f\upsilon_{\mu b}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{\varepsilon}}{\xi_{max}}
\end{array}\right) \\
+\frac{2\tau^{\varepsilon}}{\xi_{max}}\left(\begin{array}{l}
\upsilon_f\frac{\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}\omega^{\varepsilon}\mu^{\varepsilon} - \upsilon_f\frac{\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{2\varepsilon}\mu^{2\varepsilon}}{\xi_{max}} \\
+\upsilon_{\mu b}\Gamma(1+2\varepsilon)\omega^{2\varepsilon} - \upsilon_{\mu b}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^3(1+\varepsilon)}\omega^{3\varepsilon}\mu^{\varepsilon}
\end{array}\right)
\end{array}\right] \\
& - D\upsilon_f\frac{\Gamma^2(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\frac{\omega^{\varepsilon}\tau^{\varepsilon}}{\xi_{max}}.
\end{aligned} \tag{6.21}
$$

Proceeding in a similar way, we obtain the fractal series solution of (6.12) as

$$\xi(\omega,\mu,\tau) = \lim_{n\to\infty}\xi_n(\omega,\mu,\tau). \tag{6.22}$$

Figure 2 shows the graphs for the aforementioned approximations with parameter $\varepsilon = \ln 2/\ln 3$. These graphical depictions of the approximations of solution show that, for a given 2D fractal LWR model, the non-differentiable traffic density function $\xi(\omega,\mu,\tau)$ evolves dynamically under the influence of lateral dynamics constrained by the road width ($\mu = 10m$), and reflecting drivers' preference to stay on the road surface in fractal vehicular traffic flow. Moreover, visualisations that are displayed make it evident that these iterations are converging near the solution's surface.

Example 6.3. Consider a 2D local fractional LWR model

$$\frac{\partial^{\varepsilon}}{\partial\tau^{\varepsilon}}\xi + \frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}(\xi\upsilon_{\omega}) + \frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}(\xi\upsilon_{\mu}) = D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi,\ \forall\tau > 0, -\infty < \omega < \infty, -b \leqslant \mu \leqslant b, 0 < \varepsilon \leqslant 1, \tag{6.23}$$

with longitudinal υ_{ω} and lateral υ_{μ} speed

$$\upsilon_{\omega} = \upsilon_f\left(1 - \frac{\xi}{\xi_{max}}\right), \tag{6.24}$$

$$\upsilon_{\mu} = \upsilon_{\mu b}E_{\varepsilon}(\mu^{\varepsilon})\left(1 - \frac{\xi}{\xi_{max}}\right), \tag{6.25}$$

