Math. & Comput. Sci., 6(2) 2025, 72–76
Online: ISSN 2717-2708

Mathematics and Computational Sciences

Journal Homepage: http://mcs.qut.ac.ir/mcs

Introduction of some one-searchable graphs

Mohammad Taati[a,*]

[a]Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Abstract

Search Graph problems are usually modeled in the form of On-Graph games between one escaping thief and some police officers. Police intends to arrest the thief while the real goal of robber is prevention from arrest. As it has been evaluated in this paper, it is known as Cops & Robber Game. There are two specific characteristics for this play which may separate it from others. Firstly it has unlimited speed and secondly all players have special playing time.

Keywords: Graph, Search number, Game.

2020 MSC: 05C30, 05C38, 05C57

1. Introduction

Search on-graph issues are mostly modeled in the form of on-graph games between an escaping thief and some cops. Cops intend to arrest the thief while the real goal of robber is prevention from arrest. This game is also named as Chase. It is practically applicable in any discussions about control, robotic, sensing networks and network safety. There are also some theoretical interests for studying these games because of interesting relations among them and also various parameters in graph theories like tree width. Upon determining different conditions for location of players (on vertexs or wings), definition of arrest, tangible or intangible and speed of players, there are different forms of this game as well. There are two important parameters in these types such as minimum number of required officers and required time for arresting. Police is obliged to have guaranteed strategies which means to make any functions with final arrest. This paper is about a form of this relation named as Cops & Robber. Unlimited speed and playing time are two specific characteristics of this play which may separate it from the others. This game was studied for the first time in 1978 ([6]). There are also three international workshops about in-graph search through the years 2006, 2008 and 2009 ([4], [2], [5] & [3]). A review of different definitions from viewpoint of Graph Theory

Definition 1.1. Graph: It means a non-empty set of vertices named as vertexs and a non-sequential set of pairs named as edges. $V(G)$ illustrates the set of vertexs of graph and $E(G)$ illustrates the set of graph edges.

*Corresponding author
Email address: taati@pnu.ac.ir (Mohammad Taati)

doi: 10.30511/mcs.2025.2048243.1268

Received: 16 December 2024 Accepted: 24 April 2025

Subgraph: It means a graph in which its set of vertexs is a sub-collection of $V(G)$ and its set of edges is a substitute of $E(G)$.

Path: Graph G is a path when there is not a repeated vertex or edge.

Cycle: Cycle means a closed way in graph G like $v_0e_1v_1...v_{k-1}e_kv_k$ while $v_0 = v_k$ is a path.

Connected graph: Graph G is named as connected graph when there is a path between its both vertexs.

Tree: Tree is the name of a connected graph without any cycles.

Bipartite graph: It means a graph in which total vertexs are partitioned into two non-empty sets of $v_1 and v_2$ while there are not two vertexs in neighboring to V1 and not two vertexs close to v_2. A complete bipartite graph is a part of a bipartite graph while the vertex of v_1 is close to each vertex in v_2.

Complete graph: A complete graph has an edge between two vertexs. Complete graph n illustrates a vertex of K_n.

Directed graph: A directed graph has two sets of $E(D) and V(D)$ and a sequential pair (u, v) is named as a directed edge from u to v. It is illustrated by a directed line from u to v.

Flat graph: It is named as Flat graph when there is a vertex at the cross point of two edges.

Definition 1.2. **(Game)** There are two players (Cops and Robber) and also an officer c who may play this game by a connected graph without any direction and/or edge. Firstly police officer is located on a graph vertex. Thief is located on another graph vertex. Then it is the start of play with some cycles. Firstly thief and police start to move in each cycle. Police is entitled just to remain in its place and/or move towards a neighbor vertex. It is no matter to have some officers on a special vertex at the beginning. Thief is entitled to move towards a neighboring vertex or remain in its position. In case the thief and one of the officers are located in a same vertex, it is the time of arresting the thief and game is over. The game is played with complete information. This means that all players are informed about the location of officers and thief.

Definition 1.3. **(Search number)** Assume c as a police officer. In case of a police strategy for arresting thief in each movement, it is said that c is able to arrest thief. The minimum number of officers for arresting of thief for each graph G is named as search G number which is illustrated by $cn(G)$. If $cn(G) \leqslant c$ then we name G graph as $c - searchable$.

In this [1] paper, for the first time, the concept of search number was presented.

2. Main results

The first papers published on the cop-robber game were [7] and [1], where (independently) a game with an agent was proposed and graphs where the cop could win were specified. Therefore determining of 1-searchable graphs are the oldest issue in Cops & Robber Game. Hereinafter we will explain how to solve this problem.

