



Presenting a mathematical approach to developing the internal rate of Return method: An application of Horner's theorem in econometrics

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Abstract

Economics is the development of a collection of Mathematical Methods (MMs) for the economic evaluation of investment projects or plans. Two main methods for this type of valuation are the internal rate of return method and the present value method. For various reasons, directors and investors are more likely to use the Internal Rate of Return (IRR) method, even though the present value method produces trustworthy results. This is the case if using the internal rate of return method presents significant problems. In recent years, numerous articles have been published to address the problems with this method. It is safe to say that much of this work is based on Hazen's method. The current study aims to illustrate Hazen's strategy using Horner's method in a simple way that ultimately leads to fewer calculations. If a financial project presents a negative rate of return or multiple rates of return (whether negative or positive), it can confuse decision-makers and hinder their ability to make effective choices. This study introduces a new approach that, rather than focusing solely on a single rate of return, considers two different indicators: a rate of return and virtual cash flow. This approach aids in making sound decisions. The approach proposed in this study comprises several math theorems, which provide a simpler and more understandable process than the Hazen Approach (HA). Also, a Decision Tree (DT) has been presented in the proposed approach that can be used to determine whether the financial project under evaluation is profitable or unprofitable.

Keywords: Engineering economics, Horner's theorem, NPV, IRR, Project analysis

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1. Introduction

Capital is indeed a word defined as buying or holding any cash or non-cash asset that has financial value [48, 49]. The available cash in bank accounts, factories, machinery, buildings, housing, land, etc., for instance [35]. Investment literary means that we buy all assets with financial value by accepting certain risks and expect them to generate revenues for us in the future [59, 46].

In terms of accounting, these two words can be defined as follows: all monetary resources and assets that can be changed to money are called capital, and investment is defined as the use of capital in a way

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that enhances capital [43]. Economics is defined as a social science that addresses the production, distribution, and consumption of goods and services and determines how individuals, occupations, governments, and nations choose to allocate resources [30].

Economy focuses on the actions of people, assuming that humans perform based on rational behavior and look for the most optimum level of profit or utility [57]. Labor and trade studies are the constituent units of the economy, and since there are numerous applications of human labor and various ways to earn resources, the economics' task is to determine which one of the methods provides better results [6].

In general, the economy can be divided into a macroeconomy that concentrates on the economic behavior as a whole and a microeconomy that focuses on the persons and individual jobs [4].

Econometrics is a branch of economics that measures and analyzes the relationships between economic variables. Econometrics deals with the systematic study of economic phenomena by using the observed data [63]. In other words, econometrics is the science of statistical analysis of economic models. As Ragnar Frisch explains in the first issue of *Econometrica* Journal, there is economic theory and math that create econometrics [17]. Although many econometric methods express the application of statistical modes, some specific characteristics of economic data distinguish econometrics from other branches of statistics [2].

Economic data are many based on the observation, and not retrieved from control experiments [36]. Since economic units act within the interaction with each other, the observed data imply a complicated economic equilibrium, not resulting from a simple relational behavior of priority or technology. Thus, econometrics would create some methods to detect and estimate models having several unknowns. These methods allow the researcher to present a casual inference under the circumstances difference from the controlled trial situation [32]. Now, students and many employees such as managers financial market traders, and researchers highly use econometrics [52]. Managers and practitioners often use the Internal Rate of Return (IRR) when making investment decisions [18]. Unfortunately, it has significant shortcomings, including the following:

1. It can lead to the emergence of multiple real-valued IRRs.
2. Complex-valued IRRs.
3. The Internal Rate of Return is generally inconsistent with the Net Present Value (NPV) when making acceptance or rejection decisions.
4. The Internal Rate of Return ranking is generally different from the net present value ranking.
5. The Internal Rate of Return criterion does not apply to variable capital costs.
6. It does not measure the return on the initial investment.
7. It does not mean the loss of all capital.
8. It is unable to measure the return of an arbitrage strategy [9, 15, 39, 44, 66].