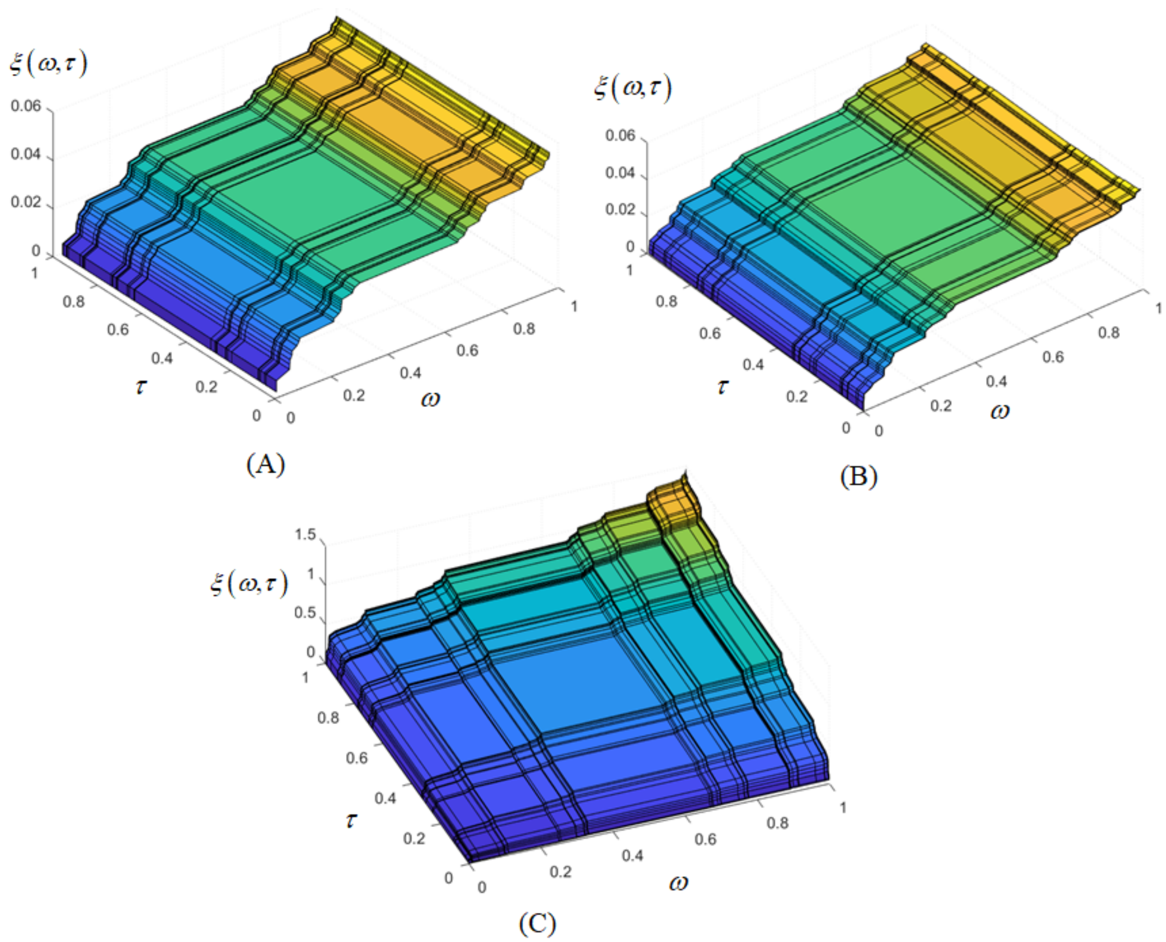

Figure 2: (A), (B) and (C) represents initial, first and second approximations of (6.12) with road width $\mu = 10m$ and parameter $\varepsilon = \ln 2/\ln 3$

