Math. & Comput. Sci., 6(2) 2025, 1–31
Online: ISSN 2717-2708
Mathematics and Computational Sciences
Journal Homepage: http://mcs.qut.ac.ir/mcs

A spectral collocation method for solving stochastic fractional integro-differential equation

Mahsa Zaboli[a], Haleh Tajadodi[a,*]

[a] Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.

Abstract

This paper presents a numerical scheme based on shifted Vieta-Lucas polynomials to solve stochastic fractional integro-differential equations (SFIDEs). The proposed method approximates Brownian motion using Gauss-Legendre quadrature, simplifying computational processes. Additionally, it employs strategically chosen collocation points to transform the target stochastic equation into a system of algebraic equations, which are solved via Newtons method. Convergence and error analyses of the method are rigorously established. The study further examines the existence and uniqueness of solutions for the considered equations. Numerical examples demonstrate the effectiveness, compatibility, and accuracy of the proposed technique, highlighting its advantages in reducing computational effort while maintaining minimal error margins.

Keywords: Stochastic fractional integro-differential equations, Shifted Vieta-Lucas polynomials, Operational matrix, Brownian motion.

2020 MSC: 47Gxx, 60G22, 26A33.

1. Introduction

Integral equations are prevalent in a wide array of physical applications and appear across various scientific disciplines and numerous practical scenarios, such as heat and mass transfer, oscillation theory, fluid dynamics, electrostatics, electrodynamics, biomechanics, control theory, electrical engineering, economics, and medicine. These equations can be categorized into several types, including Volterra integral equations (IEs), Fredholm IEs, and Volterra-Fredholm IEs.

A distinguishing feature of Fredholm integral equations is that their integral limits are constant, whereas in Volterra integral equations, the limits are variable. Both types can appear in the form of first- and second-kind equations. The evolution of integral equation theory has led to the formulation of numerous models addressing problems in engineering and mathematical physics, such as scattering in quantum mechanics, diffraction phenomena, conformal mapping, and water wave dynamics. Many initial and boundary value problems in physics and engineering can also be addressed by transforming them into integral equations, encompassing population growth models, heat transformation processes, electromagnetic and electrostatic problems, and more [24].

*Corresponding author
Email addresses: maza25124@gmail.com (Mahsa Zaboli), tajadodi@math.usb.ac.ir (Haleh Tajadodi)

doi: 10.30511/mcs.2025.2037499.1218

Received: 08 August 2024 Accepted: 19 February 2025

Integro-differential equations involve both integrals and derivatives of an unknown function and arise in diverse scientific and engineering contexts. Similar to integral equations, integro-differential equations manifest in various forms.

Fractional differential equation, fractional integral and integro-differential equations have gained significant attention due to their broad applicability in real-world problems [18, 19]. In natural dynamical processes, the future states of a system often depend on all its past states. Fractional-order operators preserve the inherited properties of given functions, providing researchers with a more comprehensive understanding of real-world phenomena. Numerous theoretical and numerical methods for solving fractional-order equations have been introduced in the literature, including the variational iteration method (Sayevand [17], 2015), Wilson wavelets approximation method (Mousavi et al. [13], 2017), least squares method (Ezzati et al. [5], 2019), Legendre spectral collocation method (Yousefi et al. [25], 2019), perturbation iteration algorithm (Senol et al. [20], 2017), Sixth-kind Chebyshev collocation method (Babaei et al. [2], 2020), and Hosoya collocation method (Jafari et al. [6], 2024).

Stochastic differential equations (SDEs) extend ordinary differential equations by incorporating a stochastic term alongside the deterministic component. The deterministic term describes the behavior of the phenomenon, while the stochastic term accounts for random perturbations or "noise" affecting the system. SDEs play a crucial role in basic science and technology, particularly when accounting for random environmental fluctuations. They are widely applied in fields such as physics, finance, medicine, and biology to accurately model phenomena influenced by random perturbations.

Since most stochastic fractional integro-differential equations (SFIDEs) lack exact analytical solutions, researchers have developed various numerical techniques to approximate their solutions. Over the past decade, several methods have been proposed to solve SFIDEs. For instance, Mirzaee et al. [11] (2017) utilized operational matrices based on orthonormal Bernstein polynomials to solve fractional stochastic integro-differential equations. Mirzaee et al. [10] (2019) employed a numerical method combining block-pulse and parabolic functions to approximate solutions for systems of nonlinear stochastic Itô-Volterra integral equations of fractional order. Sayevand et al. [16] (2020) presented a numerical technique using Bernstein operational matrices and the trapezoidal rule to address a family of stochastic fractional integro-differential equations. Masti et al. [8] (2024) applied On collocation-Galerkin method to for a class of SFIDEs. Azimi et al. [1] (2022) introduced a Tau method based on shifted Legendre polynomials for solving a class of fractional stochastic integro-differential equations. More recently, Saha Ray et al. [3] (2023) proposed an operational matrix method using shifted VietaFibonacci polynomials to numerically solve fractional-order stochastic integro-differential equations. Additionally, Saha Ray et al. [21] (2023) developed the Lerch operational matrix method to solve stochastic fractional differential equations. Other approaches have also been explored to address SFIDEs effectively.

In this paper, we consider the following stochastic fractional integral-differential equation (SFIDE):

$$D^{\alpha}\Upsilon(\chi) = g(\chi) + \int_0^{\chi} \mathcal{K}_1(\chi,\xi)\Upsilon(\xi)d\xi + \sigma\int_0^{\chi} \mathcal{K}_2(\chi,\xi)\Upsilon(\xi)dB(\xi), \quad \chi \in [0,1], \tag{1.1}$$

with initial condition

$$\Upsilon(0) = \Upsilon_0, \quad 0 < \alpha \leqslant 1, \tag{1.2}$$

where $D^{\alpha}(.)$ denotes the Caputo fractional order derivate of order α and
$\Upsilon(\chi)$, $g(\chi)$, $\mathcal{K}_i(\chi,\xi), i = 1,2$ are stochastic process defined on the same probability space (Ω, F, P), $\Upsilon(\chi)$ is unknown and $\int_0^{\chi} \mathcal{K}_2(\chi,\xi)\Upsilon(\xi)dB(\xi)$ is Itô integral. Here, σ is constant number. A real-valued stochastic process $B(\chi), \chi \in [0,T]$ is called the Brownian motion with the following properties [12]:
i) For $\chi \geqslant 0$, $B(\chi)$ is a continuous function of χ.
ii) For $0 \leqslant \chi_0 < \chi \leqslant T$, $B(\chi) - B(\chi_0)$, is independent of the past. As a results, $B(0) = 0$ (with the probability 1)
iii) For $0 \leqslant \chi_0 < \chi \leqslant T$, $B(\chi) - B(\chi_0)$ has normal distribution with mean zero and variance $\chi - \chi_0$. Such that, $B(\chi) - B(\chi_0) \sim \sqrt{\chi - \chi_0}\, \mathcal{N}(0,1)$, where $\mathcal{N}(0,1)$ denotes a normally distributed random variable with

zero mean and unit variance.
Vieta-Lucas polynomials offer several advantages in computational mathematics. First, their unique properties-including orthogonality and recursive structure-make them highly effective as basis functions for achieving precise numerical solutions. Despite these advantages, Vieta-Lucas polynomials have received limited attention in research compared to other polynomial families. A key contribution of this work lies in employing a modified set of shifted Vieta-Lucas polynomials as basis functions. By retaining only a few dominant modes, this approach enables highly accurate approximations while significantly reducing computational complexity. Furthermore, the associated truncation errors are negligible, ensuring robustness in practical applications.

The remainder of this paper is organized as follows. Section 2 introduces fundamental definitions and background in fractional and stochastic calculus necessary for the subsequent analysis. This section also details Vieta-Lucas polynomials, a class of orthogonal polynomials, and their role in function approximation. Section 3 provides a theoretical proof of the error analysis and convergence properties of the proposed scheme. Section 4 presents the integral operational matrix, derivative operational matrix, and operational matrix of product. Section 5 details the implementation of the method for solving stochastic fractional integral-differential equations. In Section 6, we establish the existence and uniqueness of solutions for these equations. Section 7 illustrates the efficacy of the scheme through numerical examples. Finally, concluding remarks are provided in Section 8.

2. Definitions and Basic concepts

In this section, we present some necessary definitions and properties of fractional operators, Vieta-Lucas polynomials.

2.1. Fractional and Stochastic calculus

Fractional calculus extends traditional integration and differentiation to non-integer orders through diverse definitions. The Riemann-Liouville formulation is primarily utilized for fractional integration, while the Caputo definition is favored for differentiation. Beyond these, alternative approaches are also developed within stochastic calculus frameworks.

Definition 2.1. The fractional Riemann-Liouville integral of order $\alpha > 0$ is exhibited as [14]

$$I^{\alpha}\Upsilon(\chi) = \frac{1}{\Gamma(\alpha)}\int_0^{\chi} \frac{\Upsilon(\xi)}{(\chi-\xi)^{1-\alpha}}d\xi. \tag{2.1}$$

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ is defined as [14]

$$D^{\alpha}\Upsilon(\chi) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\chi} \frac{\Upsilon^{(n)}(\xi)}{(\chi-\xi)^{1+\alpha-n}}d\xi, \quad n-1<\alpha<n,\ n\in N. \tag{2.2}$$

Some of most significant properties of fractional operators are as follows

I) $D^{\alpha}C = 0$, (C is a constant),

II) $D^{\alpha}\chi^{\gamma} = \begin{cases} 0, & \gamma \in N,\ \gamma < \lceil\alpha\rceil, \\ \frac{\Gamma(\gamma+1)}{\Gamma(1+\gamma-\alpha)}\chi^{\gamma-\alpha}, & \gamma\in N,\ \gamma \geqslant \lceil\alpha\rceil, \text{ or } \gamma\notin N,\ \gamma > \lfloor\alpha\rfloor, \end{cases}$

III) $D^{\alpha}I^{\alpha}\Upsilon(\chi) = \Upsilon(\chi)$,

IV) $I^{\alpha}D^{\alpha}\Upsilon(\chi) = \Upsilon(\chi) - \sum_{k=0}^{n-1} h^{(k)}(0^{+})\frac{\chi^k}{k!}, \quad n-1<\alpha\leqslant n.$

Definition 2.3. (Itô isometry): Let $B(\chi)$ be the real valued winer process and X be a stochastic process then[7]

$$E\left|\int_0^{\chi} X(\xi)dB(\xi)\right|^2 = \int_0^{\chi} E|X(\xi)|^2 d\xi, \tag{2.3}$$

where E is the mathematical expection.

Definition 2.4. (Jensen's inequality): Let $h \in L^2(0,1)$ then we have[7]

$$E\left(\left|\int_0^{\chi} h(\Upsilon(\xi))d(\xi)\right|^2\right) \leqslant E\left(\chi\int_0^{\chi} |h(\Upsilon(\xi))|^2 d\xi\right). \tag{2.4}$$

2.2. Shifted Vieta-Lucas polynomials

This section outlines the definitions and properties of shifted Vieta-Lucas polynomials (VLPs), emphasizing their classification as orthogonal polynomials. We highlight their recurrence relations and analytical formulations, which underpin the construction of the orthogonal polynomial family termed Vieta-Lucas polynomials.

Vieta-Lucas polynomials are defined in the interval $[-2, 2]$. The Vieta-Lucas polynomials of degree $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is given by[4]:

$$VL_n(\chi) = 2\cos(n\upsilon), \quad \upsilon = \arccos\left(\frac{\chi}{2}\right), \quad \upsilon \in [0, \pi],$$

The explicit forms of $VL_n(\chi)$ are obtained as follows:

$$\begin{aligned} VL_n(\chi) &= \sum_{k=0}^{\lceil \frac{n}{2} \rceil} (-1)^k \binom{n-k}{k} \frac{n}{n-k}\, \chi^{n-2k} \\ &= \sum_{k=0}^{\lceil \frac{n}{2} \rceil} (-1)^k \frac{n\Gamma(n-k)}{\Gamma(k+1)\Gamma(n+1-2k)}\, \chi^{n-2k}, \quad n \geqslant 1, \end{aligned}$$

where $\lceil \frac{n}{2} \rceil$ is the celling function.

Using $VL_0(\chi) = 2$ and $VL_1(\chi) = \chi$, we can also generate the Vieta-Lucas polynomials by the following recurrence relation:

$$VL_n(\chi) = \chi VL_{n-1}(\chi) - VL_{n-2}(\chi), \quad n \geqslant 2,$$

Furthermore, The orthogonality condition of these polynomials is given with the weight function $w(\chi) = \frac{1}{\sqrt{4-\chi^2}}$ in the following form:

$$\langle VL_n(\chi), VL_m(\chi)\rangle = \int_{-2}^{2} w(\chi) VL_n(\chi) VL_m(\chi) d\chi = \begin{cases} 4\pi, & m = n = 0, \\ 2\pi, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

Vieta-Lucas polynomials are directly linked to Chebyshev polynomials of the first kind through shared orthogonality, recurrence relations, and structural properties, differing primarily by scaling factors due to distinct normalization conventions [15].

$$VL_n(\chi) = 2T_n\left(\frac{\chi}{2}\right), \qquad n \in \mathbb{N}_0,$$

where T_n is the nth Chebyshev polynomial. To use the Vieta-Lucas polynomials on the interval $[0, 1]$, we introduce the shifted Vieta-Lucas polynomials, $VL_n^*(\chi)$ as $VL_n^*(\chi) = VL_n(4\chi - 2)$. So, the explicit forms of $VL_n^*(\chi)$ are obtained as follows:

$$VL_n^*(\chi) = 2n \sum_{k=0}^{n} (-1)^k \frac{4^{n-k}\Gamma(2n-k)}{\Gamma(k+1)\Gamma(2n-2k+1)}\, \chi^{n-k}, \quad n \geqslant 2,$$

Moreover, the following recurrence relation can be utilized for generating these polynomials:

$$VL_n^*(\chi) = (4\chi - 2)VL_{n-1}^*(\chi) - VL_{n-2}^*(\chi), \quad n \geqslant 2,$$

with

$$VL_0^*(\chi) = 2, \qquad VL_1^*(\chi) = 4\chi - 2. \tag{2.5}$$

Also, another representation of the relation (2.5) could be expressed as

$$VL_n^*(\chi) = \sum_{k=0}^{n} (-1)^{n-k} \frac{4^k (2n)\Gamma(n+k)}{\Gamma(2k+1)\Gamma(n-k+1)}\, \chi^k, \;\; n \geqslant 1. \tag{2.6}$$

The orthogonality condition of the shifted Vieta-Lucas polynomials is as follows:

$$Q = \langle VL_n^*(\chi), VL_m^*(\chi)\rangle = \int_0^1 w^*(\chi) VL_n^*(\chi) VL_m^*(\chi) d\chi = \begin{cases} 4\pi, & m = n = 0, \\ 2\pi, & m = n \neq 0, \\ 0, & m \neq n, \end{cases} \tag{2.7}$$

where $w^*(\chi) = \frac{1}{\sqrt{\chi - \chi^2}}$ is weight function.

