



On q-double modified Laplace transform

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Abstract

The Laplace transform is widely used in science and technology to deal with complex problems in stability and control systems. The modified Laplace transform has been applied in physics and mathematics to solve boundary layer equations in ordinary differential equations with variable coefficients. The q-calculus appeared as a connection between mathematics and physics. It has many applications in different mathematical areas, such as number theory, combinatory theory, orthogonal polynomials, essential hyper-geometric functions, quantum mechanics, and relativity. Laplace transform, and its several extended versions are used frequently. The double Laplace transform applies to solving some q-functional and partial q-differential equations. Q-calculus has been used to solve complex and more potentially typical problems in a larger domain to investigate the calculus without limits for getting more generalizations. In the paper, we introduce the double-modified Laplace transform in q-calculus, namely the q-double modified Laplace transform, and establish some properties. Furthermore, several propositions concerned with q-double modified Laplace transform are explored.

Keywords: q-calculus, q-modified Laplace transform, q-Integration, q- differentiation.

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1. Introduction

Researchers are actively engaged in an integral transform to develop the subject as it is applicable in describing and analyzing physical systems [1, 2, 3, 4, 7, 13, 11, 14, 15, 16, 19, 17, 21, 29, 23, 24, 25, 26, 31, 37, 38, 40, 41]. Jackson [22] introduced q-calculus, and now a days q-calculus got much significance in different branches of science and engineering. The concept q-calculus is applicable in fractional calculus, control problems, and for finding solutions of q-difference and q-integral equations [27, 32, 33, 34, 35, 36, 40]. The main advantages of the transform and the novelty of this work are related to a few relevant works in several extended versions of Laplace transformation with q- calculus to deal with complex problems in mathematics and physics. The double Laplace transform of a function of two variables, which was first introduced in 1939 by Berstein [8] which was applied in functional, integral, and partial differential equations [12]. The development of q-analysis started in the 1740s when Euler initiated the theory

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of partitions. In 1829, Jacobi presented triple product identity and principal functions, equivalent to q -analysis. The progress of q -calculus continued with the invention of the hyper-geometric series and their contiguity relations. The theory of q -analysis has been applied in the recent past in many areas, like ordinary fractional calculus, optimal control problems, quantum calculus, q -transform analysis, and in finding solutions of the q -difference and q -integral equations. In 1910, Jackson [22] presented a precise definition of the q -Jackson integral and developed q -calculus in a systematic way. In continuation of the advancement of the transform, P. Njionou Sadjang [36] has introduced four kinds of double q -Laplace transforms with applications to solve some classical partial q -differential equations. The double q -Laplace transform is a generalization of the given one [8]. Jafari et al [21] have given a new algorithm for solving dynamic equations on a time scale, called the q -difference equation, with applications for solving the strongly non-linear damped q -difference equation. Ali et al. [7] have explored an innovative analysis of the time-fractional q -deformed tanh-Gordon equation via a modified double Laplace transform method. This equation represents a significant advancement in mathematical physics due to its applications in many fields, including superconductivity and fiber optics. It has many applications in condensed matter physics and modeling physical systems exhibiting violated symmetries. Notably, this equation has not been previously solved in fractional form, which was the approach pioneering in the analysis. Ali et al. [7] solved the equation utilizing the modified double Laplace transform method, which was considered a semi-analytical technique that combined the double Laplace transform with Adomian polynomials to enable us to extract nonlinear terms. This method was renowned for its efficacy in handling fractional differential equations, evident from the results obtained by comparing the analytical solution with the approximate solution. Laplace transform, and its several extended versions are used frequently till now. So, inspired by the remarkable work of modified double Laplace transform in different fields of mathematics and physics to solve complex problems, we introduce the double-modified Laplace transform in q -calculus, namely the q -double modified Laplace transform, and establish some properties. Several propositions concerned with q -double modified Laplace transform properties are explored.

2. Remarkable Notes

We recall some well known definition and notation used in [5, 9, 12, 18, 20, 22, 28, 30, 34].