subject to the initial condition
$$\xi\left(\omega,\mu,0\right)=\omega^{\varepsilon}\mu^{\varepsilon}. \tag{6.26}$$
With reference to (5.8), the successive approximation corresponding to (6.23) is
$$M_{\varepsilon}\left\{\xi_{n+1}\left(\omega,\mu,\tau\right)\right\}=M_{\varepsilon}\left\{\xi_{n}\left(\omega,\mu,\tau\right)\right\}-\frac{1}{s^{\varepsilon}}M_{\varepsilon}\left\{\begin{array}{l}\frac{\partial^{\varepsilon}}{\partial\tau^{\varepsilon}}\xi_{n}\left(\omega,\mu,\tau\right)+\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\left(\xi_{n}\left(\omega,\mu,\tau\right)\upsilon_{n\omega}\right)\\+\frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}\left(\xi_{n}\left(\omega,\mu,\tau\right)\upsilon_{n\mu}\right)-D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_{n}\left(\omega,\mu,\tau\right)\end{array}\right\}, \tag{6.27}$$
with initial approximation
$$M_{\varepsilon}\left\{\xi_{0}\left(\omega,\mu,\tau\right)\right\}=M_{\varepsilon}\left\{\omega^{\varepsilon}\mu^{\varepsilon}\right\}=\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{s^{\varepsilon}}. \tag{6.28}$$
First approximation to $\xi\left(\omega,\mu,\tau\right)$ is given by
$$\begin{aligned}M_{\varepsilon}\left\{\xi_{1}\left(\omega,\mu,\tau\right)\right\}&=M_{\varepsilon}\left\{\xi_{0}\left(\omega,\mu,\tau\right)\right\}-\frac{1}{s^{\varepsilon}}M_{\varepsilon}\left\{\begin{array}{l}\frac{\partial^{\varepsilon}}{\partial\tau^{\varepsilon}}\xi_{0}\left(\omega,\mu,\tau\right)+\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\left(\xi_{0}\left(\omega,\mu,\tau\right)\upsilon_{n\omega}\right)\\+\frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}\left(\xi_{0}\left(\omega,\mu,\tau\right)\upsilon_{n\mu}\right)-D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_{0}\left(\omega,\mu,\tau\right)\end{array}\right\}\\
&=\xi_{0}\left(\omega,\mu,s\right)-\\
&\frac{1}{s^{\varepsilon}}\left\{\begin{array}{l}s^{\varepsilon}\xi_{0}\left(\omega,\mu,s\right)-\xi_{0}\left(\omega,\mu,0\right)+\upsilon_{f}\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\left(\xi_{0}\left(\omega,\mu,s\right)-\frac{M_{\varepsilon}\left\{\xi_{0}^{2}(\omega,\mu,\tau)\right\}}{\xi_{max}}\right)\\+\upsilon_{\mu b}\frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}E_{\varepsilon}\left(\mu^{\varepsilon}\right)\left(\xi_{0}\left(\omega,\mu,s\right)-\frac{M_{\varepsilon}\left\{\xi_{0}^{2}(\omega,\mu,\tau)\right\}}{\xi_{max}}\right)-\\D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_{0}\left(\omega,\mu,s\right)\end{array}\right\}\\
&=\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{s^{\varepsilon}}-\frac{\upsilon_{f}}{s^{\varepsilon}}\left(\Gamma\left(1+\varepsilon\right)\frac{\mu^{\varepsilon}}{s^{\varepsilon}}-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{\varepsilon}\mu^{2\varepsilon}}{s^{\varepsilon}\xi_{max}}\right)-\\
&\frac{\upsilon_{\mu b}}{s^{\varepsilon}}\left\{E_{\varepsilon}\left(\mu^{\varepsilon}\right)\left(\Gamma\left(1+\varepsilon\right)\frac{\omega^{\varepsilon}}{s^{\varepsilon}}-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{2\varepsilon}\mu^{\varepsilon}}{s^{\varepsilon}\xi_{max}}\right)+E_{\varepsilon}\left(\mu^{\varepsilon}\right)\left(\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{s^{\varepsilon}}-\frac{\omega^{2\varepsilon}\mu^{2\varepsilon}}{s^{\varepsilon}\xi_{max}}\right)\right\}\\