Thus, a function $\Upsilon(\chi) \in L^2[0,1]$ may be expressed via the shifted Vieta-Lucas polynomials as follows:

$$\Upsilon(\chi) = \sum_{i=0}^{\infty} c_i VL_i^*(\chi),$$

where

$$c_0 = \frac{1}{2\pi} \int_0^1 w^*(\chi) \Upsilon(\chi) d\chi,$$

$$c_i = \frac{1}{2\pi} \int_0^1 w^*(\chi) \Upsilon(\chi) VL_i^*(\chi) d\chi, \qquad i \in \mathbb{N}. \tag{2.8}$$

So, by truncating the infinite series in (2.2), we obtain the following approximation:

$$\Upsilon(\chi) \simeq \Upsilon_n(\chi) = \sum_{i=0}^{n} c_i VL_i^*(\chi) = C^T \psi(\chi), \tag{2.9}$$

where

$$C = [c_0, \ldots, c_n]^T, \tag{2.10}$$

and

$$\psi(\chi) = [VL_0^*(\chi), VL_1^*(\chi), \ldots, VL_n^*(\chi)]^T, \tag{2.11}$$

In the same, a function $\Upsilon(\chi, \xi)$ defined over $L^2([0,1] \times [0,1])$, can be expanded as:

$$\Upsilon(\chi, \xi) \simeq \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{c}_{i,j} VL_i^*(\chi) VL_j^*(\xi) = \psi^T(\chi) \hat{C} \psi(\xi),$$

where $\hat{C} = [\hat{c}_{i,j}]$ is an $(m+1) \times (n+1)$ matrix with

$$\hat{c}_{00} = \frac{1}{4\pi^2} \int_0^1 \int_0^1 w^*(\chi) w^*(\xi) \Upsilon(\chi, \xi) d\chi d\xi,$$

$$\hat{c}_{0j} = \frac{1}{4\pi^2} \int_0^1 \int_0^1 w^*(\chi) w^*(\xi) \Upsilon(\chi, \xi) VL_j^*(\xi) d\chi d\xi, \quad 1 \leqslant j \leqslant n,$$

$$\hat{c}_{i0} = \frac{1}{4\pi^2} \int_0^1 \int_0^1 w^*(\chi) w^*(\xi) \Upsilon(\chi, \xi) VL_i^*(\chi) d\chi d\xi, \quad 1 \leqslant i \leqslant m,$$

$$\hat{c}_{i0} = \frac{1}{4\pi^2} \int_0^1 \int_0^1 w^*(\chi) w^*(\xi) \Upsilon(\chi, \xi) VL_i^*(\chi) VL_j^*(\xi) d\chi d\xi, \quad 1 \leqslant i \leqslant m, \; 1 \leqslant j \leqslant n.$$

Remark 2.5. The Brownian motion $B(\chi)$ can be expanded as

$$B(\chi) \simeq \beta^T \psi(\chi), \tag{2.12}$$

where the undetermind coefficients $\beta_i, i = 0, 1, \cdots, n$ can be computed by using Eq. (2.8) and Gauss-Legendre quadrature reule [23]

$$\begin{aligned} \beta_i &= \frac{1}{4\pi}\int_0^1 w^*(\chi)B(\chi)VL_i^*(\chi)d\chi \\ &= \frac{1}{4\pi}\sum_{j=1}^{m} \omega_j w^*(\frac{1}{2}\eta_j+\frac{1}{2})B(\frac{1}{2}\eta_j+\frac{1}{2})VL_i^*(\frac{1}{2}\eta_j+\frac{1}{2}), \quad i = 0, 1, 2, \dots, n, \end{aligned} \tag{2.13}$$

where $\{\eta_j\}_{j=1}^m$ are zeros of Legendre polynomial P_m and $\{\omega_j\}_{j=1}^m$ are the corresponding weights as

$$\omega_j = \frac{2}{(1-\eta_j^2)(P'_m(\eta_j))^2}, \qquad j = 1, 2, \cdots, m.$$

3. Error and convergence analysis

3.1. Error bound

Theorem 3.1. *Suppose that function* $\Upsilon(\chi) \in C^{n+1}[0,1]$ *and* $\Upsilon_n(\chi) = \sum_{i=0}^{n} c_i VL_i^*(\chi)$ *be best approximation function to* $\Upsilon(\chi)$ *on the interval* $[0,1]$*. Then, the coefficients* c_i*, for* $i = 0, 1, \dots, n$ *are bounded as follows*

$$|c_i| \leqslant \frac{M_\Upsilon}{\gamma_i\sqrt{\pi}}\sum_{k=0}^{i}(-1)^{i-k}\frac{4^k(2i)\Gamma(i+k)\Gamma(k+\frac{1}{2})}{\Gamma(2k+1)\Gamma(i-k+1)\Gamma(k+1)}, \tag{3.1}$$

where

$$\gamma_i = \begin{cases} 4, & i = 0, \\ 2, & i \geqslant 1, \end{cases}$$

and M_Υ *is the maximum value of* $\Upsilon(\chi)$ *on the interval* $[0,1]$*.*

Proof. Using Eqs. (2.6) and (2.8) for $i = 0, 1, \dots, n$, we have

$$\begin{aligned} c_i &= \frac{1}{\gamma_i\pi}\int_0^1 \Upsilon(\chi)VL_i^*(\chi)w^*(\chi)d\chi = \frac{1}{\gamma_i\pi}\int_0^1 \Upsilon(\chi)\sum_{k=0}^{i}(-1)^{i-k}\frac{4^k(2i)\Gamma(i+k)}{\Gamma(2k+1)\Gamma(i-k+1)}\chi^k w^*(\chi)d\chi \\ &= \frac{1}{\gamma_i\pi}\sum_{k=0}^{i}(-1)^{i-k}\frac{4^k(2i)\Gamma(i+k)}{\Gamma(2k+1)\Gamma(i-k+1)}\int_0^1 \Upsilon(\chi)\chi^k w^*(\chi)d\chi. \end{aligned} \tag{3.2}$$

Since $\Upsilon(\chi)$ is a continuous function on the interval $[0,1]$, so it is bounded and there is a constant M_Υ such that

$$|\Upsilon(\chi)| \leqslant M_\Upsilon, \quad \forall \chi \in [0,1]. \tag{3.3}$$

By replacing Eq. (3.3) in Eq. (3.2), we have

$$\begin{aligned} |c_i| &\leqslant \frac{M_\Upsilon}{\gamma_i\pi}\sum_{k=0}^{i}(-1)^{i-k}\frac{4^k(2i)\Gamma(i+k)}{\Gamma(2k+1)\Gamma(i-k+1)}\int_0^1 \chi^k\frac{1}{\sqrt{\chi-\chi^2}}d\chi \\ &\leqslant \frac{M_\Upsilon}{\gamma_i\sqrt{\pi}}\sum_{k=0}^{i}(-1)^{i-k}\frac{4^k(2i)\Gamma(i+k)\Gamma(k+\frac{1}{2})}{\Gamma(2k+1)\Gamma(i-k+1)\Gamma(k+1)}. \end{aligned}$$

□

Theorem 3.2. *Suppose that function* $\Upsilon(\chi) \in C^{n+1}[0,1]$ *and* $\Upsilon_n(\chi)$ *be its best approximation using VLPs defined by Eq. (2.9). Then,*

$$\|\Upsilon(\chi) - \Upsilon_n(\chi)\| \leqslant \frac{M}{(n+1)!}\sqrt{\pi},$$

where

$$M = \max_{\chi \in [0,1]} \Upsilon^{(n+1)}(\chi).$$

Proof. It is easily proved by using of [22] and considering the weight function $w^*(\chi) = \frac{1}{\sqrt{\chi - \chi^2}}$. □

Theorem 3.3. *Suppose* $\mathcal{K}(\chi,\xi)$ *be a sufficiently function on* $\Omega = [0,1] \times [0,1]$, $\{VL_i^*(\chi)\}_{i=0}^{n} \subset L^2[0,1]$, $\{VL_i^*(\xi)\}_{i=0}^{n} \subset L^2[0,1]$, $\boldsymbol{X} = Span[VL_0^*(\chi), VL_1^*(\chi), \dots, VL_n^*(\chi)]$, $\boldsymbol{Y} = Span[VL_0^*(\xi), VL_1^*(\xi), \dots, VL_n^*(\xi)]$ *and* $\mathcal{K}_n(\chi,\xi)$ *be the best approximation of* $\mathcal{K}(\chi,\xi)$ *in space* $\boldsymbol{X} \times \boldsymbol{Y}$*. also, suppose*

$$\max_{(\chi,\xi)\in\Omega} \left| \frac{\partial^{n+1}\mathcal{K}(\chi,\xi)}{\partial \chi^{n+1}} \right| \leqslant \mu_1,$$
$$\max_{(\chi,\xi)\in\Omega} \left| \frac{\partial^{n+1}\mathcal{K}(\chi,\xi)}{\partial \xi^{n+1}} \right| \leqslant \mu_2,$$
$$\max_{(\chi,\xi)\in\Omega} \left| \frac{\partial^{2n+2}\mathcal{K}(\chi,\xi)}{\partial \chi^{n+1}\partial \xi^{n+1}} \right| \leqslant \mu_3,$$

Then, the error bound is as

$$\|\mathcal{K}(\chi,\xi) - \mathcal{K}_n(\chi,\xi)\|_2 \leqslant \left(\frac{\mu_1}{(n+1)!2^{2n+1}} + \frac{\mu_2}{(n+1)!2^{2n+1}} + \frac{\mu_3}{((n+1)!)^2\, 2^{4n+2}} \right)\pi.$$

Proof. It is easily proved by using of [22] and considering the weight function $w^*(\chi)w^*(\xi) = \frac{1}{\sqrt{\chi-\chi^2}}\frac{1}{\sqrt{\xi-\xi^2}}$. □

3.2. Convergence Analysis

Theorem 3.4. *Let* $\Upsilon(\chi)$ *be the exact solution of Eq. (1.1) and* $\Upsilon_n(\chi)$ *be its approximate solution. According to theorems 3.2 and 3.3, if* $\mathcal{K}_n(\chi,\xi)$ *and* $g_n(\chi)$ *are approximations of* $\mathcal{K}(\chi,\xi)$ *and* $g(\chi)$ *respectively. Furthermore suppose that if*
(1) $|\Upsilon(\chi)| \leqslant \rho, \quad \forall\, \chi \in [0,1]$,
(2) $|\mathcal{K}_i(\chi,\xi)| \leqslant \kappa_i, \ i = 1,2, \quad \forall (\chi,\xi) \in [0,1] \times [0,1]$,
(3)$4\mathcal{V}\left[\left(2\kappa_1^2 + 4\beta_1^2(n)\right) + \sigma^2\left(2\kappa_2^2 + 4\beta_2^2(n)\right)\right] < 1$
Then

$$\|e_n(\chi)\| \leqslant \sqrt{\frac{4\|Res_n(\chi)\|^2 + 4\mathcal{V}^2\gamma^2(n) + 16\mathcal{V}\rho^2(\beta_1^2(n) + \sigma^2\beta_2^2(n))}{1 - 4\mathcal{V}\left[\left(2\kappa_1^2 + 4\beta_1^2(n)\right) + \sigma^2\left(2\kappa_2^2 + 4\beta_2^2(n)\right)\right]}},$$

and also $\Upsilon_n(\chi)$ *convergence to* $\Upsilon(\chi)$ *as* $n \to \infty$ *in* L^2 *and* $\|\Upsilon\|^2 = E|\Upsilon|^2$.
For notational simplicity, $\mathcal{V}$ *denotes a positive constant dependent on* α *throughout this work.*

Proof. Consider the SFIDE as follows:

$$D^{\alpha}\Upsilon(\chi) = g(\chi) + \int_0^{\chi} \mathcal{K}_1(\chi,\xi)\Upsilon(\xi)d\xi + \sigma\int_0^{\chi} \mathcal{K}_2(\chi,\xi)\Upsilon(\xi)dB(\xi), \quad \chi \in [0,1], \tag{3.4}$$

and let $\Upsilon_n(\chi)$ be the approximate solution of Eq. (3.4), that is

$$D^{\alpha}\Upsilon_n(\chi) = g_n(\chi) + \int_0^{\chi} \mathcal{K}_{1_n}(\chi,\xi)\Upsilon_n(\xi)d\xi + \sigma\int_0^{\chi} \mathcal{K}_{2_n}(\chi,\xi)\Upsilon_n(\xi)dB(\xi). \tag{3.5}$$

Substracting Eq. (3.5) of Eq. (3.4) and defining the error function $|\Upsilon(\chi)-\Upsilon_n(\chi)|$, we have

$$D^{\alpha}\left(\Upsilon(\chi)-\Upsilon_n(\chi)\right)=\left(g(\chi)-g_n(\chi)\right)+\int_0^{\chi}\left(\mathcal{K}_1(\chi,\xi)\Upsilon(\xi)-\mathcal{K}_{1_n}(\chi,\xi)\Upsilon_n(\xi)\right)d\xi$$
$$+\sigma\int_0^{\chi}\left(\mathcal{K}_2(\chi,\xi)\Upsilon(\xi)-\mathcal{K}_{2_n}(\chi,\xi)\Upsilon_n(\xi)\right)dB(\xi). \tag{3.6}$$

Applying the Riemann-Liouville operator I^{α} to both sides of Eq. (3.6) yields.

$$I^{\alpha}\left(D^{\alpha}\left(\Upsilon(\chi)-\Upsilon_n(\chi)\right)\right)=I^{\alpha}\left(g(\chi)-g_n(\chi)\right)+I^{\alpha}\left(\int_0^{\chi}\left(\mathcal{K}_1(\chi,\xi)\Upsilon(\xi)-\mathcal{K}_{1_n}(\chi,\xi)\Upsilon_n(\xi)\right)d\xi\right)$$
$$+\sigma I^{\alpha}\left(\int_0^{\chi}\left(\mathcal{K}_2(\chi,\xi)\Upsilon(\xi)-\mathcal{K}_{2_n}(\chi,\xi)\Upsilon_n(\xi)\right)dB(\xi)\right). \tag{3.7}$$

By using the properties of I^{α}, Eq. (3.7) can be written as

$$\Upsilon(\chi)-\Upsilon_n(\chi)=\mathrm{Res}_n(\chi)+\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\xi)^{\alpha-1}\left(g(\xi)-g_n(\xi)\right)d\xi$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\left(\int_0^{\zeta}\left(\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)-\mathcal{K}_{1_n}(\zeta,\xi)\Upsilon_n(\xi)\right)d\xi\right)d\zeta$$
$$+\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\left(\int_0^{\zeta}\left(\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)-\mathcal{K}_{2_n}(\zeta,\xi)\Upsilon_n(\xi)\right)dB(\xi)\right)d\zeta. \tag{3.8}$$

Using the relation $\left(\sum_{i=0}^{n}\upsilon_i\right)^2\leqslant n\left(\sum_{i=0}^{n}\upsilon_i^2\right)$, the following inequality has been obtained.

$$\|e_n(\chi)\|^2=\|\Upsilon(\chi)-\Upsilon_n(\chi)\|^2\leqslant 4\|\mathrm{Res}_n(\chi)\|^2+4\left\|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\xi)^{\alpha-1}\left(g(\xi)-g_n(\xi)\right)d\xi\right\|^2$$
$$+4\left\|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\left(\int_0^{\zeta}\left(\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)-\mathcal{K}_{1_n}(\zeta,\xi)\Upsilon_n(\xi)\right)d\xi\right)d\zeta\right\|^2$$
$$+4\left\|\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\left(\int_0^{\zeta}\left(\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)-\mathcal{K}_{2_n}(\zeta,\xi)\Upsilon_n(\xi)\right)dB(\xi)\right)d\zeta\right\|^2. \tag{3.9}$$

Using Cauchy-Schwartz inequality and the theorem 3.2, we have

$$J_1=\left\|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\xi)^{\alpha-1}\left(g(\xi)-g_n(\xi)\right)d\xi\right\|^2=E\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\xi)^{\alpha-1}\big(g(\xi)-g_n(\xi)\big)d\xi\right|^2$$
$$\leqslant E\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\xi)^{\frac{\alpha-1}{2}}(\chi-\xi)^{\frac{\alpha-1}{2}}\left(g(\xi)-g_n(\xi)\right)d\xi\right|^2$$
$$\leqslant\frac{1}{\Gamma^2(\alpha)}\Big(\int_0^{\chi}(\chi-\xi)^{\alpha-1}d\xi\Big)\Big(\int_0^{\chi}(\chi-\xi)^{\alpha-1}\big\|g(\xi)-g_n(\xi)\big\|^2d\xi\Big)$$
$$\leqslant\mathcal{V}^2\big\|g(\xi)-g_n(\xi)\big\|^2\leqslant\mathcal{V}^2\gamma^2(n), \tag{3.10}$$

where $\gamma(n)=\frac{M}{(n+1)!}\sqrt{\pi}$ and $M=\max_{\chi\in[0,1]}g^{(n+1)}(\chi)$.