Over the years, a tremendous amount of work has been done to address these issues due to the efforts of management scientists and economists to provide trustworthy project returns [40].

The fact that financial practice has lagged behind theory is a factor contributing to the ongoing debate. Although banks and large corporations use both metrics, they often prefer the Internal Rate of Return, perhaps due to its easy comparability to the cost of capital [47].

2. Literature Review

Chisini (1929) proposed a method called Chisini Mean (CM) to eliminate the multi-rate nature of projects. Consistency with the present value method was among the features of this rate of return (CM) [14].

Some authors, such as Soper (1959) and Gronchi (1986) examined the Internal Rate of Return within the range of $[+1, +\infty)$. These researchers believe that those rates out of this range are not significant, so cannot be used for the assessment of projects [20, 62].

According to Norstrom (1972), multiple or nonexistent internal interest rates are not incompatible, meaningless, or unreliable as rates of return [45].

Asquith and Bethel (1995) used heuristic techniques to measure the Internal Rate of Return (IRR) [7].

Payne et al. (1999) and Kester et al. are two surveys from the last ten years. Who discuss practices that vary from country to country; Sometimes the Net Present Value is preferred, and sometimes the Internal Rate of Return (IRR). While their study of Fortune 100 companies shows that NPV is slightly preferred over IRR [33, 50].

Sanchez et al. (1999) presented a technique for the measurement of the rate of return of capital under indeterministic (fuzzy) conditions regarding the point that cash flows are not always deterministic [58].

According to surveys by Graham and Harvey (2001) Brounen et al. (2004) and Liljeblom and Vahkoski (2004), Net Present Value is used more frequently than the Internal Rate of Return [11, 19, 37].

Ryan and Ryan (2002) report studies conducted between 1960 and 1996 that show that the Internal Rate of Return dominated corporate practice during this period [54].

Brounen et al. (2004) apply Graham and Harvey's methodology to European companies and show that the Internal Rate of Return is more commonly used in the UK and France, while Net Present Value is more commonly used in Germany and the Netherlands. A project with one or more interest rates can be recognized in the interval $(-1, +\infty)$ if its cash flow has a single condition regarding a brand change [12].

Vukeviand Vujadinovi (2008) introduced equilibrium techniques, and also the Internal Rate of Return (IRR) method [65].

ILLÉS (2012) considered some computational errors for the present value method and introduced a more accurate equation for measuring the present value of a cash flow with a known cost of capital [26].

Pressacco et al. (2014) presented a new method in a paper for the measurement of capital Internal Rate of Return. They have named this rate as the Pseudo Internal Rate of Return [51].

Desai (2015) suggests the polynomial decomposition method for the analysis of projects by examining some points about the Internal Rate of Return method [16].

According to Russell and Rickard, the multi-rate of projects is detectable regarding their cash flow, and economic projects must follow some patterns in terms of transactions (flows) to have a unique rate, which such case is not possible in general mode [53].

Regardless of what interest rate is applied, we mean that the decision to accept or reject the cash flow is the same and consistent with the net present value [24]. The studies are summarized in Table 1 below.

It should be noted that the work presented in the works of Magni (2010) and Moradi et al. (2012) and Hazen (2009) and Liu, Zhou, and Wang (2014) are based on Hazen's methodology [38, 40, 43]. To achieve the main goal of this work, namely simplifying the calculations through a schematic model, we first discuss the internal return on capital method and Hazen's technique before illustrating this approach using Horner's Method (HM).e. the Horner Table (HT) itself.

3. Necessity of Research

Evaluating investments without appropriate financial criteria can be challenging, as it risks capital loss and potential bankruptcy. In other words, if an organization selects an unprofitable financial project for investment, its survival may be at risk [21]. Investors and company managers typically use two criteria to estimate whether an investment in a project is cost-effective: Internal Rate of Return (IRR) and Net Present Value (NPV). These criteria measure the expected returns on investments, enabling financial managers to make more informed decisions [41]. The present research focuses on defining and developing the IRR criterion and emphasizes its importance in finance. Additionally, it describes the calculation of this criterion and how it differs from other methods. The findings of this research aim not only to contribute to theoretical knowledge but also to assist financial managers across organizations in evaluating financial projects with greater confidence and accuracy.