&=\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{s^{\varepsilon}}-\frac{\upsilon_{f}\mu^{\varepsilon}}{s^{2\varepsilon}}\left(\Gamma\left(1+\varepsilon\right)-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{\xi_{max}}\right)-\\
&\frac{\upsilon_{\mu b}}{s^{2\varepsilon}}E_{\varepsilon}\left(\mu^{\varepsilon}\right)\omega^{\varepsilon}\left(\Gamma\left(1+\varepsilon\right)-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{\xi_{max}}+\mu^{\varepsilon}-\frac{\omega^{\varepsilon}\mu^{2\varepsilon}}{\xi_{max}}\right).\end{aligned} \tag{6.29}$$
This implies
$$\begin{aligned}\xi_{1}\left(\omega,\mu,\tau\right)&=M_{\varepsilon}^{-1}\left\{\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{s^{\varepsilon}}\right\}-M_{\varepsilon}^{-1}\left\{\frac{\upsilon_{f}\mu^{\varepsilon}}{s^{2\varepsilon}}\left(\Gamma\left(1+\varepsilon\right)-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{\xi_{max}}\right)\right\}-\\
&M_{\varepsilon}^{-1}\left\{\frac{\upsilon_{\mu b}}{s^{2\varepsilon}}E_{\varepsilon}\left(\mu^{\varepsilon}\right)\omega^{\varepsilon}\left(\Gamma\left(1+\varepsilon\right)-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{\xi_{max}}+\mu^{\varepsilon}-\frac{\omega^{\varepsilon}\mu^{2\varepsilon}}{\xi_{max}}\right)\right\}\\
&=\omega^{\varepsilon}\mu^{\varepsilon}-\frac{\upsilon_{f}\mu^{\varepsilon}\tau^{\varepsilon}}{\Gamma\left(1+\varepsilon\right)}\left(\Gamma\left(1+\varepsilon\right)-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{\xi_{max}}\right)-\\
&\frac{\upsilon_{\mu b}}{\Gamma\left(1+\varepsilon\right)}E_{\varepsilon}\left(\mu^{\varepsilon}\right)\omega^{\varepsilon}\tau^{\varepsilon}\left(\Gamma\left(1+\varepsilon\right)-\frac{\Gamma\left(1+2\varepsilon\right)}{\Gamma\left(1+\varepsilon\right)}\frac{\omega^{\varepsilon}\mu^{\varepsilon}}{\xi_{max}}+\mu^{\varepsilon}-\frac{\omega^{\varepsilon}\mu^{2\varepsilon}}{\xi_{max}}\right).\end{aligned} \tag{6.30}$$
Second approximation to $\xi\left(\omega,\mu,\tau\right)$ is given by
$$\begin{aligned}M_{\varepsilon}\left\{\xi_{2}\left(\omega,\mu,\tau\right)\right\}&=M_{\varepsilon}\left\{\xi_{1}\left(\omega,\mu,\tau\right)\right\}-\frac{1}{s^{\varepsilon}}M_{\varepsilon}\left\{\begin{array}{l}\frac{\partial^{\varepsilon}}{\partial\tau^{\varepsilon}}\xi_{1}\left(\omega,\mu,\tau\right)+\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\left(\xi_{1}\left(\omega,\mu,\tau\right)\upsilon_{n\omega}\right)\\+\frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}\left(\xi_{1}\left(\omega,\mu,\tau\right)\upsilon_{n\mu}\right)-D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_{1}\left(\omega,\mu,\tau\right)\end{array}\right\}\\
&=\xi_{1}\left(\omega,\mu,s\right)-\\
&\frac{1}{s^{\varepsilon}}\left\{\begin{array}{l}s^{\varepsilon}\xi_{1}\left(\omega,\mu,s\right)-\xi_{1}\left(\omega,\mu,0\right)+\upsilon_{f}\frac{\partial^{\varepsilon}}{\partial\omega^{\varepsilon}}\left(\xi_{1}\left(\omega,\mu,s\right)-\frac{M_{\varepsilon}\left\{\xi_{1}^{2}(\omega,\mu,\tau)\right\}}{\xi_{max}}\right)\\+\upsilon_{\mu b}\frac{\partial^{\varepsilon}}{\partial\mu^{\varepsilon}}E_{\varepsilon}\left(\mu^{\varepsilon}\right)\left(\xi_{1}\left(\omega,\mu,s\right)-\frac{M_{\varepsilon}\left\{\xi_{1}^{2}(\omega,\mu,\tau)\right\}}{\xi_{max}}\right)-\\D\frac{\partial^{2\varepsilon}}{\partial\mu^{2\varepsilon}}\xi_{1}\left(\omega,\mu,s\right)\end{array}\right\}\end{aligned} \tag{6.31}$$

