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Numerical solution of volterra integral equations with weakly singular kernel using legendre wavelet method

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Abstract: The presented paper investigates a new numerical method based on the characteristics of Legendre wavelet for solving Volterra Integral equations In this method, with the help of block-pulse functions and their characteristics, we obtain the fractional integral operational matrices corresponding to these wavelets. Then, by introducing collocation points, we use them to convert the desired equations into a system of algebraic equations. After solving the system, the approximate solution of the equation is easily calculated. Finally, we provide some numerical examples to demonstrate the accuracy and efficiency of the proposed methods.

Keywords: Volterra Integral equations, Weakly singular kernel, Legendre wavelet.
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1 Introduction

The name integral equations was first proposed in 1988 by Boyce-Raymond. According to some sources, integral equations first appeared in the research work of Laplace in 1782 [13]. In 1900-1903, a mathematician named Fredholm used integral equations to prove the existence of the solutions to Dirichlet's problem [5]. Volterra integral equations (VIEs) find application in many disciplines, such as electromagnetic scattering, demography, viscœlastic materials, insurance mathematics, etc. They have been subject of many theoretical and numerical investigations. Among the numerical methods for VIEs, the spectral approximations have been attracting more attention recently. The recent progress in the numerical methods for VIEs, includes, but is not limited to: the collocation methods for the Volterra integral and related functional equations, (see, e.g.,) [3], the Jacobi spectral-collocation method for VIEs with a weakly singular kernel [4], spectral Petrov–Galerkin methods for the second kind Volterra type integro-differential equations [28], a spectral Jacobi-collocation approximation for VIEs with Abel type singular kernel [18], a spectral collocation method for weakly singular VIEs with pantograph delays [31], a spectral method for Volterra functional integro-differential equations with delays and smooth kernel [26].In particular, Chen and Tang proposed and analyzed a Jacobi-collocation spectral method for the second kind VIEs with a weakly singular kernel.Zhang et al. investigated the VIEs of second kind with weakly singular kernel and pantograph delays.
There are different methods for numerically solving the singular Volterra integral equation. Among them, the product integration method based on Newton-Cotts laws[11], the Hermitian collocation method[12], the spline collocation

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method, and the iterative collocation method [10] of Laplace transform and Taylor series[30], etc.
In recent years, the application of wavelet theory has expanded in various scientific and engineering fields. Also, many articles have been published with the aim of providing methods to solve all kinds of equations using Legendre wavelets, such as the solution of Volterra's nonlinear integral equation by means of the Legendre wavelet operational matrix[25], Legendre wavelets for solving linear and nonlinear integro-differential equations[21], etc. In this paper, we present a method to solve Volterra's integral equation which is defined as follows:

$$f(x) - \lambda \int_0^x \frac{k(x,t)F(f(t))}{(x-t)^{1-\alpha}} dt = g(x), \quad 0 \leq \ x \leq \ 1. \tag{1.1}$$

where $f(t)$ is the unknown of the problem and the functions $k(x,t)$ ⊠ $g(x)$ and F are known. the real numbers λ and $\alpha > 0$ define a specific type of equation.
In equation (1.1) when $0 < \alpha < 1$, this equation is presented as an integral equation with a weakly singular kernel. The above equation for $\alpha = 1$ and $F(f(t)) = f(t)$ reduces to a linear Volterra integral equation. Especially if $k(x,t) = 1$ and $0 < \beta = 1 - \alpha < 1$, then we will have Abel's integral equation as follows:

$$f(x) - \lambda \int_0^x \frac{f(t)}{(x-t)^{\beta}} dt = g(x), \quad 0 < \beta < 1. \tag{1.2}$$

In this article, using the Legendre wavelet, we present a method for solving equation (1.1) numerically.

2 Legendre Wavelet

Wavelets are powerful tools in approximation theory and numerical analysis of the Hilbert space $L^2(\mathbb{R})$ [6].

There are several bases for wavelets, such as Haar wavelet, Daubechies wavelets, Chebyshev wavelets, Legendre wavelets, and so on [2, 9, 1, 22, 23, 29].