The q -derivatives is given in [22]

$$(D_q \mathfrak{h})(\nu) = \frac{\mathfrak{h}(\nu) - \mathfrak{h}(q\nu)}{(1-q)\nu}, \text{ if } (\nu \neq 0)$$

$$(D_q \mathfrak{h})(0) = \mathfrak{h}'(0) \text{ exists.}$$

The q -derivative of the product

$$D_q(\mathfrak{h} \cdot \varphi)(\nu) = \varphi(\nu)D_q \mathfrak{h}(\nu) + \mathfrak{h}(q\nu)D_q \varphi(\nu).$$

The q -Jackson integral in [18, 22]

$$\int_0^\kappa \mathfrak{h}(\nu) d_q \nu = (1-q)\kappa \sum_{n=0}^{\infty} \mathfrak{h}(\nu q^n) q^n,$$

$$\int_0^\infty \mathfrak{h}(\nu) d_q \nu = (1-q) \sum_{n=-\infty}^{\infty} \mathfrak{h}(q^n) q^n.$$

A q -analogue of integration is

$$\int_{\kappa}^{\rho} \varphi(\nu) D_q \mathfrak{h}(\nu) d_q \nu = \mathfrak{h}(\rho) \varphi(\rho) - \mathfrak{h}(\kappa) \varphi(\kappa) - \int_{\kappa}^{\rho} \mathfrak{h}(q\nu) D_q \varphi(\nu) d_q \nu.$$

In [20, 27, 28, 30, 32], we have

$$E_q^{\mathfrak{U}} = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{\mathfrak{U}^n}{[n]!} = (-(1-q)\mathfrak{U}; q)_{\infty}, \quad (2.1)$$

$$e_q^{\mathfrak{U}} = \sum_{n=0}^{\infty} \frac{\mathfrak{U}^n}{[n]!} = \frac{1}{((1-q)\mathfrak{U}; q)_{\infty}}, \quad \mathfrak{U} < \frac{1}{1-q}. \quad (2.2)$$

Thus from (2.1) and (2.2), we have

$$D_q e_q^{\mathfrak{U}} = e_q^{\mathfrak{U}}, \quad D_q E_q^{\mathfrak{U}} = E_q^{q\mathfrak{U}},$$

and

$$e_q^{\mathfrak{U}} E_q^{-\mathfrak{U}} = E_q^{-\mathfrak{U}} e_q^{\mathfrak{U}} = 1.$$

In [9, 18, 34, 35, 36], we have

$$\Gamma(\psi) = \int_0^{\infty} \nu^{\psi-1} e^{-\nu} d\nu \quad \text{by}$$

$$\Gamma_q(\psi) = \frac{(q; q)_{\infty}}{(q^{\psi}; q)_{\infty}} (1-q)^{\psi-1}, \quad \psi \neq 0, -1, -2, \dots$$

Therefore

$$\Gamma_q(\psi + 1) = [\psi]_q \Gamma_q(\psi), \quad \Gamma_q(1) = 1,$$

and

$$\lim_{q \rightarrow 1^-} \Gamma_q(\psi) = \Gamma(\psi), \quad \operatorname{Re}(\psi) > 0.$$

We have

$$\Gamma_q(\Omega) = \int_0^{\frac{1}{1-q}} \psi^{\Omega-1} E_q^{-q\psi} d_q \psi = \int_0^{\frac{\infty}{1-q}} \psi^{\Omega-1} E_q^{-q\psi} d_q \psi.$$

In [5, 12, 20, 28, 30, 36, 39, 40, 41, 42], we have :

For all $\Omega, \psi > 0$, we have

$$\Gamma_q(\Omega) = K_q(\Omega) \int_0^{\frac{\infty}{1-q}} \nu^{\Omega-1} e_q^{-\nu} d_q \nu,$$

and

$$B_q(\psi, \Omega) = K_q(\psi) \int_0^\infty v^{\psi-1} \frac{(-vq^{\Omega+\psi}; q)_\infty}{(-v; q)_\infty} d_q v,$$

where,

$$K_q(\psi) = \frac{(-q, -1; q)_\infty}{(-q^\psi, -q^{1-\psi}; q)_\infty}.$$

3. q - double modified Laplace transform

Definition 3.1. The q - double modified Laplace transform of a function $\mathfrak{h}(\tau, \kappa)$ is defined by

$$L_q[\mathfrak{h}(\tau, \kappa)] = G(\omega, \lambda) = \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau + \lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q \tau d_q \kappa. \quad (3.1)$$

Property 3.2. (Linearity property)

$$L_q\{M\mathfrak{h}(\tau, \kappa) + Ng(\tau, \kappa)\} = M L_q[\mathfrak{h}(\tau, \kappa)] + N L_q\{g(\tau, \kappa)\}.$$

Proof. $L_q\{M\mathfrak{h}(\tau, \kappa) + Ng(\tau, \kappa)\}$

$$\begin{aligned} &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty \{M\mathfrak{h}(\tau, \kappa) + Ng(\tau, \kappa)\} e_q^{-(\omega\tau + \lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q \tau d_q \kappa \\ &= \frac{M}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau + \lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q \tau d_q \kappa \\ &\quad + \frac{N}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau + \lambda\kappa)\log a} g(\tau, \kappa) d_q \tau d_q \kappa \\ &= M L_q[\mathfrak{h}(\tau, \kappa)] + N L_q[g(\tau, \kappa)]. \end{aligned}$$