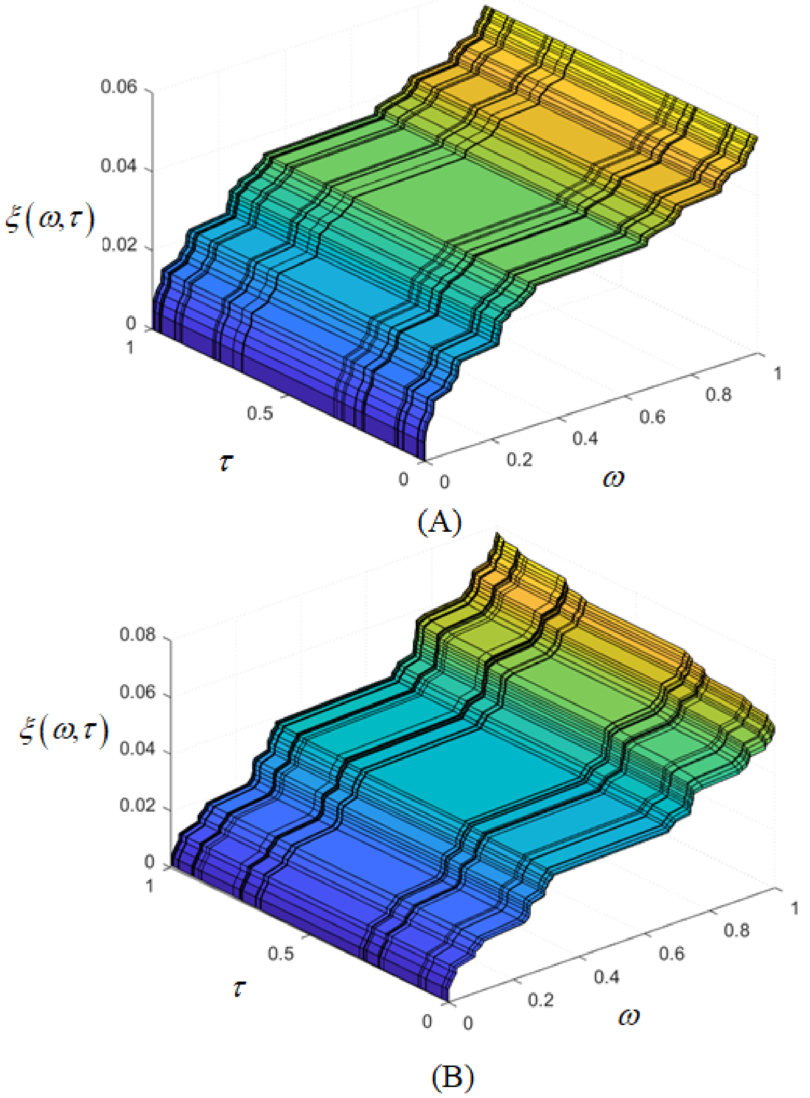

Figure 3: (A) and (B) represent initial and first approximations of (6.23) with road width $\mu = 10m$ and parameter $\varepsilon = \ln 2/\ln 3$

Proceeding in the same way, we get the fractal series solution of (6.23) by

$$M_{\varepsilon}\{\xi(\omega,\mu,\tau)\} = \lim_{n\to\infty} M_{\varepsilon}\{\xi_n(\omega,\mu,\tau)\}, \tag{6.32}$$

and thus, we have

$$\xi(\omega,\mu,\tau) = \lim_{n\to\infty} M_{\varepsilon}^{-1}\{M_{\varepsilon}\{\xi_n(\omega,\mu,\tau)\}\}. \tag{6.33}$$

With parameter $\varepsilon = \ln 2/\ln 3$, the graphical results for the above performed iterations are displayed in Figure 3 and depict the behaviour by which non-differentiable traffic density function $\xi(\omega,\mu,\tau)$ evolves dynamically with an influence of road width $(\mu = 10m)$. Furthermore, the evolution of density function w.r.t road width for road length 1000m, 3000m and 5000m, respectively, is shown by Figure 4.

7. Conclusion

This study proposed a 2D local fractional LWR model of fractal vehicular traffic flow, and non-differentiable solutions are derived by using LFVIM. Numerous research studies have been carried out in the literature to investigate the one-dimensional LWR model. Nonetheless, this study highlighted on