In the continuation of the proof, we let

$$
\begin{aligned}
J_2 &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^{\chi} (\chi-\zeta)^{\alpha-1} \left(\int_0^{\zeta} (\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{1_n}(\zeta,\xi)\Upsilon_n(\xi))\, d\xi \right) d\zeta \right\|^2 \\
&= \frac{1}{\Gamma^2(\alpha)} E \left| \int_0^{\chi} (\chi-\zeta)^{\alpha-1} \left(\int_0^{\zeta} (\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{1_n}(\zeta,\xi)\Upsilon_n(\xi))\, d\xi \right) d\zeta \right|^2 \\
&= \frac{1}{\Gamma^2(\alpha)} E \left| \int_0^{\chi} (\chi-\zeta)^{\frac{\alpha-1}{2}} \left(\int_0^{\zeta} (\chi-\zeta)^{\frac{\alpha-1}{2}} \left(\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{1_n}(\zeta,\xi)\Upsilon_n(\xi)\right) d\xi \right) d\zeta \right|^2 .
\end{aligned}
$$

Using the Cauchy - Schwarz inequality, the theorem 3.3 and $\left(\sum_{i=0}^{n} \upsilon_i\right)^2 \leqslant n\left(\sum_{i=0}^{n} \upsilon_i^2\right)$. we have

$$
\begin{aligned}
J_2 &\leqslant \frac{1}{\Gamma^2(\alpha)} \int_0^{\chi} (\chi-\zeta)^{\alpha-1} d\zeta E \left(\int_0^{\chi} \left| \int_0^{\zeta} (\chi-\zeta)^{\frac{\alpha-1}{2}} \left(\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{1_n}(\zeta,\xi)\Upsilon_n(\xi)\right) d\xi \right|^2 d\zeta \right) \\
&\leqslant \mathcal{V} E \left(\chi \int_0^{\chi} \left(\zeta \int_0^{\zeta} (\chi-\zeta)^{\alpha-1} \left| \mathcal{K}_1(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{1_n}(\zeta,\xi)\Upsilon_n(\xi) \right|^2 d\xi \right) d\zeta \right) \\
&\leqslant \mathcal{V} E \Bigg(\chi \int_0^{\chi} \Bigg(\zeta \int_0^{\zeta} (\chi-\zeta)^{\alpha-1} \Big| \mathcal{K}_1(\zeta,\xi)(\Upsilon(\xi)-\Upsilon_n(\xi)) + (\mathcal{K}_1(\zeta,\xi) - \mathcal{K}_{1_n}(\zeta,\xi))(\Upsilon_n(\xi) \\
&\quad - \Upsilon(\xi) + \Upsilon(\xi)) \Big|^2 d\xi \Bigg) d\zeta \Bigg) \\
&\leqslant \mathcal{V} E \Bigg(\chi \int_0^{\chi} \Bigg(\zeta \int_0^{\zeta} (\chi-\zeta)^{\alpha-1} \Big(2\left|\mathcal{K}_1(\zeta,\xi)(\Upsilon(\xi)-\Upsilon_n(\xi))\right|^2 + 4\Big|(\mathcal{K}_1(\zeta,\xi) - \mathcal{K}_{1_n}(\zeta,\xi)) \\
&\quad \times (\Upsilon_n(\xi) - \Upsilon(\xi))\Big|^2 + 4\left|(\mathcal{K}_1(\zeta,\xi) - \mathcal{K}_{1_n}(\zeta,\xi))(\Upsilon(\xi))\right|^2 \Big) d\xi \Bigg) d\zeta \Bigg) \\
&\leqslant \mathcal{V} \Bigg(\chi^2 \int_0^{\chi} \Bigg(\int_0^{\zeta} (\chi-\zeta)^{\alpha-1} \Big(2\left|\mathcal{K}_1(\zeta,\xi)\right|^2 E\left|(\Upsilon(\xi)-\Upsilon_n(\xi))\right|^2 + 4E\left|(\mathcal{K}_1(\zeta,\xi) - \mathcal{K}_{1_n}(\zeta,\xi))\right|^2 \\
&\quad \times E\left|(\Upsilon_n(\xi) - \Upsilon(\xi))\right|^2 + 4E\left|(\mathcal{K}_1(\zeta,\xi) - \mathcal{K}_{1_n}(\zeta,\xi))\right|^2 \left|\Upsilon(\xi)\right|^2 \Big) d\xi \Bigg) d\zeta \Bigg) \\
&\leqslant \mathcal{V} \int_0^{\chi} \left(\int_0^{\zeta} (\chi-\zeta)^{\alpha-1} \left(2\kappa_1^2 E\left|e_n(\xi)\right|^2 + 4\beta_1^2(n) E\left|e_n(\xi)\right|^2 + 4\beta_1^2(n)\rho^2 \right) d\xi \right) d\zeta.
\end{aligned}
$$

By changing the integration order and substituting $(\chi-\xi)^{\alpha} < 1$, we derive

$$
\begin{aligned}
J_2 &\leqslant \mathcal{V} \Bigg(\left(2\kappa_1^2 + 4\beta_1^2(n)\right) \int_0^{\chi} E\left|e_n(\xi)\right|^2 \left(\int_{\xi}^{\chi} (\chi-\zeta)^{\alpha-1} d\zeta \right) d\xi \\
&\quad + 4\beta_1^2(n)\rho^2 \int_0^{\chi} \left(\int_{\xi}^{\chi} (\chi-\zeta)^{\alpha-1} d\zeta \right) d\xi \Bigg) \\
&\leqslant \mathcal{V} \left(\left(2\kappa_1^2 + 4\beta_1^2(n)\right) \int_0^{\chi} E\left|e_n(\xi)\right|^2 (\chi-\xi)^{\alpha} d\xi + 4\beta_1^2(n)\rho^2 \int_0^{\chi} (\chi-\xi)^{\alpha} d\xi \right) \\
&\leqslant \mathcal{V} \left(\left(2\kappa_1^2 + 4\beta_1^2(n)\right) \left(\int_0^{\chi} \left\|e_n(\xi)\right\|^2 d\xi \right) + 4\beta_1^2(n)\rho^2 \right) \\
&\leqslant \mathcal{V} \left(\left(2\kappa_1^2 + 4\beta_1^2(n)\right) \left\|e_n(\xi)\right\|^2 + 4\beta_1^2(n)\rho^2 \right),
\end{aligned} \tag{3.11}
$$

where $\beta_i(n) = \left(\frac{\mu_1}{(n+1)!2^{2n+1}} + \frac{\mu_2}{(n+1)!2^{2n+1}} + \frac{\mu_3}{((n+1)!)^2 2^{4n+2}}\right)\pi,\ i = 1, 2.$
Again, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
J_3 &= \left\| \frac{\sigma}{\Gamma(\alpha)} \int_0^{\chi} (\chi-\zeta)^{\alpha-1} \int_0^{\zeta} (\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{2_n}(\zeta,\xi)\Upsilon_n(\xi))\, dB(\xi) d\zeta \right\|^2 \\
&= \frac{\sigma^2}{\Gamma^2(\alpha)} E\left| \int_0^{\chi} (\chi-\zeta)^{\alpha-1} \left(\int_0^{\zeta} (\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{2_n}(\zeta,\xi)\Upsilon_n(\xi))\, dB(\xi) \right) d\zeta \right|^2 \\
&= \frac{\sigma^2}{\Gamma^2(\alpha)} E\left| \int_0^{\chi} (\chi-\zeta)^{\frac{\alpha-1}{2}} \left(\int_0^{\zeta} (\chi-\zeta)^{\frac{\alpha-1}{2}} (\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{2_n}(\zeta,\xi)\Upsilon_n(\xi)) dB(\xi) \right) d\zeta \right|^2 \\
&\leqslant \frac{\sigma^2}{\Gamma^2(\alpha)} \left(\int_0^{\chi} (\chi-\zeta)^{\alpha-1} d\zeta \right) \\
&\times \left(\chi \int_0^{\chi} E\left| \int_0^{\zeta} (\chi-\zeta)^{\frac{\alpha-1}{2}} (\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{2_n}(\zeta,\xi)\Upsilon_n(\xi)) dB(\xi) \right|^2 d\zeta \right).
\end{aligned}$$

Now, by using the Itô isometry property and the relation $\left(\sum_{i=0}^{n} \upsilon_i\right)^2 \leqslant n\left(\sum_{i=0}^{n} \upsilon_i^2\right)$. we have

$$\begin{aligned}
J_3 &\leqslant \sigma^2 \mathcal{V} \left(\chi \int_0^{\chi} \left(\int_0^{\zeta} (\chi-\zeta)^{\alpha-1} E\left| \mathcal{K}_2(\zeta,\xi)\Upsilon(\xi) - \mathcal{K}_{2_n}(\zeta,\xi)\Upsilon_n(\xi) \right|^2 d\xi \right) d\zeta \right) \\
&\leqslant \sigma^2 \mathcal{V} \left(\chi \int_0^{\chi} \left(\int_0^{\zeta} (\chi-\zeta)^{\alpha-1} E\left| \mathcal{K}_2(\zeta,\xi)(\Upsilon(\xi) - \Upsilon_n(\xi)) + (\mathcal{K}_2(\zeta,\xi) - \mathcal{K}_{2_n}(\zeta,\xi))(\Upsilon_n(\xi) \right.\right.\right. \\
&\left.\left.\left. - \Upsilon(\xi) + \Upsilon(\xi)) \right|^2 d\xi \right) d\zeta \right) \\
&\leqslant \sigma^2 \mathcal{V} \left(\chi \int_0^{\chi} \left(\int_0^{\zeta} (\chi-\zeta)^{\alpha-1} \Big(2\left|\mathcal{K}_2(\zeta,\xi)\right|^2 E\left|(\Upsilon(\xi) - \Upsilon_n(\xi))\right|^2 + 4E\left|(\mathcal{K}_2(\zeta,\xi) - \mathcal{K}_{2_n}(\zeta,\xi))\right|^2 \right.\right. \\
&\left.\left. \times E\left|(\Upsilon_n(\xi) - \Upsilon(\xi))\right|^2 + 4E\left|(\mathcal{K}_2(\zeta,\xi) - \mathcal{K}_{2_n}(\zeta,\xi))\right|^2 \left|\Upsilon(\xi)\right|^2 \Big) d\xi \right) d\zeta \right) \\
&\leqslant \sigma^2 \mathcal{V} \left(\int_0^{\chi} \left(\int_0^{\zeta} (\chi-\zeta)^{\alpha-1} \left(2\kappa_2^2 E\left|e_n(\xi)\right|^2 + 4\beta_2^2(n) E\left|e_n(\xi)\right|^2 + 4\beta_2^2(n)\rho^2 \right) d\xi \right) d\zeta \right).
\end{aligned}$$

By using the theorem 3.3 and by changing the order of integration. Since $(\chi-\xi)^{\alpha} < 1$, we have

$$\begin{aligned}
J_3 &\leqslant \sigma^2 \mathcal{V} \left(\left(2\kappa_2^2 + 4\beta_2^2(n)\right) \int_0^{\chi} E\left|e_n(\xi)\right|^2 \left(\int_{\xi}^{\chi} (\chi-\zeta)^{\alpha-1} d\zeta \right) d\xi \right. \\
&\left. + 4\beta_2^2(n)\rho^2 \int_0^{\chi} \left(\int_{\xi}^{\chi} (\chi-\zeta)^{\alpha-1} d\zeta \right) d\xi \right) \\
&\leqslant \sigma^2 \mathcal{V} \left(\left(2\kappa_2^2 + 4\beta_2^2(n)\right) \int_0^{\chi} E\left|e_n(\xi)\right|^2 (\chi-\xi)^{\alpha} d\xi + 4\beta_2^2(n)\rho^2 \int_0^{\chi} (\chi-\xi)^{\alpha} d\xi \right) \\
&\leqslant \sigma^2 \mathcal{V} \left(\left(2\kappa_2^2 + 4\beta_2^2(n)\right) \left(\int_0^{\chi} \left\|e_n(\xi)\right\|^2 d\xi \right) + 4\beta_2^2(n)\rho^2 \right) \\
&\leqslant \sigma^2 \mathcal{V} \left(\left(2\kappa_2^2 + 4\beta_2^2(n)\right) \left\|e_n(\xi)\right\|^2 + 4\beta_2^2(n)\rho^2 \right). \qquad (3.12)
\end{aligned}$$

Now, by substituting Eqs. (3.10), (3.11), (3.12) into Eq. (3.9)

$$\begin{aligned}\|e_n(\chi)\|^2 \leqslant 4\|\text{Res}_n(\chi)\|^2 + 4\mathcal{V}^2\gamma^2(n) + 4\mathcal{V}\left[\left(2\kappa_1^2 + 4\beta_1^2(n)\right)\|e_n(\xi)\|^2 + 4\rho^2\beta_1^2(n)\right] \\ + 4\sigma^2\mathcal{V}\left[\left(2\kappa_2^2 + 4\beta_2^2(n)\right)\|e_n(\xi)\|^2 + 4\rho^2\beta_2^2(n)\right].\end{aligned}$$

Then,

$$\|e_n(\chi)\| \leqslant \sqrt{\frac{4\|\text{Res}_n(\chi)\|^2 + 4\mathcal{V}^2\gamma^2(n) + 16\mathcal{V}\rho^2(\beta_1^2(n) + \sigma^2\beta_2^2(n))}{1 - 4\mathcal{V}\left[\left(2\kappa_1^2 + 4\beta_1^2(n)\right) + \sigma^2\left(2\kappa_2^2 + 4\beta_2^2(n)\right)\right]}}.$$

It implies that, if $n \to \infty$ then $\|e_n(\chi)\| \to 0$ and therefore $\Upsilon_n(\chi)$ convergence to $\Upsilon(\chi)$ in L^2. □

4. Operational matrices based on shifted Vieta-Lucas polynomials (SVLP)

This section derives integral and fractional derivative operational matrices, along with a product matrix, for Vieta-Lucas polynomials.