Definition 2.1. **(Surrounded vertex & Peripheral vertex)** The vertex u in graph G is named as a surrounded one when there is adjacent vertex like v out of G. Therefore the vertex v is named as peripheral (or limited) vertex of u.

Example 2.2. In figure 1, the vertex v_6 is surrounded by v_5. Also v_1 is limited by v_2. But v_4 is not limited by v_3. If $cn(G) = 1$, then G has at least one surrounded vertex.

Definition 2.3. **(contraction)**. A subgraph H of G is called a contraction of G if there exists a function $f : V(G) \longrightarrow V(H)$ such that:

1) For each $u \sim v$, $f(u) = f(v)$ or adjacent.

2) For each $u \in V(H)$ we have $f(u) = u$

Theorem 2.4. *If* H *is a contraction of* G *then* $cn(H) \leqslant cn(G)$.

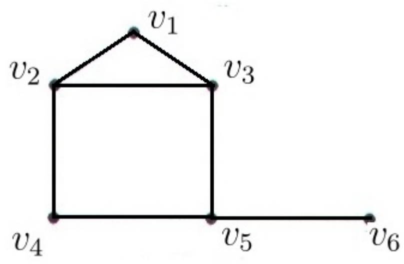

Figure 1: example 2.2.

Proof. Let $f : V(G) \longrightarrow V(H)$ be a contraction function and $c < cn(H)$. We show that $c < cn(G)$. It is enough to show an escape strategy for stealing on G. At any moment when the agents are at the vertices $v_1, v_2, ..., v_c \in V(G)$, he imagines that the agents are actually on$f(v_1), f(v_2), ..., f(v_c) \in V(H)$ are With this idea, the thief plays his escape strategy on H. Note that if the agents make a move on G, their image under f makes a valid move on H (due to the first contraction property). As a result, the thief can easily play his escape strategy on H. Because of $c < cn(H)$, the thief can play such that he never ends up in a vertex like (v_i). As a result, since he never leaves H and f is the same function on H, we conclude that the thief is never caught. So$c < cn(G)$. Since this is true for every $c < cn(H)$, we have $cn(H) \leqslant cn(G)$. □

Definition 2.5. $N(u)$ is the set of neighboring vertices of u and $N[u] = u \cup N(u)$.

Lemma 2.6. *If u is a surrounded vertex of* G, *and* $cn(G) = 1$ *if and only if* $cn(G - u) = 1$.

Proof. Assume v has no more peripheral head u. It is possible to have $f : V(G) \longrightarrow V(G - u)$ in which may connect all heads to themselves except for u. It is connected to v as a contraction. According to Theorem 2.4, we have $cn(G - u) \leqslant cn(G)$. Also, If $cn(G) = 1$, then $cn(G - u) = 1$. Now, if $cn(G - u) = 1$, then we may prove that an officer is able to arrest robber at G point. Assume that R is location of robber. An officer can arrest $f(R)$ in $G - u$.. After that, either he R has arrested or robber is present at u and officer at $f(u) = v$. As a result, since we have $N[u] \subset N[v]$, any movements of robber will be resulted in arresting in next round. □

Example 2.7. Assume there is a graph like the same in figure 1. The vertex of V1 is a surrounded one. Then we have $cn(G) = 1$ if and only if $cn(G - v_1) = 1$. In addition, $v_6$has been surrounded at $G - v_1$, then we have $cn(G - v_1) = 1$ if and only if $cn(G - \{v_1, v_6\}) = 1$. But please note that we have $G - \{v_1, v_6\} \cong C_4$, then it is not a $1 - \mathtt{searchable}$. As a result, G is not a $1 - \mathtt{searchable}$ as well. On the other hand , we have $cn(G) \leqslant 1$, then we may conclude that $cn(G) = 2$.

The result of above-mentioned example is that $cn(G) = 1$ if and only if G has surrounded vertex like u while $cn(G - u) = 1$. In order to provide a more regular definition for this result, we use the following definition.

Definition 2.8. (Sortable graph) A graph is named as Sortable when it is possible to organize its vertexs as $(v_1, v_2, ..., v_n)$ while the vertex v_i is surrounded by $\{v_i, ..., v_n\}$ below the induced graph.

Example 2.9. In case of numbering all vertexs in graph of figure 2 as mentioned below, we will have a Sortable graph .