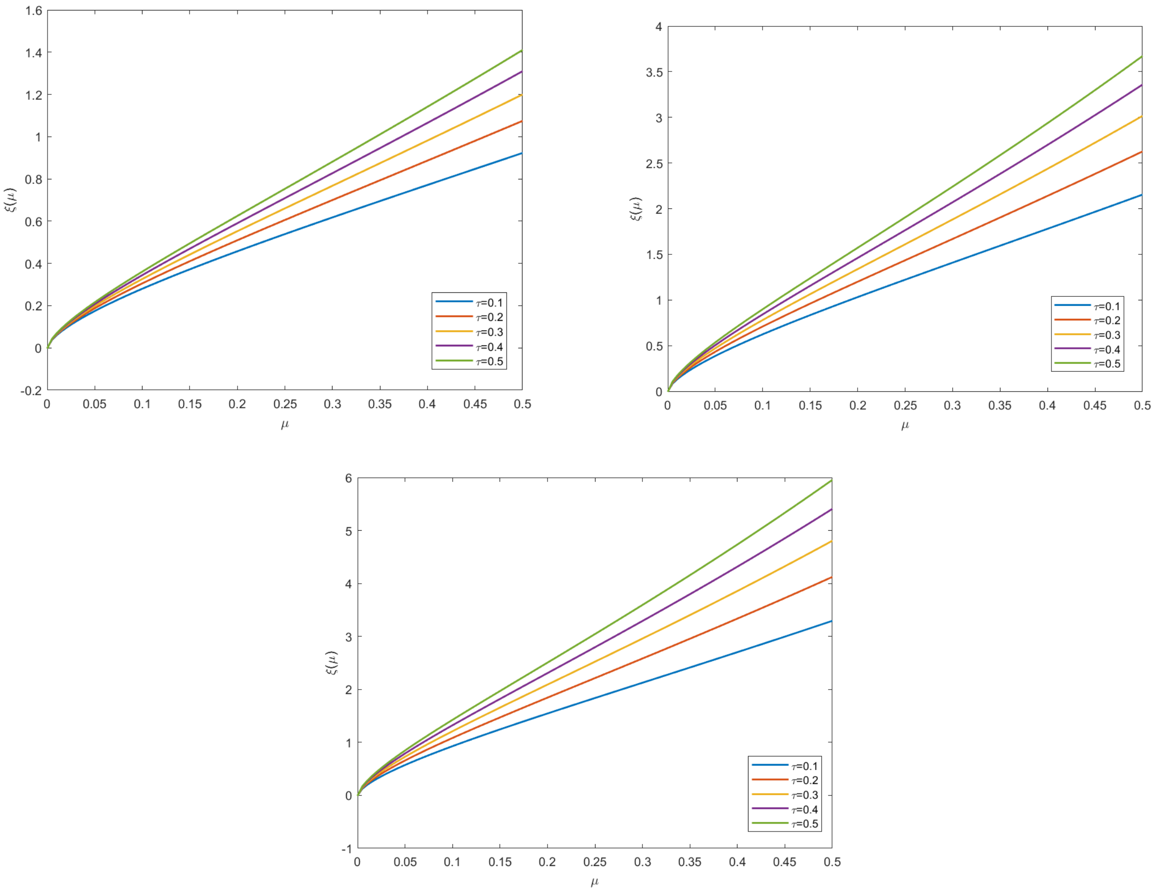

Figure 4: (a), (b) and (c) represent the evolution of density w.r.t road width for road length 1000m, 3000m and 5000m respectively for model eqn (6.23) with parameter $\varepsilon = \ln 2/\ln 3$

the two-dimensional local fractional LWR model to address the non-differentiable traffic parameters and include lateral dynamics stimulated by boundary repulsion and diffusion. This resulted in a novel and comprehensive fractal order 2D-directed lane-free fractal vehicular traffic model. Firstly, the discussion has focused on the existence and uniqueness of the solution of proposed model. Illustrative instances are provided to demonstrate the usefulness of implementing LFLVIM in the presented 2D fractal model. Further, numerical simulations for each case have also been shown. It has been demonstrated through the graphic representations of the successive approximations of solutions that the non-differentiable traffic density function $\xi(\omega, \mu, \tau)$ for the 2D fractal LWR model dynamically evolves with the effect of lateral dynamic restricted by the road width, which reflects drivers' desire to remain on the road surface in fractal vehicular traffic flow. Furthermore, it is observed that these approximations are seen to be converging close to the solution surface. The results of the study illustrate the effectiveness and efficiency of the model in describing the fractal flow of traffic. Also, it is found that the local fractional iterative technique is useful to derive the non-differentiable solution for the 2D local fractional LWR model. As part of the future scope of study, the proposed 2D local fractional LWR model can be extended to n-dimensions and examined using numerous iterative methodologies for discovering new knowledge and advancements.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of Interest

This work does not have any conflicts of interest.

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References

[1] Agrawal, S., Kanagaraj, V., & Treiber, M. (2023). Two-dimensional LWR model for lane-free traffic. Physica A: Statistical Mechanics and its Applications, 625, 128990. 1, 3, 6

[2] Amin, S., Andrews, S., Apte, S., Arnold, J., Ban, J., Benko, M., ... & Tinka, A. (2008, November). Mobile century-using GPS mobile phones as traffic sensors: a field experiment. In 15th World Congress on Intelligent Transportation Systems (Vol. 2008, p. 18). IEEE. 1

[3] Aw, A. A. T. M., & Rascle, M. (2000). Resurrection of" second order" models of traffic flow. SIAM journal on applied mathematics, 60(3), 916-938. 1, 1

[4] Babakhani, A., & Daftardar-Gejji, V. (2002). On calculus of local fractional derivatives. Journal of Mathematical Analysis and Applications, 270(1), 66-79. 1