In this section, we explain how to build scale functions and Legendre wavelet [17, 24, 15].and we consider the Legendre wavelets,which are an orthonormal set of functions with respect to the weight function $w(t) = 1$, on the interval [0,1), as follows:

$$\psi_{nm}(t) = \begin{cases} \sqrt{\dfrac{2m+1}{2}}.2^{(\dfrac{k}{2})} P_m(2^k t - 2n + 1), & \dfrac{n-1}{2^{K-1}} \leq t < \dfrac{n}{2^{K-1}}. \\ 0, & otherwise \end{cases} \tag{2.1}$$

where $n = 1, ..., 2^{k-1}$, k is an integer,m is the degree of Legendre polynomial P_m,$m = 0, 1, ...M - 1$, for some positive integer M. A function $f \in L^2(0,1)$, can be represented as series of Legendre wavelets

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \psi_{nm}(t). \tag{2.2}$$

Suppose

$V_{k-1}^M = \left\{ \psi_{nm} : n = 1, ..., 2^{k-1}, m = 0, 1, ..., M-1 \right\}, P_{k-1}^M(f(t)) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} f_{nm} \psi_{nm}(t)$.
then, we have the following theorem about the error of the approximated solution in subspaces V_{k-1}^M[19].

Theorem 2.1. *Let $f \in C^M[0,1]$ and $P_{k-1}^M(f(t)) \in V_{k-1}^M$, then*

$$|f(t) - P_{k-1}^M(f(t))| \leq M_1 2^{-M(k+1)} max|f^{(M)}(\xi)|, \ \xi \in [0,1]$$

where M_1,is a constant.

Proof. See Theorem 2.4 of [19].

□

Theorem 2.2. *Let* $f \in C^M[l, l+1]$, $V^M_{k-1,l} := \{\psi_{nm}(t-l) : n = 1, ..., 2^{k-1}, m = 0, 1, ..., M-1\}$, *and*

$$P^M_{k-1}(f(t)) := \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} f^{(l)}_{nm} \psi_{nm}(t-l) \in V^M_{k-1,l},$$

where

$$f^{(l)}_{nm} = \int_{l+\frac{n-1}{2^{k-1}}}^{l+\frac{n}{2^{k-1}}} f(\tau)\psi_{nm}(\tau - l)d\tau,$$

then

$$|f(t) - P^M_{k-1}(f(t))| \leq M_1 2^{-M(k+1)} max|f^{(M)}(\xi)|, \ \xi \in [l, l+1],$$

where M_1*,is constant.*

Proof. A multiresolution analysis framework developed by Mallat [20],Meyer [7], and discussed at length by Daubechies [27] shows that $\overline{V^M} = L^2[0,1]$, where $V^M := \cup_{k=1}^{\infty} V^M_{k-1}$. Similar analysis shows that $\overline{V^{M,l}} = L^2[l, l+1]$, where $V^{M,l} := \cup_{k=1}^{\infty} V^M_{k-1,l}$.Application of Theorem 1 for $\overline{V^M_{k-1,l}}$ insted of V^M_{k-1} forces the statement.

□

3 Approximation of functions by the Legendre wavelet basis

In this section, we describe methods for calculating the integral operational matrix of multiple Legendre wavelets and explain the approximation of functions using multiple Legendre wavelets.
The contents of this section are taken from [17, 24].
In this section, we want to approximate the assumed function f by a basic set consisting of scale functions $m = 0, ..., r$, $\phi^m \in L^2(\mathbb{R})$ or Legendre wavelets $m = 0, ..., r$, $\psi^m \in L^2(\mathbb{R})$
we explain:

$$\phi^m_{j,k}(x) := 2^{(\frac{j}{2})} \phi^m(2^j x - k), \ j, k \in \mathbb{Z}, m = 0, ..., r, \tag{3.1}$$

$$\psi^m_{j,k}(x) := 2^{(\frac{j}{2})} \psi^m(2^j x - k), \ j, k \in \mathbb{Z}, m = 0, ..., r. \tag{3.2}$$

Now, any function $f(x)$ on the interval [0,1] can be approximated using scale functions as follows where we call P_j the orthogonal image of f on V_j

$$f(x) \approx P_J f(x) = \sum_{k=0}^{2^J-1} \sum_{m=0}^{r} c_{J,K} \phi^m_{J,K}(x) = C^T \Phi_J(x). \tag{3.3}$$