Table 1: Details of conducted studies

No.	Author(s)	Year	Article Topic	Key Findings
1	Chisini	1929	Multiple IRRs in projects	The Chisini mean can partially address the issue of multiple internal rates of return (IRRs).
2	Soper	1959	Evaluation of rates of return	Rates falling within the interval $(-, +1)$ are considered meaningless.
3	Gronchi	1986	Evaluation of rates of return	Rates falling within the interval $(-, +1)$ are considered meaningless.
4	Norstrom	1972	Evaluation of positive rates of return	Only rates within the interval $(0, +)$ can be evaluated.
5	Russell and Rickard	1982	Multiple IRRs in Projects	Multiple IRRs in a project can be detected based on its cash flow patterns.
6	Asquith and Bethel	1995	IRR calculation	Proposed a heuristic approach that is relatively successful in calculating IRRs.
7	Payne et al.	1999	Inconsistencies between NPV and IRR	In some cases, the results of the NPV and IRR methods may conflict.
8	Kester et al.	1999	Inconsistencies between NPV and IRR	In some cases, the results of the NPV and IRR methods may conflict.
9	Sanchez et al.	1999	Uncertain cash flows	Proposed a novel method for calculating IRR under fuzzy conditions.
10	Graham and Harvey	2001	Extent of IRR usage	During a specified time period, IRR was more commonly applied in case studies.
11	Ryan and Ryan	2002	Comparison of IRR and NPV usage	Economists generally prefer using the NPV method over the IRR method.
12	Brounen et al.	2004	Comparison of IRR and NPV usage	Economists generally prefer using the NPV method over the IRR method.
13	Liljeblom and Vaihekoski	2004	Comparison of IRR and NPV Usage	Economists generally prefer using the NPV method over the IRR method.
14	Brounen et al.	2004	Use of IRR and NPV in different countries	IRR is more frequently used in the UK and France, whereas NPV is more common in Germany and the Netherlands.
15	Vukevi and Vujadinovi	2008	Examination of the IRR method	The IRR method is explained in simple terms using several illustrative examples.
16	Illés	2012	Computational limitations of the NPV method	Introduced a more precise mathematical formula for calculating the present value of a cash flow given a known cost of capital.
17	Pressacco et al.	2014	IRR calculation	Introduced a new method for computing the internal rate of return.
18	Desai	2015	Insights into the IRR method	Employed polynomial decomposition for the economic analysis of financial projects.

4. Research Novelty

The Internal Rate of Return (IRR) is a financial metric used to assess the performance and potential rate of return of investments. It estimates the profitability of potential investments and projects and indicates the annual rate of return expected from an investment. In essence, the internal rate of return is the percentage of return that a project generates over a specified period to recover its initial capital, achieved when the Net Present Value (NPV) of the project equals zero [61]. However, if a financial project

presents a negative rate of return or multiple rates of return (whether negative or positive), it can confuse decision-makers and hinder their ability to make effective choices. This study introduces a new approach that, rather than focusing solely on a single rate of return, considers two different indicators: a rate of return and virtual cash flow. This approach aids in making sound decisions. To further clarify this method, a Decision Tree (DT) is presented, which helps organizations determine whether to accept or reject an investment in a financial project. This proposed approach is compatible with the present value method, thus alleviating potential confusion for decision-makers. The following explains and proves the various sections of this approach.

5. Research Methodology

Mathematical modeling is one of the most widely used research methods [42]. It employs variables and their relationships instead of relying on judgmental descriptions and personal opinions. High accuracy in describing and analyzing real-world problems sets this method apart [5]. More generally, mathematics is the discipline that uncovers and explains the order hidden within seemingly complex situations. This field provides tools based on principles that enable us to illustrate this order [17]. Since the present research is no exception to this rule, it utilizes mathematical modeling to address the challenges associated with financial projects that exhibit multiple rates of return. The proposed approach is validated using algebraic methods in mathematics to prove the necessary theorems. Also, MATLAB software has been used to provide accurate results in the analysis of numerical examples.