[5] Baleanu, D., & Jassim, H. K. (2019). A modification fractional homotopy perturbation method for solving Helmholtz and coupled Helmholtz equations on Cantor sets. Fractal and Fractional, 3(2), 30. 1

[6] Baleanu, D., Jassim, H. K., & Al Qurashi, M. (2019). Solving Helmholtz equation with local fractional derivative operators. Fractal and Fractional, 3(3), 43. 1

[7] Balzotti, C., & Göttlich, S. (2020). A two-dimensional multi-class traffic flow model. arXiv preprint arXiv:2006.10131. 1

[8] Blandin, S., Bretti, G., Cutolo, A., & Piccoli, B. (2009). Numerical simulations of traffic data via fluid dynamic approach. Applied Mathematics and Computation, 210(2), 441-454. 1

[9] Cannon, J. W. (1984). The fractal geometry of nature. by Benoit B. Mandelbrot. The American Mathematical Monthly, 91(9), 594-598. 1
[10] Chen, W., Sun, H., Zhang, X., & Koroak, D. (2010). Anomalous diffusion modeling by fractal and fractional derivatives. Computers & Mathematics with Applications, 59(5), 1754-1758. 1
[11] Claudel, C. G., & Bayen, A. M. (2010). LaxHopf based incorporation of internal boundary conditions into Hamil-tonJacobi equation. Part I: Theory. IEEE Transactions on Automatic Control, 55(5), 1142-1157. 1
[12] Cui, P., & Jassim, H. K. (2024). Local fractional Sumudu decomposition method to solve fractal PDEs arising in mathematical physics. Fractals, 32(04), 2440029. 1
[13] Daganzo, C. F. (2002). A behavioral theory of multi-lane traffic flow. Part I: Long homogeneous freeway sections. Transportation Research Part B: Methodological, 36(2), 131-158. 1
[14] Fan, S., Herty, M., & Seibold, B. (2013). Comparative model accuracy of a data-fitted generalized Aw-Rascle-Zhang model. arXiv preprint arXiv:1310.8219. 1
[15] Gazis, D. C., Herman, R., & Weiss, G. H. (1962). Density oscillations between lanes of a multilane highway. Operations Research, 10(5), 658-667. 1
[16] Gupta, A. K., & Dhiman, I. (2014). Analyses of a continuum traffic flow model for a nonlane-based system. International Journal of Modern Physics C, 25(10), 1450045. 1
[17] Hao, Y. J., Srivastava, H. M., Jafari, H., & Yang, X. J. (2013). Helmholtz and diffusion equations associated with local fractional derivative operators involving the Cantorian and Cantortype cylindrical coordinates. Advances in Mathematical Physics, 2013(1), 754248. 1
[18] He, J. H., Elagan, S. K., & Li, Z. B. (2012). Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. Physics letters A, 376(4), 257-259. 1
[19] Helbing, D., & Treiber, M. (1999). Numerical simulation of macroscopic traffic equations. Computing in Science & Engineering, 1(5), 89-98. 1
[20] Herty, M., Fazekas, A., & Visconti, G. (2017). A two-dimensional data-driven model for traffic flow on highways. arXiv preprint arXiv:1706.07965. 1
[21] Jafari, H., & Jassim, H. K. (2015). Local fractional variational iteration method for solving nonlinear partial differential equations within local fractional operators. Applications and Applied Mathematics: An International Journal (AAM), 10(2), 29. 2.4, 2.5
[22] Jafari, H., Jassim, H. K., Ünlü, C., & Nguyen, V. T. (2024). Laplace decomposition method for solving the two-dimensional diffusion problem in fractal heat transfer. Fractals, 32(04), 2440026. 1
[23] Jafari, H., Jassim, H. K., Ansari, A., & Nguyen, V. T. (2024). Local fractional variational iteration transform method: A tool for solving local fractional partial differential equations. Fractals, 32(04), 2440022. 1
[24] Jafari, H., Zair, M. Y., & Jassim, H. K. (2023). Analysis of fractional NavierStokes equations. Heat Transfer, 52(3), 2859-2877. 1
[25] Jassim, H. K., & Khafif, S. A. (2021). SVIM for solving Burgers and coupled Burgers equations of fractional order. Progress in Fractional Differentiation and Applications, 7(1), 1-6. 1
[26] Kamil Jassim, H., & Vahidi, J. (2021). A new technique of reduce differential transform method to solve local fractional PDEs in mathematical physics. International Journal of Nonlinear Analysis and Applications, 12(1), 37-44. 1
[27] Laval, J. A., & Daganzo, C. F. (2006). Lane-changing in traffic streams. Transportation Research Part B: Methodological, 40(3), 251-264. 1
[28] Lebacque, J. P. (1996, July). The godunov scheme and what it means for rst order tra c ow models. In Proceedings of the 13th International Symposium on Transportation and Traffic Theory, Lyon, France, July (Vol. 2426). 1
[29] Liang, Y., Allen, Q. Y., Chen, W., Gatto, R. G., Colon-Perez, L., Mareci, T. H., & Magin, R. L. (2016). A fractal derivative model for the characterization of anomalous diffusion in magnetic resonance imaging. Communications in Nonlinear Science and Numerical Simulation, 39, 529-537. 1
[30] Liang, Y., Chen, W., Xu, W., & Sun, H. (2019). Distributed order Hausdorff derivative diffusion model to characterize non-Fickian diffusion in porous media. Communications in Nonlinear Science and Numerical Simulation, 70, 384-393. 1
[31] Lighthill, M. H., Whitham, G. B. (1961). II-A Theory of Traffic Flow on Long Crowded Roads. Proceedings of the Royal Society of London: Mathematical and physical sciences, 229, 317. 1
[32] Liu, J. G., Yang, X. J., Feng, Y. Y., & Cui, P. (2020). A new perspective to study the third-order modified KDV equation on fractal set. Fractals, 28(06), 2050110. 1
[33] Mohan, R., & Ramadurai, G. (2021). Multi-class traffic flow model based on three dimensional flowconcentration surface. Physica A: Statistical Mechanics and its Applications, 577, 126060. 1
[34] Munjal, P. K., & Pipes, L. A. (1971). Propagation of on-ramp density waves on uniform unidirectional multilane freeways. Transportation science, 5(4), 390-402. 1
[35] Richards, P. I. (1956). Shock waves on the highway. Operations research, 4(1), 42-51. 1
[36] Vikram, D., Mittal, S., & Chakroborty, P. (2022). Stabilized finite element computations with a two-dimensional continuum model for disorderly traffic flow. Computers & Fluids, 232, 105205. 1
[37] Wang, L. F., Yang, X. J., Baleanu, D., Cattani, C., & Zhao, Y. (2014). Fractal dynamical model of vehicular traffic flow within the local fractional conservation laws. In Abstract and Applied Analysis (Vol. 2014, No. 1, p. 635760).