Property 3.2. (Shifting Property)

If $\xi > 0$, $\Phi > 0$

$$L_q\{e_q^{-\xi\tau - \Phi\kappa} \mathfrak{h}(\tau, \kappa)\} = \bar{\mathfrak{h}}(\omega + \xi, \lambda + \Phi).$$

Proof. $L_q\{e_q^{-\xi\tau - \Phi\kappa} \mathfrak{h}(\tau, \kappa)\}$

$$\begin{aligned} &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau + \lambda\kappa)\log a} e_q^{-\xi\tau - \Phi\kappa} \mathfrak{h}(\tau, \kappa) d_q \tau d_q \kappa \\ &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-\tau(\omega + \xi)\log a} e_q^{-\kappa(\lambda + \Phi)\log a} \mathfrak{h}(\tau, \kappa) d_q \tau d_q \kappa \\ &= \bar{\mathfrak{h}}[(\omega + \xi)\log a, (\lambda + \Phi)\log a]. \end{aligned}$$

Property 3.3. (Change of scale property)

For $\Omega > 0$, and $\Psi > 0$, we have

$$\begin{aligned}
L_q[\mathfrak{h}(\Omega\tau)g(\Psi\kappa)] &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-\omega\tau\log a - \Lambda\kappa\log a} \mathfrak{h}(\Omega\tau) g(\Psi\kappa) d_q\tau d_q\kappa \\
\text{Put, } \Omega\tau = s \rightarrow \Omega d_q\tau &= d_qs, \quad \Psi\kappa = t, \Rightarrow \Psi d_q\kappa = d_qt \\
&= \frac{1}{(1-q)^2 \Omega\Psi} \int_0^\infty \int_0^\infty e_q^{-\omega \frac{s}{\Omega} \log a - \Lambda \frac{t}{\Psi} \log a} \mathfrak{h}(s) g(t) d_qs d_qt \\
&= \frac{1}{\Omega\Psi} \mathfrak{h}\left(\frac{\omega\log a}{\Omega}, \frac{\Lambda\log a}{\Psi}\right).
\end{aligned}$$

Property 3.4.

For $\Omega > 0$, $\Psi > 0$ we have

$$\begin{aligned}
L_q\{e_q^{-\Omega\tau - \Psi\kappa} \mathfrak{h}(\tau, \kappa)\} &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-\Omega\tau - \Psi\kappa} e_q^{-\omega\tau\log a - \Lambda\kappa\log a} d_q\tau d_q\kappa \\
&= \frac{1}{(1-q)^2} \int_0^\infty e_q^{-\tau(\omega\log a + \Omega)} d_q\tau \int_0^\infty e_q^{-\kappa(\Lambda\log a + \Psi)} d_q\kappa \\
&= \frac{1}{(1-q)^2 (\omega\log a + \Omega)(\Lambda\log a + \Psi)}.
\end{aligned}$$

3.5. q-double modified Laplace transform holds for the following:

(1) If $\mathfrak{h}(\tau, \kappa) = 1$, then

$$\begin{aligned}
L_q\{1\} &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau + \Lambda\kappa)\log a} d_q\tau d_q\kappa \\
&= \frac{1}{(1-q)} \int_0^\infty e_q^{-(\omega\tau)\log a} d_q\tau \int_0^\infty \frac{1}{(1-q)} e_q^{-(\Lambda\kappa)\log a} d_q\kappa \\
&= \frac{1}{(1-q)^2 \omega\log a \Lambda\log a}.
\end{aligned}$$

(2) If $\mathfrak{h}(\tau, \kappa) = e_q^{(\Omega\tau + \Psi\kappa)}$, then

$$\begin{aligned}
L_q\{e_q^{(\Omega\tau + \Psi\kappa)}\} &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{(\Omega\tau + \Psi\kappa)} e_q^{-(\omega\tau + \Lambda\kappa)\log a} d_q\tau d_q\kappa \\
&= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{(\Omega\tau + \Psi\kappa)} e_q^{-(\omega\tau + \Lambda\kappa)\log a} d_q\tau d_q\kappa \\
&= \frac{1}{(1-q)} \int_0^\infty e_q^{-(\omega\log a - \Omega)\tau} d_q\tau \int_0^\infty e_q^{-(\Lambda\log a - \Psi)\kappa} d_q\kappa \\
&= \frac{1}{(1-q)^2 (\omega\log a - \Omega)(\Lambda\log a - \Psi)}.
\end{aligned}$$