MATLAB have high accuracy and speed in data processing and allow for visual interpretation of the model results [3, 34]. In addition to its high computational accuracy, this software provides high-quality and comprehensible graphical outputs [29, 64].

6. Internal Rate of Return Method

In the long run, these changes are more noticeable, but the Time Value of Money (TVM) is not always constant and fluctuates over time [1, 8, 55, 56, 60]. A finite or infinite sequence of monetary values, $X = (x_0, x_1, \dots)$, is called a cash flow stream. After period t , the amount received is x_t , while the amount originally received is x_0 . For a finite stream $X = (x_0, x_1, \dots, x_T)$, we assume the horizon T is chosen so that $x_T \neq 0$ [31].

The net present value $PV(X|r)$ of a cash flow stream X at interest rate r is given by [22]:

$$PV(X|r) = \sum_{t=0}^T \frac{x_t}{(1+r)^t} \quad (6.1)$$

defined for proper interest rates $-1 < r$. Assume that IRR_X is the sum of all interest rates r for a cash flow stream X such that $PV(X|r) = 0$. (Note that IRR_X cannot contain -1 because $PV(X|r = -1)$ is undefined.) For finite streams $X = (x_0, x_1, \dots, x_T)$, the present value function $PV(X|r)$ is a degree- T polynomial in $(1+r)^{-1}$, so IRR_X can contain anywhere from 0 to T distinct values. We will call r the internal rate of return for X if it is $r \in IRR_X$ [27].

As is well known, the internal rate of return exists and varies for traditional cash flows X , which are negative in the first few periods but positive thereafter. Additionally, the highest interest rate at which cash flow has a discounted net profit is the internal rate of return. Therefore, the investment that generates cash flow X is worthwhile if IRR_X is greater than the available market interest rate (r) and $PV(X|r) > 0$. On the other hand, it is better to invest at the market interest rate r if the internal rate of return is lower than this interest rate [28]. This serves as the main argument for the use of the internal rate of return [30].

7. Hazen's Approach

It is still unknown which of the IRRs is the economically correct rate of return, although any IRR can be used to make decisions. According to Hazen, $C = (c_0, c_1, \dots, c_{T-1})$ is an investment stream that yields a cash flow stream $X = (x_0, x_1, \dots, x_T)$ at (constant) if the following equations hold per period) return k [23]:

$$\begin{cases} x_0 = -c_0, \\ x_t = (1+k)c_{t-1} - c_t \quad \text{for } t = 1, 2, 3, \dots, T-3, T-2, T-1, \\ x_T = (1+k)c_{T-1}. \end{cases} \quad (7.1)$$

To put it another way, X is the cash flow stream that results from an investor investing c_t in a project at the end of each period $t < T$ and receiving a return of $(1+k) \times c_t$ once period $t+1$ has passed. The quantity c_t is the increment of capital invested at time t , and $(1+k) \times c_{t-1}$ is the increment of capital recovered at time t . We also permit the values of c_t to be negative. In this case, $-c_t$ is the increment of capital borrowed at time t and $(1+k) \times c_{t-1}$ is the increment of capital repaid at time t [23].