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[38] Yang, A. M., Cattani, C., Zhang, C., Xie, G. N., & Yang, X. J. (2014). Local fractional Fourier series solutions for nonhomogeneous heat equations arising in fractal heat flow with local fractional derivative. Advances in Mechanical Engineering, 6, 514639. 1, 1, 2.8, 2.9

[39] Yang, X. J. (2012). Advanced local fractional calculus and its applications. 1, 2.1, 2.2, 2.3, 2.6, 2.7, 5

[40] Yang, X. J., Baleanu, D., & Tenreiro Machado, J. A. (2013). Systems of NavierStokes equations on Cantor sets. Mathematical Problems in Engineering, 2013(1), 769724. 1, 1

[41] Yang, X. J. (2011). Local Fractional Functional Analysis & Its Applications (Vol. 1). Hong Kong: Asian Academic Publisher Limited. 1, 2.8, 2.9

[42] Zayir, M. Y., & Kamil Jassim, H. (2022). A Fractional Variational Iteration Approach for Solving Time-Fractional Navier-Stokes Equations. Mathematics and Computational Sciences, 3(2), 41-47. 1

[43] Zhang, H. M. (2002). A non-equilibrium traffic model devoid of gas-like behavior. Transportation Research Part B: Methodological, 36(3), 275-290. 1

[44] Zhao, Y., Baleanu, D., Cattani, C., Cheng, D. F., & Yang, X. J. (2013). Maxwells equations on Cantor sets: a local fractional approach. Advances in high energy physics, 2013(1), 686371. 1, 1