Also, this approximation in terms of wavelet functions can be written as follows:

$$f(x) \approx P_J f(x) = \sum_{m=0}^{r} (c^m_{0,0} \phi^m_{0,0}(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d^m_{j,k} \psi^m_{j,k}(x)) = C^T \Phi_J(x)) = D^T \Psi_J(x). \tag{3.4}$$

where in

$$c_{j,k}^{m}=\int_{0}^{1}f(x)\phi_{j,k}^{m}(x)d(x), \tag{3.5}$$

$$d_{j,k}^{m}=\int_{0}^{1}f(x)\psi_{j,k}^{m}(x)d(x). \tag{3.6}$$

C and D the next $n\times 1$ vectors with $n=(r+1)2^{J}$ are given as follows:

$$D=\left[c_{0,0}^{0},...c_{0,0}^{r}|...|d_{J-1,0}^{0},...d_{J-1,0}^{r}|,...d_{J-1,2^{J-1}-1}^{0},...d_{J-1,2^{J-1}-1}^{r}\right]^{T}, \tag{3.7}$$

$$C=\left[c_{J,0}^{0},...c_{J,0}^{r}|...|c_{J,2^{J}-1}^{0},...c_{J,2^{J}-1}^{r}\right]^{T}. \tag{3.8}$$

4 Solving Volterra integral equation with single weak kernel

Consider the integral equation (1.1), where the function $f(x)$ is the unknown function and the rest are known values. For ease of analysis, we assume:

$$F(f(x))=(f(x))^{q}. \tag{4.1}$$

where q is a positive integer.
We approximate the functions $f(x)$, $g(x)$ and $k(x,t)$ as below.

$$f(x)\simeq\ F^{T}\Psi(x), \tag{4.2}$$

$$g(x)\simeq\ G^{T}\Psi(x), \tag{4.3}$$

$$k(x,t)\simeq\ \Psi(x)^{T}K\Psi(t). \tag{4.4}$$

where vector $(\Psi(t))\Psi(x)$ is the vector of Legendre functions. F and G are the vector of Legendre cœfficients corresponding to the functions $f(x)$ and $g(x)$. The matrix K is also the matrix of Legendre cœfficients. We can write:

$$f(x)\simeq\ F^{T}\phi_{m'\times\ m'}B_{m'}(x). \tag{4.5}$$

we explain:

$$A=[a_{0},a_{1},...a_{m'-1}]=F^{T}\phi_{m'\times\ m'}. \tag{4.6}$$

Therefore, equation (4.5) becomes the following relation.

$$f(x)\simeq\ AB_{m'}(x). \tag{4.7}$$

After calculations and simplification, we arrive at the following linear equation system.

$$F^{T}-\lambda\,\Gamma(\alpha)F^{T}P_{m'\times\ m'}^{\alpha}\approx\ G^{T}. \tag{4.8}$$

5 Numerical examples

In this section, in order to better understand the presented content and check the accuracy of the proposed method, we solve a numerical example with this method.

Example 5.1. *Consider the following Abel integral equation:*

$$f(x) = 1 - \int_0^x \frac{f(x)}{(x-t)^{0.5}} dt. \tag{5.1}$$

In Eq (5.1) $\lambda = 1$, $\alpha = 0.5$, g(x)=1.

As an example, for M=2 and K=1(m'=2) we form the device obtained from equation (4.8).
For each $0 \le t < 1$,

$$\Psi_{nm}(x) = \begin{cases} \sqrt{\frac{2m+1}{2}}.2^{(\frac{k}{2})} L_m(2^k x - n), & \frac{n-1}{2^K} \le x < \frac{n+1}{2^K} \\ 0, & otherwise \end{cases} \tag{5.2}$$

where $\widehat{n} = 2n - 1$, $n = 1, 2, ..., 2^{k-1}$ and $m = 0, 1, ..., M - 1$ then $n = 1$, $\widehat{n} = 1$ and $m = 0, 1$

$L_0(t) = 1$

$L_1(t) = t$

$\Psi_{10}(t) = \sqrt{\frac{1}{2}}.2^{(\frac{1}{2})} L_0(2t-1) \Rightarrow \Psi_{10}(t) = 1$

$\Psi_{11}(t) = \sqrt{\frac{3}{2}}.2^{(\frac{1}{2})} L_1(2t-1) = \sqrt{3}(2t-1)$

$F^T - \lambda\, \Gamma(\alpha) F^T P^{\alpha}_{m' \times\ m'} \approx\ G^T$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Psi(t) = \begin{bmatrix} \Psi_{10}(t) \\ \Psi_{11}(t) \end{bmatrix}$