(3) If $\mathfrak{h}(\tau, \kappa) = e_q^{i(\Omega\tau + \Psi\kappa)}$, then

$$\begin{aligned}
L_q\{e_q^{i(\Omega\tau + \Psi\kappa)}\} &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau + \Lambda\kappa)\log a} e_q^{i(\Omega\tau + \Psi\kappa)} d_q\tau d_q\kappa \\
&= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-\tau(\omega\log a - i\Omega)} e_q^{-\kappa(\Lambda\log a - i\Psi)} d_q\tau d_q\kappa \\
&= \frac{1}{(1-q)^2 (\omega\log a - i\Omega)(\Lambda\log a - i\Psi)}.
\end{aligned}$$

$$\begin{aligned}
 &= \frac{(\omega \log a + i\Omega)(\Lambda \log a + i\Psi)}{(1-q)^2(\omega^2(\log^2 a + \Omega^2)(\Lambda^2 \log^2 a + \Psi^2)} \\
 &= \frac{\omega \Lambda \log^2 a + i\omega \Psi \log a + i\omega \Lambda \log a - \Omega \Psi}{(1-q)^2(\omega^2(\log^2 a + \Omega^2)(\Lambda^2 \log^2 a + \Psi^2)} \\
 &= \frac{(\omega \Lambda \log^2 a - \Omega \Psi) + i(\omega \Psi \log a + \omega \Lambda \log a)}{(1-q)^2(\omega^2(\log^2 a + \Omega^2)(\Lambda^2 \log^2 a + \Psi^2)}
 \end{aligned}$$

Consequently, $L_q\{\cos(\Omega\tau + \Psi\kappa)\} = \frac{(\omega \Lambda \log^2 a - \Omega \Psi)}{(1-q)^2(\omega^2(\log^2 a + \Omega^2)(\Lambda^2 \log^2 a + \Psi^2)}$

$L_q\{\sin(\Omega\tau + \Psi\kappa)\} = \frac{(\omega \Psi \log a + \omega \Lambda \log a)}{(1-q)^2(\omega^2(\log^2 a + \Omega^2)(\Lambda^2 \log^2 a + \Psi^2)}$.

(4) If $\hbar(\tau, \kappa) = \cosh(\Omega\tau + \Psi\kappa)$, where Ω, Ψ are any constant

then $L_q\{\cosh(\Omega\tau + \Psi\kappa)\} = \frac{1}{2} [L_q\{e^{\Omega\tau + \Psi\kappa}\} + L_q\{e^{-\Omega\tau - \Psi\kappa}\}]$

$$\begin{aligned}
 &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty \frac{1}{2} \left[e_q^{\Omega\tau + \Psi\kappa} e_q^{-(\omega\tau + \Lambda\kappa)\log a} + e_q^{-\Omega\tau - \Psi\kappa} e_q^{-(\omega\tau + \Lambda\kappa)\log a} \right] \\
 &= \frac{1}{2(1-q)^2} \left[\frac{1}{(\Omega - \omega \log a)\tau} \cdot \frac{1}{(\Psi - \Lambda \log a)\kappa} + \frac{1}{(\Omega + \omega \log a)\tau} \cdot \frac{1}{(\Psi + \Lambda \log a)\kappa} \right].
 \end{aligned}$$

Similarly, we can obtain $L_q\{\sinh(\Omega\tau + \Psi\kappa)\} = \frac{1}{2(1-q)^2} \left[\frac{1}{(\Omega - \omega \log a)\tau} \cdot \frac{1}{(\Psi - \Lambda \log a)\kappa} - \frac{1}{(\Omega + \omega \log a)\tau} \cdot \frac{1}{(\Psi + \Lambda \log a)\kappa} \right]$.

Theorem 3.6 If $\hbar(\tau, \kappa) = g(\tau)h(\kappa)$, then

$$L_q\{\hbar(\tau, \kappa)\} = L_q[g(\tau)] L_q[h(\kappa)].$$

Proof. $L_q[\hbar(\tau, \kappa)] = \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau + \Lambda\kappa)\log a} \hbar(\tau, \kappa) d_q\tau d_q\kappa$

$$\begin{aligned}
 &= \left[\frac{1}{(1-q)} \int_0^\infty e_q^{-(\omega\tau)\log a} g(\tau) d_q\tau \right] \\
 &\quad \left[\frac{1}{(1-q)\Lambda} \int_0^\infty e_q^{-(\Lambda\kappa)\log a} h(\kappa) d_q\kappa \right] \\
 &= L_q[g(\tau)] L_q[h(\kappa)].
 \end{aligned}$$