Theorem 2.10. *Graph* G *is* $1 - \mathtt{searchable}$ *if and only if it is a Sortable.*

Proof. We proof the order by induction on n. It is completely clear for $n = 1, 2$. Assume $cn(G) = 1$, then G has a surrounded vertex like v_1. According to the previous idea, also $G - v_1$ is a $1 - \mathtt{researchable}$. According to the induction idea of vertexs $G - v_1$, it is possible to regulate it as $(v_2, v_3, ..., v_n)$ while v_i

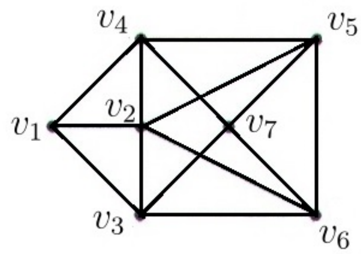

Figure 2: Figure 2 example 2.9.

has been surrounded by $\{v_i, ..., v_n\}$ below the induced graph. After adding v_1 to the beginning of this sequence, we may find out a suitable sequence for G. Then G is a Sortable.

Again we may prove the order by focusing on fixed n. It is completely clear and correct for $n = 1, 2$. $(v_2, v_3, ..., v_n)$ is a sequence forG $- v_1$ to show that G $- v_1$ is Sortable . Therefore according to the hypothesis, it is $1 - \mathrm{researchable}$ as well. Now because v_1 is a surrounded vertex at G, therefore it is also a $1 - \mathrm{researchable}$. □

Definition 2.11. Bridged graph If C is a cycle, the real meaning of a bridge on C is a path between both vertexs of u, v with a length of certainly lower than $d_c(u, v)$. If graph G is a bridge holder, it should bear a bridge in any cycles more than three.

Example 2.12. There is a bridge holder graph in figure 3.

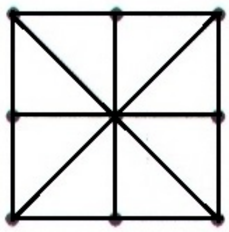

Figure 3: example 2.12.

Corollary 2.13. *All bridge holder graphs are* $1 - \mathrm{researchable}$.

Definition 2.14. Triangulated graph A graph G is called triangulated if it does not have any induced cycle of length more than three.

Example 2.15. A tree is a triangulated graph because of any lack of cycles.

Example 2.16. Figure 4 illustrates a sting graph.

Corollary 2.17. *A triangulated graph is* $1 - \mathrm{researchable}$.

Conclusion : As it was stated, this game was defined in 1978. By the way all important results are related to recent years. It is a sign that no more discussions are available about this discussion. It is not really difficult to find out new results. This game was firstly studied on random graphs. After that lots of papers have been written in this regard. (For more information about the results, refer to [BPW09]). Therefore it is really interesting to work on random graphs with further results. Also there is little studies about any plays on directed graphs. It is a clear idea for the present research.

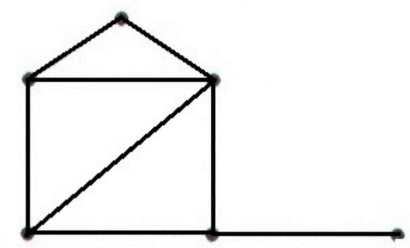

Figure 4: example 2.16.

There are smart conclusions about complexity of searching digit. It is a sign that finding out newer results is probably more difficult. Regarding finding any limitation for searching digit, it is assumed that deeper studies are useful. A presented idea in this regard is the provided G graph for making H graph through multiplying of various simple graphs. G should be set in H while searching digit H is the upper limit for searching digit G as well.

Regarding 1-researchable graphs, it seems that no more dark point are there, but only studies about 2-researchable graphs are interesting. There is not enough study about searching time of graphs. Also it is useful to work on time calculation for searching specific graphs. (For instance multiplying graphs). There are little number of studies about quicker rubbers and invisible ones. Perhaps we may find more results by making more studies. Some of random strategies need more proofs which are not as simple as two previous examples.

References

[1] Aigner, M. & Fromme, M., (1984). A game of cops and robbers, Discrete Appl. Math. 8 , 1-12. 1, 2
[2] Bonato, A., Hahn, G. & Wang, C., (2007). The cop density of a graph, Contrib. Discrete Math. 2 ,133-144. 1
[3] Bonato, A., Pralat, P. & Wang, C., (2009). Pursuit-evasion in models of complex networks, Internet Math., 419-436. 1
[4] Clarke, N. E. & Connon, E. L., (2006). Cops, robber, and alarms, Ars Combin. 81 , 283-296. 1
[5] Fomin, F. V., Golovach, P. A. & Kratochv l, J., (2008).On tractability of cops and robbers game, in: Proc.5th IFIP International Conference on Theoretical Computer Science (TCS 2008), Springer-Verlag IFIP International Federation for Information Processing, vol. 237, pp. 171-185. 1
[6] Frankl, P., (1987). On a pursuit game on Cayley graphs, Combinatorica 7 , 67-70. 1
[7] Nowakowski, R. & Winkler, P., (1983). Vertex-to-vertex pursuit in a graph, Discrete Math. 43 , 235-239. 2