Remark 4.1. The vector $\psi(\chi)$ defined in (2.11) can be approximated based on the vector of the monomial basis function

$$\psi(\chi) = \mathbb{G}\mathbb{T}_n(\chi). \tag{4.1}$$

Here, $\mathbb{T}_n(\chi)$ is defined as

$$\mathbb{T}_n(\chi) = \begin{bmatrix}1 & \chi & \cdots & \chi^n\end{bmatrix}^T, \tag{4.2}$$

and $\mathbb{G} = [g_{i,j}]$ in (4.1) is an $(n+1)$ order lower triangular matrix with

$$g_{i,j} = \begin{cases} 2, & i = j = 1, \\ \frac{(-1)^{i-j}4^{j-1}(2i-2)(i+j-3)!}{(i-j)!(2j-2)!}, & 2 \leqslant i \leqslant n+1, \quad 1 \leqslant j \leqslant i, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.2. The inverse of matrix $\mathbb{G}$ is matrix $\mathbb{G}^{-1} = [\hat{g}_{i,j}]$, which is defined as follows

$$\hat{g}_{i,j} = \begin{cases} \frac{1}{2}, & i = j = 1, \\ \frac{1}{2}\frac{\Gamma(i-\frac{1}{2})}{\sqrt{\pi}\Gamma(i)}, & 2 \leqslant i \leqslant n+1, \quad j = 1, \\ \frac{\Gamma(i)\Gamma(i-\frac{1}{2})}{\sqrt{\pi}\Gamma(i+j-1)\Gamma(i-j+1)}, & 2 \leqslant i \leqslant n+1, \quad 2 \leqslant j \leqslant i, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

4.1. *Operational matrix of integral*

The integral of the vector $\psi(\chi)$ defined in (2.11) is expressed by the VLPs as follows

$$\int_0^\chi \psi(s)ds = \mathbb{I}\,\psi(\chi), \tag{4.4}$$

where $\mathbb{I}$ represents the $(n+1) \times (n+1)$ operational matrix of the integral. First, we calculate the integral of the vector $\mathbb{T}_n(\chi)$ defined in (4.2). Therefore, we have

$$\int_0^\chi \mathbb{T}_n(\xi)d\xi \simeq \hat{\jmath}\,\mathbb{T}_n(\chi), \tag{4.5}$$

where $\hat{\mathcal{J}} = [\hat{p}_{i,j}]$ and

$$\hat{p}_{i,j} = \begin{cases} \frac{1}{i}, & 1 \leqslant i \leqslant n, \ \ 2 \leqslant j \leqslant i+1, \ \ j-i=1, \\ 0, & \text{otherwise}. \end{cases}$$

By using Eqs. (4.1) and (4.5), we conclude

$$\int_0^{\chi} \psi(\xi) d\xi = \mathbb{G} \int_0^{\chi} \mathbb{T}_n(\xi) d\xi = \mathbb{G}\hat{\mathcal{J}}\, \mathbb{T}_n(\chi) = \mathbb{G}\hat{\mathcal{J}}\mathbb{G}^{-1}\, \psi(\chi),$$

where $\mathbb{I} = \mathbb{G}\hat{\mathcal{J}}\mathbb{G}^{-1}$.

4.2. Operational matrix of derivative

The differentiation of the vector $\psi(\chi)$ can be expressed by the VLPs as follows

$$\frac{d}{d\chi}\psi(\chi) = \mathbb{D}^{(1)}\psi(\chi), \tag{4.6}$$

where $\mathbb{D}^{(1)}$ is operational matrix of derivative. According to section 4.1, we calculate the differentiation of the vector $\mathbb{T}_n(\chi)$ defined in (4.2). Then, we obtain

$$\mathbb{D}^{(1)} = \mathbb{G}\hat{\mathcal{D}}\mathbb{G}^{-1}, \tag{4.7}$$

where $\hat{\mathcal{D}} = [\hat{d}_{i,j}]$ is an $(n+1)$ order square matrix, in which

$$\hat{d}_{i,j} = \begin{cases} 0, & i=1, \ \ 1 \leqslant j \leqslant n+1, \\ i-1, & 2 \leqslant i \leqslant n+1, \ \ 1 \leqslant j \leqslant i-1, \ \ i-j=1, \\ 0, & \text{otherwise}. \end{cases}$$

4.3. Operational matrix of fractional derivative

The fractional derivative of the vector $\psi(\chi)$ can be expressed by the VLPs as follows

$$D^{\alpha}\psi(\chi) = \mathbb{D}^{\alpha}\psi(\chi), \tag{4.8}$$

where $\mathbb{D}^{\alpha}$ represents the $(n+1) \times (n+1)$ operational matrix of fractional derivative of order $0 < \alpha < 1$. By applying the Caputo derivative of order α to the vector defined in (4.2), we have

$$D^{\alpha}\mathbb{T}_n(\chi) = \begin{bmatrix} 0 \\ \frac{1!}{\Gamma(2-\alpha)}\,\chi^{1-\alpha} \\ \frac{2!}{\Gamma(3-\alpha)}\,\chi^{2-\alpha} \\ \vdots \\ \frac{n!}{\Gamma(n+1-\alpha)}\,\chi^{n-\alpha} \end{bmatrix} = \Lambda\hat{\mathbb{T}}_n(\chi), \tag{4.9}$$

in which Λ is an $(n+1)$ order square diagonal matrix as follows

$$\Lambda = \text{diag}\left[0, \frac{1!}{\Gamma(2-\alpha)}, \frac{2!}{\Gamma(3-\alpha)}, \cdots, \frac{n!}{\Gamma(n+1-\alpha)}\right],$$

and $\hat{\mathbb{T}}_n(t) = [0\ \chi^{1-\alpha}\ \chi^{2-\alpha}\ \cdots\ \chi^{n-\alpha}]^T$. Now, we approximate $\chi^{\sigma-\alpha}$ in terms of the vector defined in (4.2)

$$\chi^{\sigma-\alpha} \simeq \sum_{k=0}^{n} \theta_{\sigma,k}\chi^k = [\theta_{\sigma,0}\ \theta_{\sigma,1}\ \theta_{\sigma,2}\ \cdots\ \theta_{\sigma,n}]\ \mathbb{T}_n(\chi), \quad \sigma = 1, 2, \cdots, n.$$
$$= \Omega^T \mathbb{T}_n(\chi). \tag{4.10}$$

Let Ω be a matrix with the zero vector in the first column and the vector $\theta_{\sigma,k}$ in the σth column. By substituting (4.10) into (4.9), we have

$$D^\alpha \mathbb{T}_n(\chi) = \Lambda\Omega^T\ \mathbb{T}_n(\chi). \tag{4.11}$$

By using Eqs. (4.1) and (4.11), we obtain

$$D^\alpha\psi(\chi) = \mathbb{G}\ D^\alpha\mathbb{T}_n(\chi) = \mathbb{G}\Lambda\Omega^T\ \mathbb{T}_n(\chi) = \mathbb{G}\Lambda\Omega^T\mathbb{G}^{-1}\ \psi(\chi).$$

where $\mathbb{D}^\alpha = \mathbb{G}\Lambda\Omega^T\mathbb{G}^{-1}$ is the operational matrix of fractional derivative of order α.

4.4. Operational matrix of product

Let C is an arbitrary $(n+1)\times 1$ matrix, then the operational matrix of product for the Vieta-Lucas polynomials is given as

$$C^T\psi(\chi)\psi(\chi)^T \simeq \psi(\chi)^T\tilde{C}, \tag{4.12}$$

where $\tilde{C}$ is the $(n+1)\times(n+1)$ operational matrix of product.
To calculate the product of $\psi(\chi)$ and $\psi(\chi)^T$ in an arbitrary vector C^T, by using Eq. (4.1) have that:

$$\begin{aligned} C^T\psi(\chi)\psi(\chi)^T &= C^T\psi(\chi)(\mathbb{T}_n^T(\chi)\mathbb{G}^T) \\ &= \left[\sum_{i=0}^{n} c_i VL_i^*(\chi), \sum_{i=0}^{n} c_i\chi VL_i^*(\chi), \ldots, \sum_{i=0}^{n} c_i\chi^n VL_i^*(\chi)\right]\mathbb{G}^T. \end{aligned}$$

Taking $e_{k,i} = [e^0_{k,i}, e^1_{k,i}, \ldots, e^n_{k,i}]^T$ and approximating $\chi^k VL_i^*(\chi) \simeq e^T_{k,i}\psi(\chi)$, $i, k = 0, 1, \ldots, n$, we obtain

$$\begin{aligned} e_{k,i} &= Q^{-1}\int_0^1 \chi^k VL_i^*(\chi)\psi(\chi)d\chi \\ &= Q^{-1}\left[\int_0^1 \chi^k VL_i^*(\chi)VL_0^*(\chi)d\chi, \int_0^1 \chi^k VL_i^*(\chi)VL_1^*(\chi)d\chi, \ldots, \int_0^1 \chi^k VL_i^*(\chi)VL_n^*(\chi)d\chi\right]^T, \end{aligned}$$

we have

$$\begin{aligned} \sum_{i=0}^{n} c_i\chi^k VL_i^*(\chi) \simeq \sum_{i=0}^{n} c_i\left(\sum_{j=0}^{n} e^j_{k,i}VL_j^*(\chi)\right) &= \sum_{j=0}^{n} VL_j^*(\chi)\left(\sum_{i=0}^{n} c_i e^j_{k,i}\right) \\ &= \psi(\chi)^T\left[\sum_{i=0}^{n} c_i e^0_{k,i}, \sum_{i=0}^{n} c_i e^1_{k,i}, \ldots, \sum_{i=0}^{n} c_i e^n_{k,i}\right]^T \\ &= \psi(\chi)^T\tilde{E}C \\ &= \psi(\chi)^T F, \end{aligned}$$

where $\tilde{E}$ is an $(n+1)\times(n+1)$ and the vectors $e_{k,i}$ for $k = 0, 1, 2, \ldots, n$ are the columns of $\tilde{E}$.

$$\begin{aligned} C^T\psi(\chi)\psi(\chi)^T &= \left[\sum_{i=0}^{n} c_i VL_i^*(\chi), \sum_{i=0}^{n} c_i\chi VL_i^*(\chi), \ldots, \sum_{i=0}^{n} c_i\chi^n VL_i^*(\chi)\right]\mathbb{G}^T \\ &= \psi(\chi)^T F\mathbb{G}^T \\ &= \psi(\chi)^T\tilde{C}. \end{aligned}$$

5. Description of the proposed method

This section approximates the solution of Eq. (1.1) via a shifted Vieta-Lucas collocation approach, expanding $\Upsilon(\chi)$, $g(\chi)$, $\mathcal{K}_1(\chi,\xi)$, $\mathcal{K}_2(\chi,\xi)$ and $B(\chi)$ using SVLP for computational resolution.
Let

$$\Upsilon(\chi) \simeq \Upsilon_n(\chi) = C^T\psi(\chi) = \psi^T(\chi)C, \tag{5.1}$$

$$\mathcal{K}_i(\chi,\xi) \simeq \psi^T(\chi)K_i\psi(\xi) = \psi^T(\xi)K_i^T\psi(\chi), \quad i=1,2, \tag{5.2}$$

$$B(\chi) \simeq B^T\psi(\chi) = \psi^T(\chi)B, \tag{5.3}$$

$$g(\chi) \simeq G^T\psi(\chi) = \psi^T(\chi)G. \tag{5.4}$$

By substituting Eqs. (5.1), (5.2), (5.3) and (5.4) into Eq. (1.1), we have

$$D^\alpha C^T\psi(\chi) = G^T\psi(\chi) + \int_0^\chi \psi^T(\chi)K_1\psi(\xi)\psi^T(\xi)C d\xi + \sigma\int_0^\chi \psi^T(\chi)K_2\psi(\xi)\psi^T(\xi)C d(\psi^T(\xi)B). \tag{5.5}$$

By using from Eqs. (4.8) and (4.12), we get

$$C^T\mathbb{D}^\alpha\psi(\chi) = G^T\psi(\chi) + \psi^T(\chi)K_1\tilde{C}^T\int_0^\chi \psi(\xi)d\xi + \sigma\psi^T(\chi)K_2\tilde{C}^T\int_0^\chi \psi(\xi)\psi^T(\xi)\mathbb{D}^T B d(\xi), \tag{5.6}$$

where $\tilde{C}^T$ is the operational matrix of product. According to Eq. (4.12), we obtain

$$\psi(\xi)\psi^T(\xi)\mathbb{D}^T B = \tilde{C}_B^T\psi(\xi), \tag{5.7}$$

where $\tilde{C}_B^T$ is the operational matrix of product. By substituting Eq. (5.7) into Eq. (5.6), we have

$$\begin{aligned} C^T\mathbb{D}^\alpha\psi(\chi) &= G^T\psi(\chi) + \psi^T(\chi)K_1\tilde{C}^T\int_0^\chi \psi(\xi)d\xi + \sigma\psi^T(\chi)K_2\tilde{C}^T\int_0^\chi \tilde{C}_B^T\psi(\xi)d(\xi) \\ &= G^T\psi(\chi) + \psi^T(\chi)K_1\tilde{C}^T\int_0^\chi \psi(\xi)d\xi + \sigma\psi^T(\chi)K_2\tilde{C}^T\tilde{C}_B^T\int_0^\chi \psi(\xi)d(\xi). \end{aligned} \tag{5.8}$$

By substituting Eq. (4.4) into Eq. (5.8), we rewrite this equation

$$C^T\mathbb{D}^\alpha\psi(\chi) = G^T\psi(\chi) + \psi^T(\chi)K_1\tilde{C}^T\mathbb{I}\,\psi(\chi) + \sigma\psi^T(\chi)K_2\tilde{C}^T\tilde{C}_B^T\mathbb{I}\,\psi(\chi). \tag{5.9}$$

To calculate the unknown SVLP coefficients c_i, $i = 0,1,2,\cdots,n$. We use appropriate collocation points defined by

$$\chi_p = \frac{1}{2}\left(1-\cos(\frac{(2p+1)\pi}{2n+2})\right), \qquad p=0,1,\cdots,n.$$

The problem reduces to solving a system of $(n+1)$ algebraic equations for the $(n+1)$ unknown SVLP coefficients; substituting the computed vector $C=[c_0,c_1,\cdots,c_n]^T$ into Eq. (2.9) yields the approximate solution of Eq. (1.1).

6. Existence and Uniquenss of solutions

This section establishes existence and uniqueness of solutions for the stochastic Eq. (1.1) via a Banach space framework of continuous functions equipped with the norm.

$$\|\Upsilon\|_{\hat{B}} := \sup_{\chi\in[0,1]}\|\Upsilon(\chi)\|_{L^2} = \sup_{\chi\in[0,1]}\left(E|\Upsilon(\chi)|^2\right)^{\frac{1}{2}} < \infty.$$

Lemma 6.1. *Suppose $\Upsilon(\chi)$ is a solution of the equation (1.1) with the condition (1.2) if only if it is a solution of the following stochastic integral equation*

$$\Upsilon(\chi) = \Upsilon(0) + I_{\chi}^{\alpha}\Big(g(\chi) + \int_0^{\chi} \mathcal{K}_1(\chi,\xi)\Upsilon(\xi)d\xi + \sigma\int_0^{\chi}\mathcal{K}_2(\chi,\xi)\Upsilon(\xi)dB(\xi)\Big). \tag{6.1}$$

Proof. It can be easily proved by using the properties of fractional operators.
First we consider the following assumption:
(H1) suppose that there exists the constant $m > 0$ such that $|g(\chi)| < m$. □

Theorem 6.2. *Suppose that the assumption (1)-(2) of theorem 3.4 hold, then the stochastic equation (1.1) has at least one solution.*