3.7 For a real number K ,

$$\begin{aligned}
 L_q[K\hbar(\tau, \kappa)](\omega, \Lambda) &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty K e_q^{-(\omega\tau + \Lambda\kappa)\log a} \hbar(\tau, \kappa) d_q\tau d_q\kappa \\
 &= K L_q[\hbar(\tau, \kappa)](\omega, \Lambda).
 \end{aligned}$$

Theorem 3.8. If $\hbar(\tau, \kappa)$ is periodic function periods h and k , exists, then

$$L_q\{\hbar(\tau, \kappa)\} = \frac{[1 - e_q^{-(\omega h - \Lambda k)\log a}]^{-1}}{(1-q)^2} \int_0^h \int_0^k e_q^{-(\omega\tau + \Lambda\kappa)\log a} \hbar(\tau, \kappa) d_q\tau d_q\kappa.$$

Proof.

$$\begin{aligned} L_q\{\mathfrak{h}(\tau, \kappa)\} &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q\tau d_q\kappa \\ &= \frac{1}{(1-q)^2} \int_0^h \int_0^k e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q\tau d_q\kappa \\ &\quad + \frac{1}{(1-q)^2} \int_h^\infty \int_k^\infty e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q\tau d_q\kappa \end{aligned}$$

put $\tau = n + h$, $\kappa = m + k$

$$\begin{aligned} &= \frac{1}{(1-q)^2} \int_0^h \int_0^k e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q\tau d_q\kappa + \\ &\quad \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-\omega(n+h)\log a - \Lambda(m+k)\log a} \\ &\quad \mathfrak{h}(n+h, m+k) d_qn d_qm \\ &= \frac{1}{(1-q)^2} \int_0^h \int_0^k e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q\tau d_q\kappa \\ &\quad + \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(un)\log a - (vm)\log a} \mathfrak{h}(m, n) d_qm d_qn \\ &\Rightarrow [1 - e_q^{-(\omega h + \Lambda k)\log a}]_q L_a[\mathfrak{h}(\tau, \kappa)] = \frac{1}{(1-q)^2} \int_0^h \int_0^k e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q\tau d_q\kappa \\ &\Rightarrow L_q\{\mathfrak{h}(\tau, \kappa)\} = \frac{[1 - e_q^{-(\omega h + \Lambda k)\log a}]^{-1}}{(1-q)^2} \int_0^h \int_0^k e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau, \kappa) d_q\tau d_q\kappa. \end{aligned}$$

Theorem 3.9. If the q -double modified Laplace transform of a function $\mathfrak{h}(\tau, \kappa)$ exists, then

$$L_q\{\mathfrak{h}(\tau - M, \kappa - N)H((\tau - M, \kappa - N))\} = e_q^{-\omega A - \Lambda B}\{\mathfrak{h}(\tau - M, \kappa - N)\}.$$

Where $\mathfrak{h}(\tau, \kappa)$ is a Heaviside unit step function defined by the

$$H(\tau - M, \kappa - N) = 1, \text{ where } \tau > M \text{ and } \kappa > N,$$

$$H(\tau - M, \kappa - N) = 0 \text{ where } \tau < M \text{ and } \kappa < N.$$

Proof. We have by definition,

$$\begin{aligned} &L_q\{\mathfrak{h}(\tau - M, \kappa - N)H((\tau - M, \kappa - N))\} \\ &= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau - M, \kappa - N) \\ &\quad H(\tau - M, \kappa - N) d_q\tau d_q\kappa \\ &= \frac{1}{(1-q)^2} \int_0^\eta \int_0^\kappa e_q^{-(\omega\tau+\Lambda\kappa)\log a} \mathfrak{h}(\tau - M, \kappa - N) d_q\tau d_q\kappa \\ &\quad \text{let } \tau - M = A, \kappa - N = B \\ &L_q\{\mathfrak{h}(\tau - M, \kappa - N)\} = \frac{1}{(1-q)^2 \omega \Lambda} \int_0^\infty \int_0^\infty e_q^{-\omega[(A+M)\log a] + \Lambda[(N+B)]\log a} \\ &\quad \mathfrak{h}(A, B) d_qA d_qB \end{aligned}$$

$$\begin{aligned}
&= \frac{e_q^{-(\omega M + \wedge N) \log a}}{(1-q)^2} \int_0^\infty \int_0^\infty e_q^{-(\omega A + \wedge B) \log a} \mathfrak{h}(A, B) d_q A d_q B \\
&= e_q^{-(\omega A + \wedge B) \log a} L_q\{\mathfrak{h}(A, B)\}.
\end{aligned}$$

3.10. q-double modified Laplace transform convolution product:

Definition: The convolution of two functions $\mathfrak{h}(\tau, \kappa)$ and $g(\tau, \kappa)$ is defined by $(\mathfrak{h} * * g)$ and defined as

$$(\mathfrak{h} * * g)(\tau, \kappa) = \frac{1}{(1-q)^2} \int_0^\tau \int_0^\kappa \mathfrak{h}(\alpha, \beta) g(\tau - \alpha, \kappa - \beta) d_q \alpha d_q \beta.$$

Convolution theorem: Let $\mathfrak{h}(\tau, \kappa)$ and $g(\tau, \kappa)$ have q-double modified Laplace transforms, thus

$$L_q\{\mathfrak{h} * * g\}(\tau, \kappa) = L_q\{\mathfrak{h}(\tau, \kappa)\} L_q\{g(\tau, \kappa)\}.$$

Proof.

$$\begin{aligned}
&L_q\{(\mathfrak{h} * * g)(\tau, \kappa)\} \\
&= \frac{1}{(1-q)^4} \int_0^\infty \int_0^\infty e_q^{-(\omega \tau + \wedge \kappa) \log a} (\mathfrak{h} * * g)(\tau, \kappa) d_q \tau d_q \kappa \\
&= \frac{1}{(1-q)^4} \int_0^\infty \int_0^\infty e_q^{-(\omega \tau + \wedge \kappa) \log a} \left(\int_0^\tau \int_0^\kappa \mathfrak{h}(\alpha, \beta) g(\tau - \alpha, \kappa - \beta) d_q \alpha d_q \beta \right) d_q \tau d_q \kappa
\end{aligned}$$

Let $\tau - \alpha = A$, $\kappa - \beta = M$ and extend upper bound of integrals to $\tau \rightarrow \infty$, $\kappa \rightarrow \infty$, it implies that

$$\begin{aligned}
L_q\{(\mathfrak{h} * * g)(\tau, \kappa)\} &= \frac{1}{(1-q)^4} \int_0^\infty \int_0^\infty e_q^{-(\omega \alpha + \wedge \beta) \log a} d_q \alpha d_q \beta \\
&\quad \int_{-A}^\infty \int_{-M}^\infty e_q^{-(\omega A + \wedge M) \log a} g(A, M) d_q A d_q M
\end{aligned}$$

the functions $\mathfrak{h}(\tau, \kappa)$, $g(\tau, \kappa)$ have zero values for $\tau < 0$ and $\kappa < 0$, therefore

$$\begin{aligned}
L_q\{(\mathfrak{h} * * g)(\tau, \kappa)\} &= \frac{1}{(1-q)^4} \int_0^\infty \int_0^\infty e_q^{-(\omega \alpha + \wedge \beta) \log a} d_q \alpha d_q \beta \\
&\quad \int_0^\infty \int_0^\infty e_q^{-(\omega A + \wedge M) \log a} g(A, M) d_q A d_q M,
\end{aligned}$$

then

$$L_q\{\mathfrak{h} * * g\}(\tau, \kappa) = L_q\{\mathfrak{h}(\tau, \kappa)\} L_q\{g(\tau, \kappa)\}.$$

4. Discussion and Conclusion

Most significant work has been performed in conventional case and q-calculus on the Laplace transform and its extended versions. Solutions to initial value problems are obtained by utilizing the Laplace transform and its several generalized versions [6, 36, 44]. For instance, q-Laplace transformation are defined by Bohner and Guseinov [10]. The significant difference between regular and q-integral is found at the time of calculation. We apply the definition of q-integral in calculus, especially in quantum calculus; the concept is used to investigate calculus without limits in a general sense. The q-image of the Laplace transform is employed to solve certain q-difference, q-integral, and q-differential equations. The

q-calculus appeared as a connection between mathematics and physics. It has many applications in different mathematical areas, such as number theory, combinatorics, orthogonal polynomials, essential hypergeometric functions, and other sciences, such as quantum mechanics and relativity. The applications of double Laplace transform to solve some q-functional equations and partial q-differential equations. Q-calculus has been given to solve complex and more potential, typical problems in a larger domain to investigate calculus without limits for getting more generalization. Now a days many literatures [43, 44] have shown the importance of quantum calculus and its developments in geometric function theory. Quantum calculus is the latest development in the research concerning complex-valued functions from the perspective of geometric function theory. Contributions were sought regarding any aspect of subordination and super ordination, different types of operators specific to the research in this field, and special functions involved in univalent function theory. Quantum calculus plays a vital role in the field of classical mathematical analysis. It focuses on helpful generalization of the integral and differential operators. It is significant to note that quantum calculus has drawn the attention of modern mathematicians in the last several decades due to its broad application areas. Recently, there has been a fast growth of activities in the area of q-calculus and its applications in various fields such as mathematics, mechanics, and physics. More recently, in the geometric theory of analytic and harmonic univalent functions, the application of the q-calculus has been of utmost importance to research. As a matter of fact, basic (or q-) hyper-geometric functions are used in many diverse fields, including combinatorial analysis, finite vector spaces, Lie theory, particle physics, nonlinear electric circuit theory, mechanical engineering, theory of heat conduction, quantum mechanics, cosmology and statistics.