$\phi_{m' \times\ m'} = \begin{bmatrix} \Psi(\frac{1}{4}) & \Psi(\frac{3}{4}) \end{bmatrix} = \begin{bmatrix} \psi_{10}(\frac{1}{4}) & \psi_{10}(\frac{3}{4}) \\ \psi_{11}(\frac{1}{4}) & \psi_{11}(\frac{3}{4}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$g_{10}(t) = \int_0^1 w_1(t)\Psi_{10}(t)g(t)dt = \int_0^1 \sqrt{1-(2t-1)^2}\ dt = \frac{\pi}{4} = 0.7854$

$g_{11}(t) = \int_0^1 w_1(t)\Psi_{11}(t)g(t)dt = \int_0^1 \sqrt{3}(2t-1)\sqrt{1-(2t-1)^2}\ dt = 0$

$G = \begin{bmatrix} g_{10} \\ g_{11} \end{bmatrix} = \begin{bmatrix} 0.7854 \\ 0 \end{bmatrix}$

$F^{\frac{1}{2}} = \begin{bmatrix} 0.5319 & 0.4407 \\ 0 & 0.5319 \end{bmatrix}$

$P^{\frac{1}{2}}_{2\times2} = \begin{bmatrix} 1 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0.5319 & 0.4407 \\ 0 & 0.5319 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 0.7522 & 0.2544 \\ -0.1908 & 0.3115 \end{bmatrix}$

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} + \sqrt{\pi} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 0.7522 & 0.2544 \\ -0.1908 & 0.3115 \end{bmatrix} = \begin{bmatrix} 0.7854 & 0 \end{bmatrix}$$

$$\begin{cases} 2.3332 f_1 - 0.3382 f_2 = 0.7854 \\ 0.4509 f_1 + 1.5521 f_2 = 0 \end{cases} \tag{5.3}$$

$\Rightarrow f_1 = 0.3230, f_2 = -0.0938$

$$f(x) = 0.3230\Psi_{10}(x) - 0.0938\Psi_{11}(x) \Rightarrow f(x) = 0.3230 - 0.1625(2x-1).$$

With this equation, the value of the function $f(x)$ can be obtained at any arbitrary point $x \in [0,1)$.
For example, by using this relationship we have:

f(0.1)=0.453

The error is calculated as the difference between the exact and approximate solutions.

We solved this problem using MATLAB with M=2 and K=3,4,5. The absolute error at the points 0.1,0.2,...,0.9 is given in Table 1.

Table 1: Error Analysis for example 5.1.

x_i	K=3,M=2	K=4,M=2	K=5,M=2
0.1	3.9467×10^{-2}	6.8032×10^{-3}	1.4565×10^{-3}
0.2	1.0431×10^{-2}	1.4870×10^{-3}	1.8362×10^{-4}
0.3	1.4632×10^{-3}	5.2631×10^{-4}	1.2199×10^{-4}
0.4	1.3070×10^{-3}	2.7040×10^{-4}	6.8015×10^{-5}
0.5	2.8847×10^{-3}	8.2240×10^{-4}	2.2135×10^{-4}
0.6	3.9269×10^{-4}	1.6084×10^{-4}	2.6100×10^{-5}
0.7	5.0770×10^{-4}	7.3140×10^{-5}	2.9371×10^{-5}
0.8	3.4487×10^{-4}	4.6880×10^{-5}	2.4149×10^{-6}
0.9	1.5759×10^{-4}	7.3694×10^{-5}	8.3615×10^{-6}

In order to intuitively check the accuracy of the method, a comparison of the exact and approximate solution for M=2 , K=4 is shown in Fig 1.

Example 5.2. *Consider the following nonlinear Volterra integral equation with its singular kernel:*

$$f(x) - \int_0^x xt(x-t)^{0.5} dt = x^3 - 4096x^{8.5}. \tag{5.4}$$

The exact answer to this equation is $f(x) = x^3$. Using the computer program of the presented algorithms, we have solved this problem and summarized the results in a table and figure. table 2 shows the error values for M=2 and K=3,4,5. Figure 2 compares the exact and approximate solutions for M=2 and K=4 and shows that the obtained solutions are very close to the true solutions of the equation.

Table 3 shows the exact answer and approximate answers example 5.1) at points 0.1, 0.2, ..., 0.5 for K=1 , M=2 using the presented method and also using the Collocation method.
Table 3 shows the better performance of the proposed method.