Theorem 7.1. *If investment stream $C = (c_0, c_1, \dots, c_{T-1})$ yields $X = (x_0, x_1, \dots, x_T)$ at constant per-period rate of return k , then for $r \neq -1$ [23]:*

$$PV(X|r) = \frac{k-r}{1+r} PV(C|r). \quad (7.2)$$

Theorem 7.2. *For a cash flow stream X , the value k represents the internal rate of return if and only if there is an investment stream C that generates X at a constant rate of return k per period [23].*

Theorem 7.3. *Assume that, at a proper internal rate of return k , X represents the yield of a pure investment or pure borrowing stream C [23]. The only appropriate internal rate of return for X is then k , and:*

- a: *If X is a pure borrowing stream, then $PV(X|r) \geq 0$ if and only if $k \leq r$.*
- b: *If X is a pure investment stream, then $PV(X|r) \geq 0$ if and only if $k \geq r$.*

Theorem 7.4. *Assume that the cash flow stream X has an internal rate of return k , and let C be the investment stream that generates X at a constant rate of return k per period [23].*

- a) *If $PV(C|r) > 0$ (that is, C is a net investment) then $PV(X|r) \geq 0$ if and only if $k \geq r$.*
- b) *If $PV(C|r) = 0$, then $PV(X|r) = 0$.*
- c) *If $PV(C|r) < 0$ (that is, C is a net borrowing) then $PV(X|r) \geq 0$ if and only if $k \leq r$.*

8. Applying Hazen's method to the actual internal rate of return or return on assets rates to assess the economic quality of a project

Step1: Solve the IRR Equation and pick any one of the IRRs.

Step2: Determine the corresponding capital stream C^{k_i} and its present value $PV(C^{k_i}|r)$ to determine whether it is an investment or a loan.

Step3: If the project involves borrowing or investing, approve it only if the Internal Rate of Return (IRR) is higher than the market rate. For observation proof of the Theorems, You can see [23].

9. Introducing a new approach to using return rates

Before presenting the new methodology, it appears necessary to give a basic definition for the differentiation of plans (projects). Thus, after defining classification, we will examine Horner's theorem before applying it to demonstrate the novel strategy.

9.1. Classification of the economic plans and projects

Without accurate scientific definitions, human knowledge could not advance. It is clear that definitions based on superficial attitudes not only do not contribute to the advancement of knowledge but often also interfere with the encoding of concepts. Scholars use the term economic projects to analyze projects from an economic perspective in a variety of economic studies, including books and articles. This sentence indicates that the projects are being examined from an economic perspective. In addition, profitable projects are again referred to as “economic projects” after the assessments are completed. To clarify these ambiguities, we will classify the projects using various definitions. In general, plans and projects studied in economics can be divided into three categories, which are discussed below.

The Non-Economic Class: The C^- symbol is used to represent this class (or group) of projects or plans, all of which are loss-making.

The Economic Class: It refers to the class (or set) of projects (plans) in which each member is profitable; the C^+ symbol is used to represent this class.

The Indifference Class: The C^0 symbol is used to represent that class (or group) of projects or plans in which none of the members are either profitable or losing money.

It is enough to use accounting methods such as PV to determine which of these classes the projects or belong to. In simple terms, cash flow X could be expressed as an economic projects with interest rate r :

$$\begin{cases} C^- = \{X \mid PV(X|r) < 0\} \\ C^0 = \{X \mid PV(X|r) = 0\} \\ C^+ = \{X \mid PV(X|r) > 0\} \end{cases} \tag{9.1}$$

Given that Horner’s method of decomposing polynomials forms the basis of this new approach, Horner’s theorem will be explained first, followed by the new approach, i.e. h. We will talk about Hazen’s improved strategy.

9.2. Horner’s theorem

If P is a polynomial of degree T as the following [10, 13]:

$$P_T(x) = p_T x^T + p_{T-1} x^{T-1} + \dots + p_3 x^3 + p_2 x^2 + p_1 x + p_0 ; \quad p_T \neq 0 \tag{5}$$

and a is a given number (whether real or complex), then P could be decomposed as:

$$P_T(x) = (x - a) Q_{T-1}(x) + R \tag{6}$$

where $R = P_T(a)$ is the remainder and $Q_{T-1}(x)$ is a polynomial of $(T - 1)$ degree.

9.3. Horner’s algorithm

One can utilize Horner’s calculation to decide the coefficients of the Q polynomials in Equation (6). On the off chance that we expected conditions, deciding the coefficients of Q polynomials would be Equation (9.2) and made Table 2, a table with $(T + 2)$ columns and three columns [25].