In the study, we introduced the q-double modified Laplace transform and established some properties. Furthermore, several propositions concerned with q-double modified Laplace transform properties are explored. The work q-double modified Laplace transform helps solve complex problems effectively. Quantum calculus and its applications related to geometric function theory will also expected to provide interesting outcomes. The results obtained by using this technique can be applied in complex analysis, and its applications will also be encouraged in different fields of science and engineering.

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References

- [1] Aboodh, K.S., Farah, R.A., Almardy, I.A., Mostafa, F.A. (2017). Solution of Telegraph by using double Aboodh transform, *Elixir. Appl. Math.*, 110, 48313-48317. [1](#)
- [2] Agarwal, P., Jain, S. Choi, J. (2017). Certain q-series identities. *RACSAM* 111, 139-146. [1](#)
- [3] Agarwal, P., Dragomir, S.S., Park, J. et al.(2015). q-Integral inequalities associated with some fractional q-integral operators. *J Inequal Appl* 2015, 345 . [1](#)
- [4] Andrews, L.C., Phillips, R.L., (2003). *Mathematical Techniques for Engineers and Scientists*. SPIE Publications. [1](#)
- [5] Alidema, A.F., Makolli, S.V., (2022). On the q-Sumudu transform with two variables and some properties, *J. Math. Computer Sci.*, 25, 167-175. [2](#), [2](#)
- [6] Alp, N., Sarikaya, M.Z. (2023). q-Laplace transform on quantum integral, *Kra. J. Math.*, 47(1), 153164. [4](#)
- [7] Ali, Khalid., Mohamed, M., Alharbi, W.G., Maneea, M. (2024). Innovative analysis to the time-fractional q-deformed tanh-Gordon equation via modified double Laplace transform method. *Open Physics* 22, 20240094. [1](#)
- [8] Berstein, D.L. (1939). *The double Laplace integral*. Dissertation, Brown University. [1](#)
- [9] Brahim, K., Riahi, L, (2018) Two Dimensional Mellin Transform in Quantum Calculus, *Acta. Math. Sci.*, 32B(2), 546-560. [2](#), [2](#)
- [10] Bohner, M., Guseinov, G. (2010). The h-Laplace and q- Laplace transforms. *J. Math. Anal. Appl.*, 365, 75-92. [4](#)
- [11] Choi, J., Agarwal, P. and Jain, S. (2015). Certain fractional integral operators and extended generalized Gauss hypergeometric functions, *Kyungpook Mathematical Journal*, 55(3), 695–703. [1](#)
- [12] Chung, W.S., Kim, T., Kwon, H.I. (2014). On the q-Analogue of the Laplace Transform, *Russ. J. Math. Phys.*, 21, 156-168. [1](#), [2](#), [2](#)
- [13] Deakin, M.A.B. (1981). The Development of the Laplace Transform. *Arch. Hist. Exact. Sci.*, 25(4), 343-390. [1](#)
- [14] Debnath, L. (2016). The double Laplace transforms and their properties with applications to functional, integral and partial differential equations. *Int. J. Appl. Comput. Math.*, 2, 223-241. [1](#)