Proof. we consider the operator $\Theta : \hat{B} \to \hat{B}$ as:

$$\Theta\Upsilon(\chi) = \Upsilon_0 + I_{\chi}^{\alpha}g(\chi) + I_{\chi}^{\alpha}\Big(\int_0^{\chi}\mathcal{K}_1(\chi,\xi)\Upsilon(\xi)d\xi\Big) + I_{\chi}^{\alpha}\Big(\int_0^{\chi}\mathcal{K}_2(\chi,\xi)\Upsilon(\xi)dB(\xi)\Big), \tag{6.2}$$

we consider the set $Y := \{\Upsilon \in \hat{B} : \|\Upsilon - \Upsilon_0\|_{\hat{B}} \leqslant r,\ r > 0\}$. It is evident that Y is a closed and convex subset of the Banach space of all continuous function with the norm $\|\Upsilon\|_{\hat{B}}$. Choose $\mathcal{V}^2m^2 < \frac{r}{3}$, $\mathcal{V}\kappa_1^2\rho^2 < \frac{r}{3}$ and $\sigma^2\mathcal{V}\kappa_2^2\rho^2 < \frac{r}{3}$.
Fristly, we need to show that the operator Θ is continuous. To represent the continuous of Θ, we consider a sequence Υ_n that converges to $\Upsilon \in Y$. using the relation $\left(\sum_{i=0}^{n} \upsilon_i\right)^2 \leqslant n\left(\sum_{i=0}^{n}\upsilon_i^2\right)$, we obtain

$$\begin{aligned}
\left|\Theta\Upsilon_n(\chi) - \Theta\Upsilon(\chi)\right|^2 &\leqslant 2\Bigg(\Big|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\Big(\int_0^{\zeta}(\mathcal{K}_1(\zeta,\xi)\Upsilon_n(\xi) - \mathcal{K}_1(\zeta,\xi)\Upsilon(\xi))\,d\xi\Big)\,d\zeta\Big|^2 \\
&\quad + \Big|\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\Big(\int_0^{\zeta}(\mathcal{K}_2(\zeta,\xi)\Upsilon_n(\xi) - \mathcal{K}_2(\zeta,\xi)\Upsilon(\xi))\,dB(\xi)\Big)\,d\zeta\Big|^2\Bigg) \\
&\leqslant 2\Bigg(\Big|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\frac{\alpha-1}{2}}\Big(\int_0^{\zeta}(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_1(\zeta,\xi)\,(\Upsilon_n(\xi) - \Upsilon(\xi))\,d\xi\Big)\,d\zeta\Big|^2 \\
&\quad + \Big|\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\frac{\alpha-1}{2}}\Big(\int_0^{\zeta}(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_2(\zeta,\xi)\,(\Upsilon_n(\xi) - \Upsilon(\xi))\,dB(\xi)\Big)\,d\zeta\Big|^2\Bigg).
\end{aligned} \tag{6.3}$$

Using Cauchy-Schwartz inequality and the assumption (2) of theorem 3.4, we get

$$\begin{aligned}
\left|\Theta\Upsilon_n(\chi) - \Theta\Upsilon(\chi)\right|^2 &\leqslant 2\Bigg(\frac{1}{\Gamma^2(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta \\
&\times\Big(\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\Big|\int_0^{\zeta}\mathcal{K}_1(\zeta,\xi)\,(\Upsilon_n(\xi) - \Upsilon(\xi))\,d\xi\Big|^2 d\zeta\Big) \\
&+ \frac{\sigma^2}{\Gamma^2(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta\Big(\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\Big|\int_0^{\zeta}\mathcal{K}_2(\zeta,\xi)\,(\Upsilon_n(\xi) - \Upsilon(\xi))\,dB(\xi)\Big|^2 d\zeta\Big)\Bigg).
\end{aligned} \tag{6.4}$$

Now, applying Jensen's inequality and also Itô isometry, we obtain

$$\begin{aligned}
E\left|\Theta\Upsilon_n(\chi)-\Theta\Upsilon(\chi)\right|^2 &\leqslant 2\Bigg(\mathcal{V}\chi\int_0^\chi\Big(\zeta\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_1(\zeta,\xi)\left(\Upsilon_n(\xi)-\Upsilon(\xi)\right)\right|^2d\xi\Big)d\zeta \\
&\quad+\sigma^2\mathcal{V}\chi\int_0^\chi\Big(\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_2(\zeta,\xi)\left(\Upsilon_n(\xi)-\Upsilon(\xi)\right)\right|^2d\xi\Big)d\zeta\Bigg) \\
&\leqslant 2\Bigg(\mathcal{V}\kappa_1^2\int_0^\chi\Big(\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\left(\Upsilon_n(\xi)-\Upsilon(\xi)\right)\right|^2d\xi\Big)d\zeta \\
&\quad+\sigma^2\mathcal{V}\kappa_2^2\int_0^\chi\Big(\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\left(\Upsilon_n(\xi)-\Upsilon(\xi)\right)\right|^2d\xi\Big)d\zeta\Bigg).
\end{aligned} \tag{6.5}$$

By changing the integration order and substituting $(\chi-\xi)^\alpha<1$, we yield

$$\begin{aligned}
E\left|\Theta\Upsilon_n(\chi)-\Theta\Upsilon(\chi)\right|^2 &\leqslant 2\Bigg(\mathcal{V}\kappa_1^2\int_0^\chi E\left|\left(\Upsilon_n(\xi)-\Upsilon(\xi)\right)\right|^2\Big(\int_\xi^\chi(\chi-\zeta)^{\alpha-1}d\zeta\Big)d\xi \\
&\quad+\sigma^2\mathcal{V}\kappa_2^2\int_0^\chi E\left|\left(\Upsilon_n(\xi)-\Upsilon(\xi)\right)\right|^2\Big(\int_\xi^\chi(\chi-\zeta)^{\alpha-1}d\zeta\Big)d\xi\Bigg) \\
&\leqslant 2\Bigg(\mathcal{V}\kappa_1^2\int_0^\chi\left\|\Upsilon_n(\xi)-\Upsilon(\xi)\right\|^2(\chi-\xi)^\alpha d\xi \\
&\quad+\sigma^2\mathcal{V}\kappa_2^2\int_0^\chi\left\|\Upsilon_n(\xi)-\Upsilon(\xi)\right\|^2(\chi-\xi)^\alpha d\xi\Bigg) \\
&\leqslant\Big(2\mathcal{V}\kappa_1^2+2\sigma^2\mathcal{V}\kappa_2^2\Big)\left\|\Upsilon_n-\Upsilon\right\|^2.
\end{aligned} \tag{6.6}$$

Therefore

$$\left\|\Theta\Upsilon_n-\Theta\Upsilon\right\|_{\hat{B}}=\sup_{\chi\in[0,1]}E\left|\Theta\Upsilon_n(\chi)-\Theta\Upsilon(\chi)\right|^2=\Big(2\mathcal{V}\kappa_1^2+2\sigma^2\mathcal{V}\kappa_2^2\Big)\left\|\Upsilon_n-\Upsilon\right\|_{\hat{B}}. \tag{6.7}$$

Since Υ_n is convergent to Υ, i.e. $\|\Upsilon_n-\Upsilon\|\to 0$. Therefore, the operator Θ is continuous.
Next, we prove that $\Theta(Y)\subset Y$. For any $\Upsilon\in Y$, we have

$$\begin{aligned}
\left|\Theta\Upsilon(\chi)-\Upsilon_0\right|^2 &\leqslant 3\Bigg(\Big|\frac{1}{\Gamma(\alpha)}\int_0^\chi(\chi-\xi)^{\alpha-1}g(\xi)d\xi\Big|^2 \\
&\quad+\Big|\frac{1}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\alpha-1}\left(\int_0^\zeta\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)d\xi\right)d\zeta\Big|^2 \\
&\quad+\Big|\frac{\sigma}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\alpha-1}\left(\int_0^\zeta\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)dB(\xi)\right)d\zeta\Big|^2\Bigg) \\
&\leqslant 3\Bigg(\Big|\frac{1}{\Gamma(\alpha)}\int_0^\chi(\chi-\xi)^{\frac{\alpha-1}{2}}(\chi-\xi)^{\frac{\alpha-1}{2}}g(\xi)d\xi\Big|^2 \\
&\quad+\Big|\frac{1}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\frac{\alpha-1}{2}}\left(\int_0^\zeta(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)d\xi\right)d\zeta\Big|^2 \\
&\quad+\Big|\frac{\sigma}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\frac{\alpha-1}{2}}\left(\int_0^\zeta(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)dB(\xi)\right)d\zeta\Big|^2\Bigg).
\end{aligned} \tag{6.8}$$

By taking expectation at both sides of Eq. (6.8). Also, using Cauchy-Schwartz inequality, Itô isometry and the assumptions (1)-(2). we have

$$\begin{aligned}
E\left|\Theta\Upsilon(\chi)-\Upsilon_0\right|^2 \leqslant 3\Bigg(&\frac{1}{\Gamma^2(\alpha)}\Big(\int_0^{\chi}(\chi-\xi)^{\alpha-1}d\xi\Big)\Big(\int_0^{\chi}(\chi-\xi)^{\alpha-1}E|g(\xi)|^2d\xi\Big)\\
&+\frac{1}{\Gamma^2(\alpha)}\Big(\int_0^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta\Big)\Big(\chi\int_0^{\chi}\Big(\zeta\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)\right|^2d\xi\Big)d\zeta\Big)\\
&+\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\int_0^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta\Big)\Big(\chi\int_0^{\chi}\Big(\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)\right|^2d\xi\Big)d\zeta\Big)\Bigg)\\
\leqslant 3\Bigg(&\mathcal{V}^2m^2+\mathcal{V}\kappa_1^2\rho^2\Big(\chi\int_0^{\chi}\Big(\zeta\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi\Big)d\zeta\Big)\\
&+\sigma^2\mathcal{V}\kappa_2^2\rho^2\Big(\chi\int_0^{\chi}\Big(\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi\Big)d\zeta\Big)\Bigg)\\
\leqslant 3\Bigg(&\mathcal{V}^2m^2+\mathcal{V}\kappa_1^2\rho^2\Big(\int_0^{\chi}\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi d\eta\Big)\\
&+\sigma^2\mathcal{V}\kappa_2^2\rho^2\Big(\int_0^{\chi}\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi d\zeta\Big)\Bigg).
\end{aligned} \tag{6.9}$$

By changing the order of integration

$$\begin{aligned}
E\left|\Theta\Upsilon(\chi)-\Upsilon_0\right|^2 &\leqslant 3\Bigg(\mathcal{V}^2m^2+\mathcal{V}\kappa_1^2\rho^2\Big(\int_0^{\chi}\int_{\xi}^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta d\xi\Big)\\
&\quad+\sigma^2\mathcal{V}\kappa_2^2\rho^2\Big(\int_0^{\chi}\int_{\xi}^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta d\xi\Big)\Bigg)\\
&\leqslant 3\Big(\mathcal{V}^2m^2+\mathcal{V}\kappa_1^2\rho^2+\sigma^2\mathcal{V}\kappa_2^2\rho^2\Big)\\
&\leqslant 3\Big(\frac{r}{3}+\frac{r}{3}+\frac{r}{3}\Big)\leqslant r.
\end{aligned} \tag{6.10}$$

So, we show that $\Theta(Y)\subset Y$. Now, we prove that the operator Θ is bounded on Y.

$$\begin{aligned}
\left|\Theta\Upsilon(\chi)\right|^2 \leqslant 4\Bigg(&\left|\Upsilon_0\right|^2+\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\xi)^{\alpha-1}g(\xi)d\xi\right|^2\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\left(\int_0^{\zeta}\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)d\xi\right)d\zeta\right|^2\\
&+\left|\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\alpha-1}\left(\int_0^{\zeta}\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)dB(\xi)\right)d\zeta\right|^2\Bigg)\\
\leqslant 4\Bigg(&\left|\Upsilon_0\right|^2+\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\xi)^{\frac{\alpha-1}{2}}(\chi-\xi)^{\frac{\alpha-1}{2}}g(\xi)d\xi\right|^2\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\frac{\alpha-1}{2}}\left(\int_0^{\zeta}(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)d\xi\right)d\zeta\right|^2\\
&+\left|\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi}(\chi-\zeta)^{\frac{\alpha-1}{2}}\left(\int_0^{\zeta}(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)dB(\xi)\right)d\zeta\right|^2\Bigg).
\end{aligned} \tag{6.11}$$

By taking expectation at both sides of Eq. (6.11). Also, using Cauchy-Schwartz inequality, Itô isometry and the assumptions (1)-(2). we have

$$\begin{aligned}
E\left|\Theta\Upsilon(\chi)\right|^2 \leqslant 4\Bigg(& E\left|\Upsilon_0\right|^2 + \frac{1}{\Gamma^2(\alpha)}\Big(\int_0^{\chi}(\chi-\xi)^{\alpha-1}d\xi\Big)\Big(\int_0^{\chi}(\chi-\xi)^{\alpha-1}E|g(\xi)|^2 d\xi\Big)\\
&+\frac{1}{\Gamma^2(\alpha)}\Big(\int_0^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta\Big)\Big(\chi\int_0^{\chi}\Big(\zeta\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)\right|^2 d\xi\Big)d\zeta\Big)\\
&+\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\int_0^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta\Big)\Big(\chi\int_0^{\chi}\Big(\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)\right|^2 d\xi\Big)d\zeta\Big)\Bigg)\\
&\leqslant 4\Bigg(\left\|\Upsilon_0\right\|^2 + \mathcal{V}^2m^2 + \mathcal{V}\kappa_1^2\rho^2\Big(\chi\int_0^{\chi}\Big(\zeta\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi\Big)d\zeta\Big)\\
&+\sigma^2\mathcal{V}\kappa_2^2\rho^2\Big(\chi\int_0^{\chi}\Big(\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi\Big)d\zeta\Big)\Bigg)\\
&\leqslant 4\Bigg(\left\|y_0\right\|_{\hat{B}} + \mathcal{V}^2m^2 + \mathcal{V}\kappa_1^2\rho^2\Big(\int_0^{\chi}\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi d\zeta\Big)\\
&+\sigma^2\mathcal{V}\kappa_2^2\rho^2\Big(\int_0^{\chi}\int_0^{\zeta}(\chi-\zeta)^{\alpha-1}d\xi d\zeta\Big)\Bigg). \qquad (6.12)
\end{aligned}$$

By changing the order of integration

$$\begin{aligned}
E\left|\Theta\Upsilon(\chi)\right|^2 \leqslant 4\Bigg(& \left\|\Upsilon_0\right\|_{\hat{B}} + \mathcal{V}^2m^2 + \mathcal{V}\kappa_1^2\rho^2\Big(\int_0^{\chi}\int_{\xi}^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta d\xi\Big)\\
&+\sigma^2\mathcal{V}\kappa_2^2\rho^2\Big(\int_0^{\chi}\int_{\xi}^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta d\xi\Big)\Bigg)\\
&\leqslant 4\Big(\left\|\Upsilon_0\right\|_{\hat{B}} + \mathcal{V}^2m^2 + \mathcal{V}\kappa_1^2\rho^2 + \sigma^2\mathcal{V}\kappa_2^2\rho^2\Big)\\
&\leqslant 4\Big(\left\|\Upsilon_0\right\|_{\hat{B}} + r\Big). \qquad (6.13)
\end{aligned}$$

Therefore

$$\left\|\Theta\Upsilon(\chi)\right\|_{\hat{B}} = \sup_{\chi\in[0,1]} E\left|\Theta\Upsilon(\chi)\right|^2 \leqslant L. \qquad (6.14)$$

We show that the operator Θ is equicontinuous on Y. For any $\chi_1, \chi_2 \in [0,1]$ such that $0 \leqslant \chi_1 < \chi_2 < 1$, we have

$$
\begin{aligned}
\Theta\Upsilon(\chi_2)-\Theta\Upsilon(\chi_1) \leqslant& \frac{1}{\Gamma(\alpha)}\int_0^{\chi_2}\frac{g(\xi)}{(\chi_2-\xi)^{1-\alpha}}d\xi-\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\frac{g(\xi)}{(\chi_1-\xi)^{1-\alpha}}d\xi\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\chi_2}\int_0^{\zeta}\frac{\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)}{(\chi_2-\zeta)^{1-\alpha}}d\xi d\zeta-\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\int_0^{\zeta}\frac{\mathcal{K}_1(\zeta,\xi)\Upsilon(\xi)}{(\chi_1-\zeta)^{1-\alpha}}d\xi d\zeta\\
&+\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi_2}\int_0^{\zeta}\frac{\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)}{(\chi_2-\zeta)^{1-\alpha}}dB(\xi)d\zeta-\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi_1}\int_0^{\zeta}\frac{\mathcal{K}_2(\zeta,\xi)\Upsilon(\xi)}{(\chi_1-\zeta)^{1-\alpha}}dB(\xi)d\zeta\\
&\leqslant -\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\xi)^{1-\alpha}}-\frac{1}{(\chi_2-\xi)^{1-\alpha}}\Big)g(\xi)d\xi+\frac{1}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\frac{g(\xi)}{(\chi_2-\xi)^{1-\alpha}}d\xi\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\left(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi\right)d\zeta\\
&+\frac{1}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\int_0^{\zeta}\frac{\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)}{\chi_2-\zeta)^{1-\alpha}}d\xi d\zeta\\
&-\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi_1}\left(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)\right)d\zeta\\
&+\frac{\sigma}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\int_0^{\zeta}\frac{\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)}{(\chi_2-\zeta)^{1-\alpha}}dB(\xi)d\zeta. \qquad (6.15)
\end{aligned}
$$