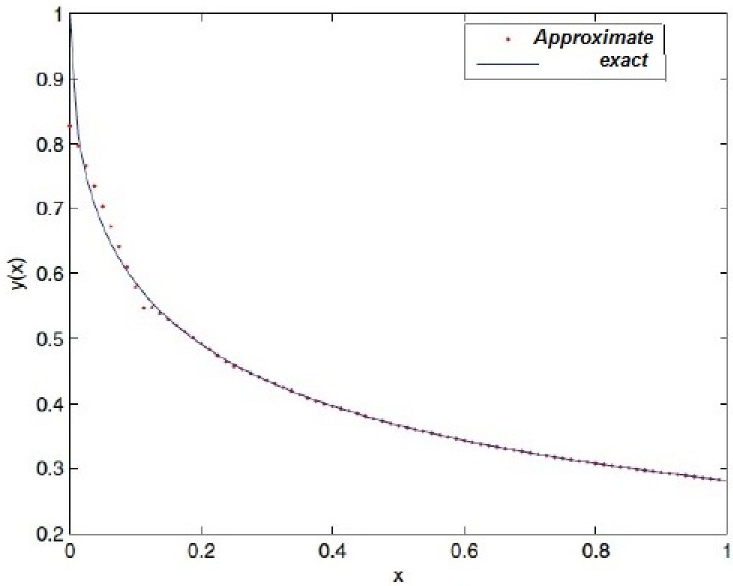

Figure 1: Comparison of exact and approximate answer for example 5.1.

Table 2: Error Analysis for example 5.2.

x_i	K=3,M=2	K=4,M=2	K=5,M=2
0.1	1.1480×10^{-3}	9.6670×10^{-5}	5.8960×10^{-5}
0.2	7.7314×10^{-4}	4.7198×10^{-4}	6.8433×10^{-5}
0.3	1.8041×10^{-3}	7.6219×10^{-4}	9.1856×10^{-5}
0.4	3.8611×10^{-3}	5.2950×10^{-4}	2.5428×10^{-4}
0.5	2.0981×10^{-2}	4.7042×10^{-3}	1.1219×10^{-3}
0.6	7.4657×10^{-3}	4.3465×10^{-4}	4.5193×10^{-4}
0.7	8.9781×10^{-5}	2.7027×10^{-5}	2.3869×10^{-6}
0.8	5.5454×10^{-4}	5.0052×10^{-5}	4.8476×10^{-6}
0.9	7.4328×10^{-2}	7.3239×10^{-3}	2.9980×10^{-3}

Table 3: Comparison of the proposed method with the Collocation method.

x_i	Proposed method	Collocation method	Exact answer
0.1	0.4530	0.4528	0.4544
0.2	0.4205	0.4201	0.4215
0.3	0.3880	0.3880	0.3885
0.4	0.3555	0.3559	0.3550
0.5	0.3230	0.3233	0.3226

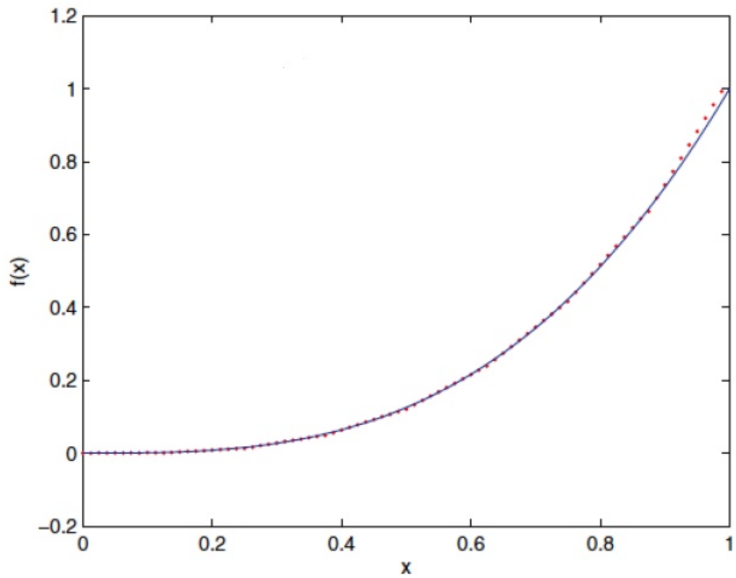

Figure 2: Comparison of exact and approximate answer for example 5.2.

6 Conclusion

The Legendre wavelet method was examined for the numerical solution of Volterra integral equations with weakly singular kernel. This method is found to be reliable, effective and straight forward to compute. MATLAB is used for all of the computations in this work.

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