- [15] Debnath, L., Bhatta, D. (2015). *Integral Transform and their Applications*, CRC Press, London. 1
- [16] Donachali, A.K., Jafari, H. (2020). A decomposition method for solving quaternion differential equations, *Int. J. Appl. Comp. Math.*, 107, 1-7. 1
- [17] Ata, E., Kymaz, O., Agarwal, P., Jain, S.(2024). Chapter 7 - A study on the properties of new generalized special functions, their integral transformations, and applications to fractional differential equations, In *Advanced Studies in Complex Systems, Fractional Differential Equations*, Academic Press, 95-114. 1
- [18] Ganie, J.A., Jain, R. (2020). On a System of q-Laplace transform of two variables with applications, *J. Comp. Appl. Math.*, 366, 112407. 2, 2
- [19] Huang, C.X., Yang, Z.C., Yi, T.S., Zou, X.F. (2014). On the Basins of Attraction for a Class of Delay Differential Equations with Non-monotone Bistable Nonlinearities, *J. Diff. Equ.*, 256(7), 2101-2114. 1
- [20] Jafari, H., Manjarekar, S. (2024). A new concept of q-calculus with respect to another function. *J. Innov. Appl. Math. Comput. Sci.*, 4(2), 113-121. 2, 2
- [21] Jafari, H., Haghbin, A., Johnston, S.J., Baleanu, D. (2017). A new algorithm for solving dynamic equations on a time scale. *J. Compu. Appl. Math.*, 312, 167-173. 1
- [22] Jackson, F.H. (1910). On a q-definite integral, *Quart. J. Appl. Math.*, 41, 193-203. 1, 2
- [23] Jain, S., Agarwal, R.P., Agarwal, P. and Singh, P.(2021). Certain Unified Integrals Involving a Multivariate Mittag-Leffler Function, *Axioms*, 10(2) 81. 1
- [24] Jain, S., Agarwal, P., Ahmad, B. and Al-Omari, S.K.Q. (2016). Certain recent fractional integral inequalities associated with the hypergeometric operators, *Journal of King Saud University - Science*, 28(1) 82–86, 1
- [25] Jain, S. and Agarwal, P. (2015). A new class of integral relations involving a general class of polynomials and I-functions, *Walailak Journal of Science and Technology*, 12(11) 1009–1018. 1
- [26] Jain, S., Goyal, R., Agarwal, P., Momani, S. (2023). Certain saigo type fractional integral inequalities and their q-analogues. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 13(1), 19. 1
- [27] Jirakulchaiwong, S., Nonlaopon, K., Tariboon, J., Ntouyas, S.K., Kim, H. (2021). On (p,q)-Analogues of Laplace-Typed Integral Transforms and Applications, *Symmetry*, 13, 1-21. 1, 2
- [28] Kac, V.G., Cheung, P. (2002). *Quantum Calculus*. Springer, New-York. 2, 2
- [29] Kilicman, A., Omran, M. (2017). On double natural transform and its applications. *J. Nonlinear Sci. Appl.* 10, 1744-1754. 1
- [30] Kilicman, A., Sinha, A., Panda, S. (2022). On a system of q-modified Laplace transform and its applications. *M.M.A.*, 45(2), 793-808. 2, 2
- [31] Mansour, E.A., Mehdi, S., Kuffi, E.A. (2021). The new integral transform and its applications, *Int. J. Nonlinear Anal. Appl.*, 12(2), 849-856. 1
- [32] Mumtaz, R., Khan, S. (2019). A determinant to q-Bessel polynomials and applications, *RACSAM* 113, 1571-1583. 1, 2
- [33] Mangontarum, M. (2014). On a q-Analogue of the Elzaki Transform Called Mangontarum q-Transform. *Discrete dynamics in nature and society 2014*, Article ID 825618. 1
- [34] Piejko, K., Sokol, J., Wieclaw, K.T. (2019). On q-Calculus and starlike functions. *Iran. Sci. Technol. Trans. Sci.* 43, 2879-2883. 1, 2, 2
- [35] Rainville, E.D. (1960). *Special Functions*. Macmillan, New-york. 1, 2
- [36] Sadjang, P.N. (2019). On double q-Laplace transform and applications. *arXiv:1905.00717*. 1, 2, 4
- [37] Saadeh, R., Qazza, A., Burqan, A. (2020). A New Integral Transform: ARA Transform and Its Properties and Applications, *Symmetry*, 2020, 1-19 (2020). 1
- [38] Sedeeg, AKH. (2016). The New Integral Transform Kamal Transform. *Adv. Theo. Appl. Math.* 11(4), 451-458. 1
- [39] A Sole, A.D., Kac, V. (2005). On Integral Representation of q-Gamma and q-Beta functions", *Rend. Math. Linecei.*, 9, 11-29. 2
- [40] Srivastava, H.M., Luo, M., Raina, R.K. (2015). A new integral transform and its applications, *Acta Mathe. Scie.*, 35(6), 1386-1400. 1, 2
- [41] Tan, Y.X., Huang, C.X., Sun, B., Wang, T. (2018). Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition, *J. Math. Anal. Appl.*, 458(2), 1115-1130. 1, 2
- [42] Uçar, F. (2014). q- Sumudu transforms of q-analogues of Bessel functions. *The Sci. World J.* 2014, Article ID 327019, 1-7. 2
- [43] Vijaya, K., Bulboaca, T., Murugusundaramoorthy, G. (2023). Quantum Calculus and its developments in geometric function theory. 4
- [44] Yilmazer, C.Y., Yilmaz, E., Goktas, S., ET, M. (2023). Multiplicative Laplace transform in q-calculus. *Filomat.* 37, 5859-5872. 4