Using the relation $\left(\sum_{i=0}^{n}\upsilon_i\right)^2 \leqslant n\left(\sum_{i=0}^{n}\upsilon_i^2\right)$ and taking expectation on both sides of the Eq. (6.15).

$$
\begin{aligned}
E\left|\Theta\Upsilon(\chi_2)-\Theta\Upsilon(\chi_1)\right|^2 \leqslant 6\Bigg(& E\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\xi)^{1-\alpha}}-\frac{1}{(\chi_2-\xi)^{1-\alpha}}\Big)g(\xi)d\xi\right|^2\\
&+E\left|\frac{1}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\frac{g(\xi)}{(\chi_2-\xi)^{1-\alpha}}d\xi\right|^2\\
&+E\left|\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\left(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi\right)d\zeta\right|^2\\
&+E\left|\frac{1}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\int_0^{\zeta}\frac{\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)}{(\chi_2-\zeta)^{1-\alpha}}d\xi d\zeta\right|^2\\
&+E\left|\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi_1}\left(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)\right)d\zeta\right|^2\\
&+E\left|\frac{\sigma}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\int_0^{\zeta}\frac{\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)}{(\chi_2-\zeta)^{1-\alpha}}dB(\xi)d\zeta\right|^2\Bigg). \qquad (6.16)
\end{aligned}
$$

First, we consider the first two expressions on the right side of Eq. (6.16). Therefore, using Cauchy-

Schwartz inequality and the hypothesis (H1), we have

$$
\begin{aligned}
&E\Big|\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\xi)^{1-\alpha}}-\frac{1}{(\chi_2-\xi)^{1-\alpha}}\Big)g(\xi)d\xi\Big|^2+E\Big|\frac{1}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\frac{g(\xi)}{(\chi_2-\xi)^{1-\alpha}}d\xi\Big|^2\\
&\leqslant \frac{1}{\Gamma^2(\alpha)}E\Big|\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\xi)^{1-\alpha}}-\frac{1}{(\chi_2-\xi)^{1-\alpha}}\Big)^{\frac{1}{2}}\Big(\frac{1}{(\chi_1-\xi)^{1-\alpha}}-\frac{1}{(\chi_2-\xi)^{1-\alpha}}\Big)^{\frac{1}{2}}g(\xi)d\xi\Big|^2\\
&+\frac{1}{\Gamma^2(\alpha)}E\Big|\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\xi)^{\frac{1-\alpha}{2}}}\frac{1}{(\chi_2-\xi)^{\frac{1-\alpha}{2}}}g(\xi)d\xi\Big|^2\\
&\leqslant \frac{1}{\Gamma^2(\alpha)}\left(\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\xi)^{1-\alpha}}-\frac{1}{(\chi_2-\xi)^{1-\alpha}}\Big)d\xi\right)\\
&\times\left(\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\xi)^{1-\alpha}}-\frac{1}{(\chi_2-\xi)^{1-\alpha}}\Big)|g(\xi)|^2d\xi\right)\\
&+\frac{1}{\Gamma^2(\alpha)}\Big(\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\xi)^{1-\alpha}}d\xi\Big)\Big(\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\xi)^{1-\alpha}}|g(\xi)|^2d\xi\Big)\\
&\leqslant \frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}((\chi_2-\chi_1)^{\alpha}-\chi_2^{\alpha}+\chi_1^{\alpha})\Big)\Big(\frac{1}{\alpha}m^2((\chi_2-\chi_1)^{\alpha}-\chi_2^{\alpha}+\chi_1^{\alpha})\Big)\\
&+\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}m^2(\chi_2-\chi_1)^{2\alpha}\Big)\\
&\leqslant \frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}m^2(\chi_2-\chi_1)^{2\alpha}\Big)+\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}m^2(\chi_2-\chi_1)^{2\alpha}\Big)\\
&\leqslant \frac{2}{\alpha^2}m^2(\chi_2-\chi_1)^{2\alpha},
\end{aligned}
\tag{6.17}
$$

where in the last inequality we have used the fact that $\frac{1}{\Gamma^2(\alpha)}\leqslant 1$ is in $(0,1]$. Now, we consider the second two expressions on the right side of Eq. (6.16). So, using Cauchy-Schwartz inequality and the assumptions (1)-(2) of theorem 3.4, we have

$$\begin{aligned}
&\mathsf{E}\Big|\frac{1}{\Gamma(\alpha)}\int_0^{\chi_1}\bigg(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi\bigg)d\zeta\Big|^2\\
&\quad+\mathsf{E}\Big|\frac{1}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi d\zeta\Big|^2\\
&\quad\leqslant\frac{1}{\Gamma^2(\alpha)}\mathsf{E}\Big|\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)^{\frac{1}{2}}\\
&\quad\times\bigg(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)^{\frac{1}{2}}\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi\bigg)d\zeta\Big|^2\\
&\quad+\frac{1}{\Gamma^2(\alpha)}\mathsf{E}\Big|\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\zeta)^{\frac{1-\alpha}{2}}}\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{\frac{1-\alpha}{2}}}\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi d\zeta\Big|^2\\
&\quad\leqslant\frac{1}{\Gamma^2(\alpha)}\Big(\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)d\zeta\Big)\\
&\quad\times\mathsf{E}\bigg(\int_0^{\chi_1}\Big|\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)^{\frac{1}{2}}\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi\Big|^2d\zeta\bigg)\\
&\quad+\frac{1}{\Gamma^2(\alpha)}\Big(\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathsf{E}\Big(\int_{\chi_1}^{\chi_2}\Big|\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{\frac{1-\alpha}{2}}}\mathcal{K}_1(\zeta,\xi)\Upsilon(\zeta)d\xi\Big|^2d\zeta\Big)\\
&\quad\leqslant\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}((\chi_2-\chi_1)^{\alpha}-\chi_2^{\alpha}+\chi_1^{\alpha})\Big)\\
&\quad\times\chi_1\int_0^{\chi_1}\bigg(\zeta\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathsf{E}\Big(|\mathcal{K}_1(\zeta,\xi)|^2|\Upsilon(\zeta)|^2\Big)d\xi\bigg)d\zeta\\
&\quad+\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\times\chi_2\int_{\chi_1}^{\chi_2}\bigg(\zeta\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\mathsf{E}\Big(|\mathcal{K}_1(\zeta,\xi)|^2|\Upsilon(\zeta)|^2\Big)d\xi\bigg)d\zeta\\
&\quad\leqslant\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\\
&\quad\times\chi_1^2\kappa_1^2\rho^2\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)d\zeta\\
&\quad+\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\times\chi_2^2\kappa_1^2\rho^2\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}d\zeta\\
&\quad\leqslant\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\Big(\frac{1}{\alpha}\kappa_1^2\rho^2((\chi_2-\chi_1)^{\alpha}-\chi_2^{\alpha}+\chi_1^{\alpha})\Big)\\
&\quad+\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}\kappa_1^2\rho^2(\chi_2-\chi_1)^{2\alpha}\Big)\\
&\quad\leqslant\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}\kappa_1^2\rho^2(\chi_2-\chi_1)^{2\alpha}\Big)+\frac{1}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}\kappa_1^2\rho^2(\chi_2-\chi_1)^{2\alpha}\Big)\\
&\quad\leqslant\frac{2}{\alpha^2}\kappa_1^2\rho^2(\chi_2-\chi_1)^{2\alpha},
\end{aligned}\tag{6.18}$$

finally, we consider the third two expressions on the right side of Eq. (6.16). Therefore, using Cauchy-

Schwartz inequality, thte Itô isometry property and the assumptions (1)-(2) of theorem 3.4, we obtain

$$\begin{aligned}
&E\Big|\frac{\sigma}{\Gamma(\alpha)}\int_0^{\chi_1}\left(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)\right)d\zeta\Big|^2\\
&+E\Big|\frac{\sigma}{\Gamma(\alpha)}\int_{\chi_1}^{\chi_2}\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)d\zeta\Big|^2\\
&\leqslant\frac{\sigma^2}{\Gamma^2(\alpha)}E\Big|\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)^{\frac{1}{2}}\\
&\times\left(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)^{\frac{1}{2}}\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)\right)d\zeta\Big|^2\\
&+\frac{\sigma^2}{\Gamma^2(\alpha)}E\Big|\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\zeta)^{\frac{1-\alpha}{2}}}\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{\frac{1-\alpha}{2}}}\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)d\zeta\Big|^2\\
&\leqslant\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)d\zeta\Big)\\
&\times E\left(\int_0^{\chi_1}\left|\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)^{\frac{1}{2}}\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)\right|^2d\zeta\right)\\
&+\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)E\Big(\int_{\chi_1}^{\chi_2}\Big|\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{\frac{1-\alpha}{2}}}\mathcal{K}_2(\zeta,\xi)\Upsilon(\zeta)dB(\xi)\Big|^2d\zeta\Big)\\
&\leqslant\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}((\chi_2-\chi_1)^{\alpha}-\chi_2^{\alpha}+\chi_1^{\alpha})\Big)\\
&\times\chi_1\int_0^{\chi_1}\left(\int_0^{\zeta}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)E\Big(|\mathcal{K}_2(\zeta,\xi)|^2|\Upsilon(\zeta)|^2\Big)d\xi\right)d\zeta\\
&+\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\times\chi_2\int_{\chi_1}^{\chi_2}\left(\int_0^{\zeta}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}E\Big(|\mathcal{K}_2(\zeta,\xi)|^2|\Upsilon(\zeta)|^2\Big)d\xi\right)d\zeta\\
&\leqslant\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\\
&\times\chi_1\kappa_2^2\rho^2\int_0^{\chi_1}\Big(\frac{1}{(\chi_1-\zeta)^{1-\alpha}}-\frac{1}{(\chi_2-\zeta)^{1-\alpha}}\Big)d\zeta\\
&+\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\times\chi_2\kappa_2^2\rho^2\int_{\chi_1}^{\chi_2}\frac{1}{(\chi_2-\zeta)^{1-\alpha}}d\eta\\
&\leqslant\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha}(\chi_2-\chi_1)^{\alpha}\Big)\Big(\frac{1}{\alpha}\kappa_2^2\rho^2((\chi_2-\chi_1)^{\alpha}-\chi_2^{\alpha}+\chi_1^{\alpha})\Big)\\
&+\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}\kappa_2^2\rho^2(\chi_2-\chi_1)^{2\alpha}\Big)\\
&\leqslant\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}\kappa_2^2\rho^2(\chi_2-\chi_1)^{2\alpha}\Big)+\frac{\sigma^2}{\Gamma^2(\alpha)}\Big(\frac{1}{\alpha^2}\kappa_2^2\rho^2(\chi_2-\chi_1)^{2\alpha}\Big)\\
&\leqslant\frac{2}{\alpha^2}\sigma^2\kappa_2^2\rho^2(\chi_2-\chi_1)^{2\alpha}.
\end{aligned}\tag{6.19}$$

By substituting Eq. (6.17), (6.18) and (6.19) in Eq. (6.16). we get

$$E\big|\Theta\Upsilon(\chi_2)-\Theta\Upsilon(\chi_1)\big|^2\leqslant 6\Big(\frac{2}{\alpha^2}m^2(\chi_2-\chi_1)^{2\alpha}+\frac{2}{\alpha^2}\kappa_1^2\rho^2(\chi_2-\chi_1)^{2\alpha}+\frac{2}{\alpha^2}\sigma^2\kappa_2^2\rho^2(\chi_2-\chi_1)^{2\alpha}\Big),\tag{6.20}$$

Hence,

$$\left\|\Theta\Upsilon(\chi_2)-\Theta\Upsilon(\chi_1)\right\|_{\hat{B}} = \sup_{\chi\in[0,1]} E\left|\Theta\Upsilon(\chi_2)-\Theta\Upsilon(\chi_1)\right|^2 \leqslant \frac{12}{\alpha^2}\left(m^2+\kappa_1^2\rho^2+\sigma^2\kappa_2^2\rho^2\right)(\chi_2-\chi_1)^{2\alpha}. \tag{6.21}$$

As the right-hand side of Eq. (6.21) vanishes with $\chi_2 \to \chi_1$, the equicontinuity of Θ on Y, via the Arzelà-Ascoli and Schauders theorems, guarantees a fixed point for operator Θ, solving stochastic Eq. (1.1). □

Theorem 6.3. *Suppose that the assumption (2) of theorem 3.4 hold, then the stochastic equation (1.1) has a unique solution if*

$$(2\mathcal{V}\kappa_1^2+2\sigma^2\mathcal{V}\kappa_2^2)<1. \tag{6.22}$$

Proof. we consider the operator $\Theta:\hat{B}\to\hat{B}$ as defined in the Eq. (6.2). using the relation $\left(\sum_{i=0}^{n}\upsilon_i\right)^2 \leqslant n\left(\sum_{i=0}^{n}\upsilon_i^2\right)$, for any $\Upsilon_1,\Upsilon_2\in\hat{B}$, we have

$$\begin{aligned}
\left|\Theta\Upsilon_2(\chi)-\Theta\Upsilon_1(\chi)\right|^2 &\leqslant 2\Bigg(\left|\frac{1}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\alpha-1}\left(\int_0^\zeta(\mathcal{K}_1(\zeta,\xi)\Upsilon_2(\xi)-\mathcal{K}_1(\zeta,\xi)\Upsilon_1(\xi))\,d\xi\right)d\zeta\right|^2\\
&\quad+\left|\frac{\sigma}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\alpha-1}\left(\int_0^\zeta(\mathcal{K}_2(\zeta,\xi)\Upsilon_2(\xi)-\mathcal{K}_2(\zeta,\xi)\Upsilon_1(\xi))\,dB(\xi)\right)d\zeta\right|^2\Bigg)\\
&\leqslant 2\Bigg(\left|\frac{1}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\frac{\alpha-1}{2}}\left(\int_0^\zeta(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_1(\zeta,\xi)\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)d\xi\right)d\zeta\right|^2\\
&\quad+\left|\frac{\sigma}{\Gamma(\alpha)}\int_0^\chi(\chi-\zeta)^{\frac{\alpha-1}{2}}\left(\int_0^\zeta(\chi-\zeta)^{\frac{\alpha-1}{2}}\mathcal{K}_2(\zeta,\xi)\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)dB(\xi)\right)d\zeta\right|^2\Bigg).
\end{aligned} \tag{6.23}$$

Using Cauchy-Schwartz inequality and the assumption (2) of theorem 3.4, we obtain

$$\begin{aligned}
\left|\Theta\Upsilon_2(\chi)-\Theta\Upsilon_1(\chi)\right|^2 &\leqslant 2\Bigg(\frac{1}{\Gamma^2(\alpha)}\int_0^\chi(\chi-\zeta)^{\alpha-1}d\zeta\\
&\quad\times\left(\int_0^\chi(\chi-\zeta)^{\alpha-1}\left|\int_0^\zeta\mathcal{K}_1(\zeta,\xi)\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)d\xi\right|^2d\zeta\right)\\
&\quad+\frac{\sigma^2}{\Gamma^2(\alpha)}\int_0^\chi(\chi-\zeta)^{\alpha-1}d\zeta\left(\int_0^\chi(\chi-\zeta)^{\alpha-1}\left|\int_0^\zeta\mathcal{K}_2(\zeta,\xi)\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)dB(\xi)\right|^2d\zeta\right)\Bigg).
\end{aligned} \tag{6.24}$$

Now, taking expectation on both sides of Eq. (6.24). by applying Jensen's inequality and the Itô isometry, we obtain

$$\begin{aligned}
E\left|\Theta\Upsilon_2(\chi)-\Theta\Upsilon_1(\chi)\right|^2 &\leqslant 2\Bigg(\mathcal{V}\chi\int_0^\chi\left(\eta\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_1(\zeta,\xi)\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)\right|^2d\xi\right)d\zeta\\
&\quad+\sigma^2\mathcal{V}\chi\int_0^\chi\left(\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\mathcal{K}_2(\zeta,\xi)\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)\right|^2d\xi\right)d\zeta\Bigg)\\
&\leqslant 2\Bigg(\mathcal{V}\kappa_1^2\int_0^\chi\left(\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)\right|^2d\xi\right)d\zeta\\
&\quad+\sigma^2\mathcal{V}\kappa_2^2\int_0^\chi\left(\int_0^\zeta(\chi-\zeta)^{\alpha-1}E\left|\left(\Upsilon_2(\xi)-\Upsilon_1(\xi)\right)\right|^2d\xi\right)d\zeta\Bigg).
\end{aligned} \tag{6.25}$$

By changing the order of integration and since $(\chi-\xi)^{\alpha}<1$, we have

$$\begin{aligned}
E\big|\Theta\Upsilon_2(\xi)-\Theta\Upsilon_1(\xi)\big|^2 &\leqslant 2\Bigg(\mathcal{V}\kappa_1^2\int_0^{\chi}E|\,(\Upsilon_2(\xi)-\Upsilon_1(\xi))\,|^2\Big(\int_{\xi}^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta\Big)d\xi \\
&\quad+\sigma^2\mathcal{V}\kappa_2^2\int_0^{\chi}E|\,(\Upsilon_2(\xi)-\Upsilon_1(\xi))\,|^2\Big(\int_{\xi}^{\chi}(\chi-\zeta)^{\alpha-1}d\zeta\Big)d\xi\Bigg) \\
&\leqslant 2\Bigg(\mathcal{V}\kappa_1^2\int_0^{\chi}\|\Upsilon_2-\Upsilon_1\|^2(\chi-\xi)^{\alpha}d\xi \\
&\quad+\sigma^2\mathcal{V}\kappa_2^2\int_0^{\chi}\|\Upsilon_2-\Upsilon_1\|^2(\chi-\xi)^{\alpha}d\xi\Bigg) \\
&\leqslant \Big(2\mathcal{V}\kappa_1^2+2\sigma^2\mathcal{V}\kappa_2^2\Big)\|\Upsilon_2-\Upsilon_1\|^2. \qquad (6.26)
\end{aligned}$$

Hence,

$$\|\Theta\Upsilon_2-\Theta\Upsilon_1\|_{\hat{B}}=\sup_{\chi\in[0,1]}E\big|\Theta\Upsilon_2(\xi)-\Theta\Upsilon_1(\xi)\big|^2=\Big(2\mathcal{V}\kappa_1^2+2\sigma^2\mathcal{V}\kappa_2^2\Big)\|\Upsilon_2-\Upsilon_1\|_{\hat{B}}. \qquad (6.27)$$

Since, $(2\mathcal{V}\kappa_1^2+2\sigma^2\mathcal{V}\kappa_2^2)<1$, therefore,the operator Θ is a contraction. So, the operator Θ has a unique fixed point. consequently, the SFTDE has a unique solution. □

7. Illustrative examples

This section presents numerical results of the method applied to benchmark/test problems.

Example 7.1. Consider the following SFIDE:

$$\begin{aligned}
D^{\alpha}\Upsilon(\chi)&=g(\chi)+\int_0^{\chi}\xi\Upsilon(\xi)d\xi+\sigma\int_0^{\chi}\Upsilon(\xi)dB(\xi), \qquad (7.1)\\
\Upsilon(0)&=0,\quad 0<\alpha<1,
\end{aligned}$$

where

$$g(\chi)=\frac{\chi^{1-\alpha}}{\Gamma(2-\alpha)}-\frac{1}{3}\chi^3-\sigma\left(\chi B(\chi)-\int_0^{\chi}B(\xi)d\xi\right),$$

which the exact solution of Eq. (7.1) is $\Upsilon(\chi)=\chi$.

We employed the given scheme in section 5 for Eq. (7.1). Numerical results are applied for the various amounts of α, σ and n. The absolute errors of the proposed method for various amounts of σ with $\alpha=0.5$ and $n=7$ have been reported in Table 1. Table 2 provides the absolute errors of our scheme in the three values, $\alpha=0.25, 0.5, 0.75$ for $n=8$ and $\sigma=1$. Graphical comparisons of our approximate solution of the problem with $n=4,6,8$, $\alpha=0.75$ and the exact solution are indicated for $\sigma=1$ in Figure 1. The curves of the absolute errors for different amount of σ for $\alpha=0.5$ and $n=7$ are depicted in Figure 2

Example 7.2. Consider the following SFIDE:

$$\begin{aligned}
{}^{c}D^{\alpha}\Upsilon(\chi)&=g(\chi)+\int_0^{\chi}e^{\chi}\xi\Upsilon(\xi)d\xi+\sigma\int_0^{\chi}e^{\chi}\xi\Upsilon(\xi)dB(\xi), \qquad (7.2)\\
\Upsilon(0)&=0\ ,\ 0<\alpha<1
\end{aligned}$$

Table 1: Absolute errors of $\Upsilon(\chi)$ for various values of σ with $n = 7$ and $\alpha = 0.5$ for Ex 7.1

χ	$\sigma = 0$	$\sigma = 1$	$\sigma = 2$	$\sigma = 3$
0.0	2.5528×10^{-17}	3.53474×10^{-17}	1.50651×10^{-16}	7.75666×10^{-17}
0.1	1.13675×10^{-13}	7.33361×10^{-10}	1.42194×10^{-9}	2.04427×10^{-9}
0.2	1.05828×10^{-13}	2.86358×10^{-10}	4.66192×10^{-10}	5.6436×10^{-10}
0.3	2.20336×10^{-13}	1.33916×10^{-9}	2.3943×10^{-9}	3.21613×10^{-9}
0.4	1.64841×10^{-13}	4.1367×10^{-10}	1.15699×10^{-9}	2.09079×10^{-9}
0.5	4.31473×10^{-13}	2.40588×10^{-9}	5.28667×10^{-9}	8.43921×10^{-9}
0.6	5.07464×10^{-14}	9.87922×10^{-10}	2.9762×10^{-9}	5.87348×10^{-9}
0.7	6.90722×10^{-13}	2.0911×10^{-9}	2.4348×10^{-9}	7.72732×10^{-10}
0.8	5.24717×10^{-14}	2.74067×10^{-10}	2.60037×10^{-9}	7.6541×10^{-9}
0.9	1.15529×10^{-12}	5.29299×10^{-9}	1.2258×10^{-8}	2.16162×10^{-8}

Table 2: Absolute errors of $\Upsilon(\chi)$ for different values of α with $n = 8$ and $\sigma = 1$ for Ex 7.1

χ	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.0	3.01473×10^{-17}	9.2961×10^{-17}	7.74929×10^{-17}
0.1	6.54125×10^{-8}	2.83203×10^{-8}	1.36059×10^{-8}
0.2	6.02485×10^{-8}	2.1649×10^{-8}	7.1716×10^{-9}
0.3	2.01952×10^{-9}	6.41758×10^{-9}	4.26687×10^{-9}
0.4	1.46462×10^{-7}	5.60737×10^{-8}	2.3125×10^{-8}
0.5	3.41838×10^{-9}	2.10429×10^{-8}	1.59788×10^{-8}
0.6	1.74736×10^{-7}	5.04423×10^{-8}	1.3149×10^{-8}
0.7	6.79306×10^{-8}	1.61719×10^{-8}	4.35102×10^{-9}
0.8	1.63536×10^{-7}	8.43322×10^{-8}	3.92666×10^{-8}
0.9	5.55358×10^{-6}	1.41649×10^{-7}	2.95484×10^{-8}

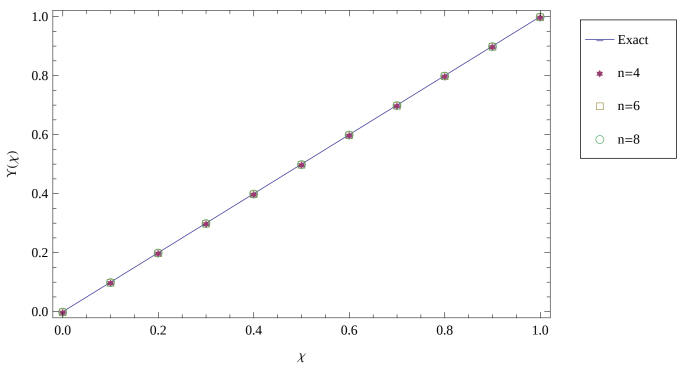

Figure 1: comparison of $\Upsilon(\chi)$ with $\alpha = 0.75$, $\sigma = 1$ for various values of n of Ex 7.1

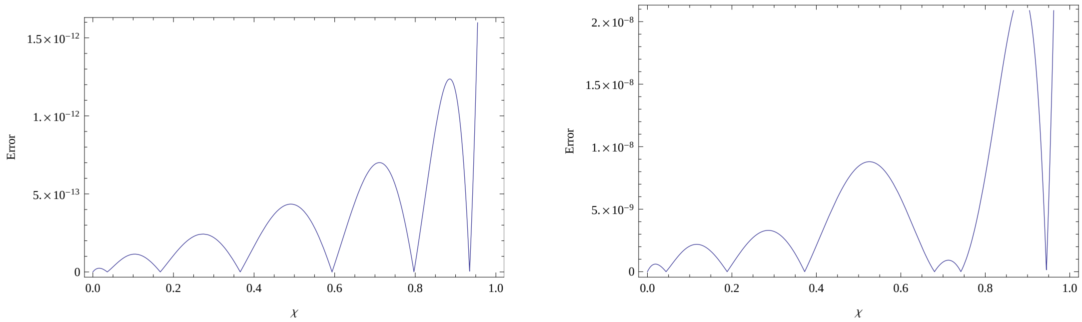

Figure 2: The left side curve of the absolute error for $\sigma = 0$ and the right side curve for $\sigma = 3$ when $\alpha = 0.5, n = 7$ of Ex 7.1

where

$$g(\chi) = -\frac{\chi^5 e^{\chi}}{5} + \frac{6\chi^{3-\alpha}}{\Gamma(4-\alpha)} - \sigma\left(\chi^4 e^{\chi} B(\chi) - \int_0^{\chi} 4\xi^3 e^{\chi} B(\xi) d\xi\right)$$

Which the exact solution of Eq. (7.2) is $\Upsilon(\chi) = \chi^3$.

The proposed method is applied for Eq. (7.2). Numerical results are employed for the different case of α, σ and n. In Table 3, the absolute error of Eq.(7.2) are compared with the obtained results in [21] and [9]. The results show that our suggested method is more careful than the two other methods. Table 4 provides the absolute error of our scheme in the three values, $n = 3, 5, 7$ with $\alpha = 0.75$ and $\sigma = 0.001$. Table 4 demonstrates that increasing the number of Vieta-Lucas basis functions n improves solution accuracy. Graphical comparisons of our approximate solution with $n = 7$, $\sigma = 1$ for values of α and the exact solution are shown in Figure 3. Figure 4 shows the graphical comparisons of approximate solution with $\alpha = 0.5$, $\sigma = 1$ for values of n and the exact solution.

Table 3: Comparison of the absolute errors for $\alpha = 0.75$ and $\sigma = 0$ of Ex 7.2 obtained by three different methods

χ	Method in [9] $n = 10$	Method in [21] $n = 7$	Present method $n = 7$
0.0	1.3724×10^{-41}	3.46129×10^{-6}	1.86238×10^{-16}
0.1	1.9763×10^{-8}	2.40488×10^{-5}	7.83721×10^{-9}
0.2	1.3097×10^{-6}	1.8821×10^{-5}	6.32056×10^{-9}
0.3	1.5733×10^{-5}	1.45089×10^{-5}	8.68267×10^{-9}
0.4	9.5404×10^{-5}	1.45188×10^{-5}	1.04524×10^{-8}
0.5	3.9813×10^{-4}	1.47196×10^{-5}	3.73287×10^{-9}
0.6	1.3125×10^{-3}	1.29151×10^{-5}	1.41256×10^{-8}
0.7	3.6800×10^{-3}	8.15217×10^{-6}	5.84437×10^{-9}
0.8	9.173×10^{-3}	7.06314×10^{-6}	8.55598×10^{-9}
0.9	2.0890×10^{-2}	5.85694×10^{-5}	1.8551×10^{-9}

Example 7.3. Consider the following SFIDE:

$$\begin{aligned} {}^cD^{\alpha}\Upsilon(\chi) &= g(\chi) + \int_0^{\chi} (\chi + \xi)\Upsilon(\xi) d\xi + \sigma \int_0^{\chi} \xi \Upsilon(\xi) dB(\xi), \\ \Upsilon(0) &= 0\,,\ 0 < \alpha < 1 \end{aligned} \tag{7.3}$$

where

$$g(\chi) = -\frac{7}{12}\chi^4 - \frac{5}{6}\chi^3 + \frac{2}{\Gamma(3-\alpha)}\chi^{2-\alpha} + \frac{1}{\Gamma(2-\alpha)}\chi^{1-\alpha} - \sigma\left((\chi^3 + \chi^2)B(\chi) - \int_0^{\chi} \left(3\xi^2 + 2\xi\right) B(\xi) d\xi\right)$$

Which the exact solution of Eq. (7.3) is $\Upsilon(\chi) = \chi^2 + \chi$.

Table 4: Absolute errors of $\Upsilon(\chi)$ for several values of n with $\alpha = 0.75$ and $\sigma = 0.001$ for Ex 7.2

χ	$n = 3$	$n = 5$	$n = 7$
0.0	0	6.93889×10^{-18}	2.84358×10^{-17}
0.1	1.79908×10^{-4}	7.97368×10^{-7}	6.37831×10^{-8}
0.2	1.20701×10^{-4}	4.01335×10^{-6}	1.23766×10^{-7}
0.3	7.35898×10^{-5}	2.51223×10^{-6}	3.93752×10^{-7}
0.4	2.98935×10^{-4}	2.90655×10^{-6}	2.64306×10^{-7}
0.5	4.51304×10^{-4}	7.44365×10^{-6}	2.15967×10^{-8}
0.6	4.26667×10^{-4}	6.23504×10^{-6}	2.22013×10^{-7}
0.7	1.20995×10^{-4}	1.71269×10^{-6}	7.73719×10^{-7}
0.8	5.69744×10^{-4}	9.58713×10^{-6}	7.55815×10^{-7}
0.9	1.74958×10^{-3}	1.16521×10^{-6}	1.33306×10^{-7}

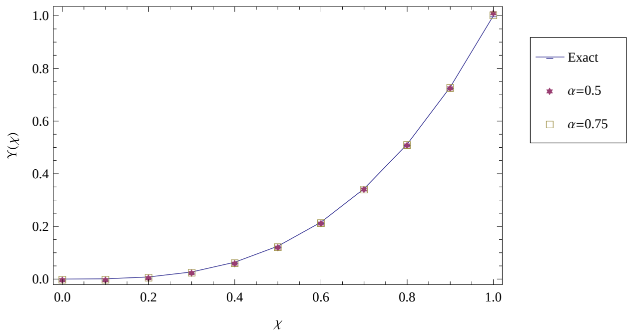

Figure 3: comparison of $\Upsilon(\chi)$ with $n = 7$, $\sigma = 1$ for various values of α of Ex 7.2

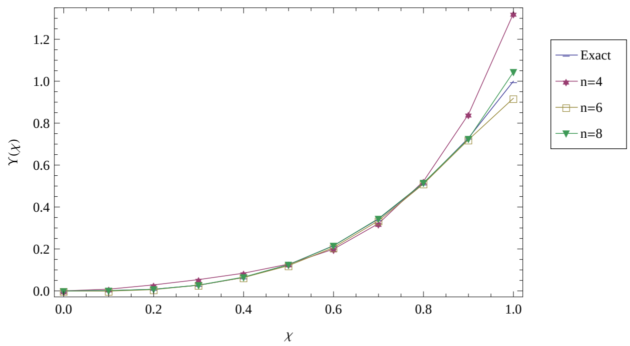

Figure 4: comparison of $\Upsilon(\chi)$ with $\alpha = 0.5$, $\sigma = 1$ for various values of n of Ex 7.2

Numerical results from the proposed method, presented in Tables 5 and 6, demonstrate improved accuracy with increasing n (Vieta-Lucas basis count) and consistent performance across varying σ. Figure 5 compares approximate solutions ($n = 7, \alpha = 0.75$) with the exact solution for different σ, showing close alignment, while Figure 6 visualizes absolute errors ($n = 7, \alpha = 0.5, \sigma = 0.01$). Table 5 and 5 confirms stability for constant χ, with approximate solutions converging to the exact solution despite σ variations.

Example 7.4. Consider the following SFIDE:

$$
\begin{aligned}
{}^{c}D^{\alpha}\Upsilon(\chi) &= g(\chi) + \int_0^{\chi} \Upsilon(\xi)d\xi + \sigma \int_0^{\chi} (\xi + \xi^2)\Upsilon(\xi)dB(\xi), \\
\Upsilon(0) &= 0\ ,\ 0 < \alpha < 1
\end{aligned}
\tag{7.4}
$$

Table 5: Absolute errors of $\Upsilon(\chi)$ for various values of σ with $n = 7$ and $\alpha = 0.25$ for Ex 7.3

χ	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$
0.0	2.72135×10^{-16}	2.64545×10^{-17}	2.18141×10^{-16}	3.1225×10^{-17}
0.1	2.0746×10^{-4}	3.88517×10^{-4}	5.45962×10^{-4}	6.82185×10^{-4}
0.2	3.26903×10^{-5}	7.33514×10^{-5}	1.20124×10^{-4}	1.71331×10^{-4}
0.3	1.7198×10^{-4}	3.25144×10^{-4}	4.61097×10^{-4}	5.81107×10^{-4}
0.4	2.45194×10^{-4}	4.87447×10^{-4}	7.24878×10^{-4}	9.55734×10^{-4}
0.5	5.48836×10^{-4}	1.08477×10^{-3}	1.60671×10^{-3}	2.11327×10^{-3}
0.6	7.52635×10^{-5}	2.0688×10^{-4}	3.93121×10^{-4}	6.32442×10^{-4}
0.7	4.9708×10^{-4}	8.6154×10^{-4}	1.08996×10^{-3}	1.17764×10^{-3}
0.8	4.31383×10^{-4}	8.94715×10^{-4}	1.41679×10^{-3}	2.0247×10^{-3}
0.9	1.24282×10^{-3}	2.35803×10^{-3}	3.39545×10^{-3}	4.39896×10^{-3}

Table 6: Absolute errors of $\Upsilon(\chi)$ with $\alpha = 0.5$ and $\sigma = 0.01$ for different values of n of Ex 7.3

χ	$n = 2$	$n = 4$	$n = 7$
0.0	0	0	1.82103×10^{-16}
0.1	4.99292×10^{-4}	5.44254×10^{-5}	2.35148×10^{-6}
0.2	9.71684×10^{-4}	2.13082×10^{-4}	3.65279×10^{-7}
0.3	1.41717×10^{-3}	3.19568×10^{-4}	1.48052×10^{-6}
0.4	1.83576×10^{-3}	2.95738×10^{-4}	2.24975×10^{-6}
0.5	2.22745×10^{-3}	1.41697×10^{-4}	5.97341×10^{-6}
0.6	2.59224×10^{-3}	6.41918×10^{-5}	2.54758×10^{-6}
0.7	2.93013×10^{-3}	1.65316×10^{-4}	3.60817×10^{-6}
0.8	3.24112×10^{-3}	7.31945×10^{-5}	2.50547×10^{-6}
0.9	3.5252×10^{-3}	9.64461×10^{-4}	1.36097×10^{-5}

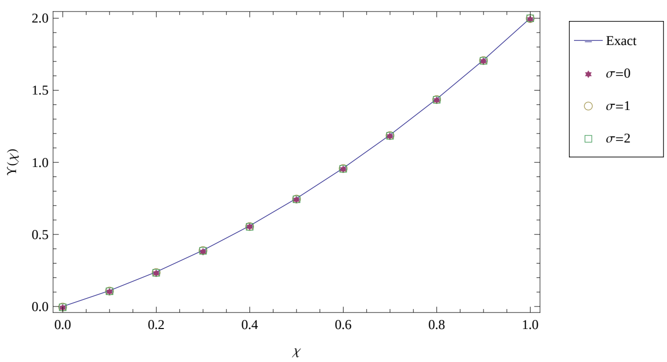

Figure 5: Comparison of $\Upsilon(\chi)$ with $n = 7$, $\alpha = 0.75$ for various values of σ of Ex 7.3

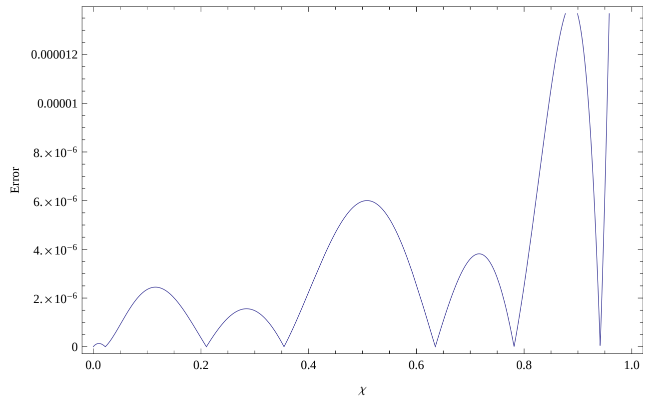

Figure 6: Absolute errors of $\Upsilon(\chi)$ with $n = 7$, $\alpha = 0.5$ and $\sigma = 0.01$ for Ex 7.3

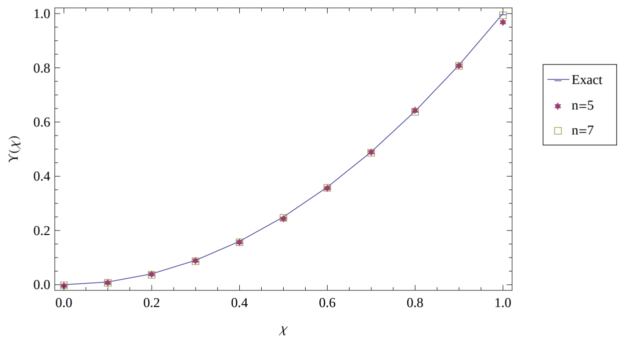

Figure 7: comparison of $\Upsilon(\chi)$ with $\alpha = 0.75$ and $\sigma = 1$ for various amounts of n of Ex 7.4

where

$$g(\chi) = \frac{2\chi^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{1}{3}\chi^3 - \sigma\left(\chi^3(\chi+1)B(\chi) - \int_0^{\chi} \xi^2(4\xi+3)B(\xi)d\xi\right)$$

which the exact solution of Eq. (7.4) is $\Upsilon(\chi) = \chi^2$.

We employed the given scheme for Eq. (7.4) at the various amounts of α, σ and n. Table 7 provides the absolute errors of our scheme in the three values, $\alpha = 0.25, 0.5, 0.75$ for $n = 7$ and $\sigma = 0.001$. Figure 7 displys the graphical comparisons of approximate solution with $\alpha = 0.75$, $\sigma = 1$ for values of n and the exact solution. Graphical comparisons of our approximate solution of the problem with $n = 5$, $\alpha = 0.25, 0.85, 0.95$ and the exact solution are indicated for $\sigma = 0.001$ in Figure 8.

Table 7: Absolute errors of $\Upsilon(\chi)$ for different values of α with $n = 7$ and $\sigma = 0.001$ for Ex 7.4

χ	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.0	1.17051×10^{-17}	3.32895×10^{-17}	8.5059×10^{-18}
0.1	6.44172×10^{-7}	3.43599×10^{-7}	1.82077×10^{-7}
0.2	2.72157×10^{-8}	1.12121×10^{-7}	1.08016×10^{-7}
0.3	2.70733×10^{-7}	8.16903×10^{-8}	6.74216×10^{-9}
0.4	8.03783×10^{-7}	4.03502×10^{-7}	2.29832×10^{-7}
0.5	1.52162×10^{-6}	8.69361×10^{-7}	5.08118×10^{-7}
0.6	2.80900×10^{-7}	4.55512×10^{-7}	4.10191×10^{-7}
0.7	1.12472×10^{-6}	2.77487×10^{-7}	7.53043×10^{-8}
0.8	1.31304×10^{-6}	4.99528×10^{-7}	2.93180×10^{-7}
0.9	3.37958×10^{-6}	1.85615×10^{-6}	1.00319×10^{-6}

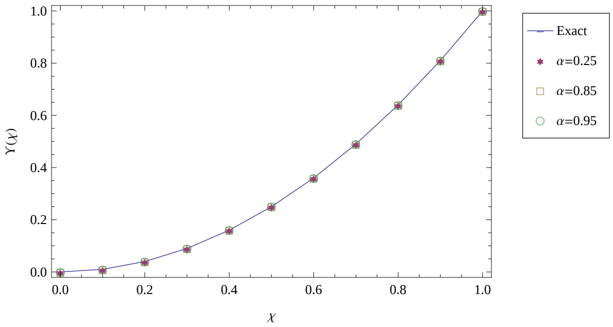

Figure 8: comparison of $\Upsilon(\chi)$ with $n = 5$ and $\sigma = 0.001$ for various amounts of α of Ex 7.4

8. Conclusion

This paper presents a numerical method for solving fractional stochastic integro-differential equations, leveraging shifted Vieta-Lucas polynomials and operational matrices to convert the problem into a linear algebraic system via collocation, with Brownian motion approximated by Gauss-Legendre quadrature. The method is supported by convergence analysis, error estimates, and proofs of existence and uniqueness of solutions. Numerical experiments demonstrate high accuracy and rapid convergence for varying parameters (α, σ, and basis size n), validated through tabular and graphical results, confirming the methods efficacy for practical applications in mathematics, physics, and biology.

References

[1] Azimi, R. & Mohagheghy Nezhad, M. & Foadian,S. (2022) Shifted Legendre Tau Method for Solving the Fractional Stochastic Integro-Differential Equations, 6 (2) 221-241. 1

[2] Babaei, A., Jafari, H. & Banihashemi, S. (2020) Numerical solution of variable order fractional nonlinear quadratic integro-differential equations based on the sixth-kind Chebyshev collocation method, Journal of Computational and Applied Mathematics, 112908. 1

[3] Gupta, R. & Saha Ray, S. (2023) A new effective coherent numerical technique based on shifted Vieta-Fibonacci polynomials for solving stochastic fractional integro-differential equation, Computational and Applied Mathematics, 1-25. 1

[4] Horadam, A. F. (2000) Vieta Polynomials, The University of New England, Armidaie, Australia. 2.2

[5] Jabari Sabegh, D., Ezzati, R. & Maleknejad, K. (2019) Approximate solution of fractional integro-differentail equations by least squares method, International Journal of Analysis and Applications, 17, 303-310. 1

[6] Jafari, H., Ganji, R.M., Salati, S. &, Johnston, S.J. (2024) A mixed-method to numerical simulation of variable order stochastic advection diffusion equations, Alexandria Engineering Journal, Volume 89, 60-70. 1

[7] Lord, G. J. & Powell, C.E. & Shardlow, T. (2014) An Introduction to Computational Stochastic PDEs, Vol. 50: Cambridge University Press. 2.3, 2.4

[8] Masti, I. & Sayevand, Kh. (2020) On collocation-Galerkin method and fractional B-spline functions for a class of stochastic fractional integro-differential equations, Mathematics and Computers in Simulation, 216, 263-287. 1

[9] Mirzaee, F. & Alipour, S. (2020) Cubic B-spline approximation for linear stochastic integro-differential equation of fractional order. J Comput Appl Math (366) 112440. 7, 3

[10] Mirzaee, F., Alipour, S. & Samadyar, N. (2019) Numerical solution based on hybrid of block-pulse and parabolic functions for solving a system of nonlinear stochastic Itô -Volterra integral equations of fractional order. J Comput Appl Math 349:157-171. 1

[11] Mirzaee, F., & Samadyar, N. (2017) Application of orthonormal Bernstein polynomials to construct a efficient scheme for solving fractional stochastic integro-differential equation. Optik 132:262-273. 1

[12] Morters, P. & Peres, Y. (2010) Brownian Motion, Cambridge University Press, Cambridge 30. 1

[13] Mousavi, B. KH., Askari Hemmat, A. & M. H. Heydari (2017). An application of Wilson system in numerical solution of Fredholm integral equations, Poincare Journal of Analysis and Applications, 2, 61-72. 1

[14] Podlubny, I. (1999) Fractional Diferential Equations, Academic Press,New York, NY, USA. 2.1, 2.2

[15] Robbins, N. (1991) Vietas triangular array and a related family of polynomials, Int. J. Math. Math. Sci. 14 (2), 239-244. 2.2

[16] Sayevand, Kh., Tenreiro Machado, J. & Masti, I. (2020) On dual Bernstein polynomials and stochastic fractional integro-differential equations, Math Meth Appl Sci., 1-20. 1

[17] Sayevand, Kh. (2015) Analytical treatment of Volterra integro-differential equations of fractional order, Applied Mathematical Modelling, 39 (15) 4330-4336. 1

[18] Sayevand, Kh. & Tenreiro Machado, J. A.(2019) A Survey on Fractional Asymptotic Expansion Method: A Forgotten Theory, 22, 1165-1176. 1

[19] Sayevand, Kh. & Rostami, M. (2022) Numerical solution of multi order fractional differential equations using Lucas polynomials, Mathematical Researches, 8 (1), 184-204. 1

[20] Senol, M., & Kasmaei, H. D. (2017) On the numerical solution of nonlinear fractional integro-differential equations, New Trends in Mathematical Sciences, 5(3), 118-127. 1

[21] Singh, P.K. & Saha Ray, S. (2023) Numerical treatment for the solution of stochastic fractional differential equation using Lerch operational matrix method, Journal of Computational and Nonlinear Dynamics, 1-27. 1, 7, 3

[22] Singh, P.K. & Saha Ray, S. (2021) Numerical solution of stochastic Itô-Volterra integral equation by using Shifted Jacobi operational matrix method. Appl Math Comput 410:126440. 3.1, 3.1

[23] Stoer, J. & Bulirsch, R. (2002) Introduction to numerical analysis. 2nd ed. Berlin: Springer. 2.5

[24] Wazwaz, A. M. (2011) Linear and Nonlinear Integral Equations Methods and Applications, Springer, New York. 1

[25] Yousefi, A., Javadi, S. & Babolian, E. (2019). A computational approach for solving fractional integral equations based on Legendre collocation method, Mathematical Sciences, 13, 231240. 1