Table 2: The matrix of variables related to Horner’s Algorithm [31].

	p_T	p_{T-1}	\dots	p_2	p_1	p_0
$x = a$	t_T	t_{T-1}	\dots	t_2	t_1	t_0
$q_T = 0$	q_{T-1}	q_{T-2}	\dots	q_1	q_0	$R = P_T(a)$

The following equations illustrate the trend of calculations:

$$\begin{cases} t_j = a \times q_j, \\ q_{j-1} = p_j + t_j, \end{cases} \quad j = T, T - 1, \dots, 2, 1 \tag{9.2}$$

Note that if a is the root of P , then the value of the remainder (R) will be equal to zero.

Example 9.1. Using $x = 2$, which is one root for P , determine the coefficients of the Q polynomial that appears in Equation (6) assuming $P_3(x)$.

Solution: Horner’s algorithm can be executed schematically and sequentially, as shown in Tables 3 to 7.

Table 3: Part of the calculations (Stage 1)

$p_3 = 1$	$p_2 = -10$	$p_1 = 31$	$p_0 = -30$
$x = 2$			
$q_3 = 0$			

Table 4: Part of the calculations (Stage 2)

	1	-10	31	-30
2	0			
0	1			

Table 5: Part of the calculations (Stage 3)

	1	-10	31	-30
2	0	2		
0	1	-8		

Table 6: Part of the calculations (Stage 4)

	1	-10	31	
2	0	2	-16	
0	1	-8	15	

Table 7: The result of calculations (Stage 5)

	1	-10	31	-30
2	0	2	-16	30
0	1	-8	15	0

According to Table 7, it may be easily stated that: $Q_2(x) = x^2 - 8x + 15$.

9.4. Theorem (Hazen’s Enhanced Approach)

There exists a cash flow such as C with a lifespan of $(T - 1)$ if X is a cash flow with lifespan n as $X = (x_0, x_1, \dots, x_T)$, k is a given internal rate of return of X , and r is assumed to be the interest rate (cost of capital) constant in various periods:

$$PV(X|r) = \frac{r - k}{1 + r} \times PV(C|r) \tag{9.3}$$

Proof. According to the definition of future value:

$$FV(X|r) = x_0(1 + r)^T + x_1(1 + r)^{T-1} + \dots + x_{T-2}(1 + r)^2 + x_{T-1}(1 + r) + x_T \tag{9.4}$$

By modifying the variables with $x = 1 + r$, $FV(X|r)$ is converted to a polynomial of degree T like P as the following, $p_j = x_{T-j}$; $j = 0, 1, \dots, T$:

$$FV(X|r) = P_T(x) = x_0x^T + x_1x^{T-1} + \dots + x_{T-2}x^2 + x_{T-1}x + x_T \tag{9.5}$$

Note: It is clear that $1 + k = a$ is its root ($P_T(a) = 0$).

Now we can say that there is a polynomial according to Horner’s theorem.

$$Q_{T-1}(x) = q_{T-1}x^{T-1} + q_{T-2}x^{T-2} + \dots + q_1x + q_0 \quad \text{so that} \quad P_T(x) = (x - a)Q_{T-1}(x) \tag{9.6}$$

Now, by restoring variables into their original states ($x = 1 + r$ and $a = 1 + k$), we have:

$$P_T(1 + r) = [(1 + r) - (1 + k)] \times [Q_{T-1}(1 + r)] \tag{9.7}$$

$$FV(X|r) = (r - k) \times [q_{T-1}(1 + r)^{T-1} + q_{T-2}(1 + r)^{T-2} + \dots + q_1(1 + r)^1 + q_0] \tag{9.8}$$

It is obvious that the result of $Q_{T-1}(1 + r)$ is equal to the future value of a flow like $C = (c_0, c_1, \dots, c_{T-1})$ (it is adequate to assume $c_j = q_{T-(j+1)}$; $j = 0, 1, 2, \dots, T - 2, T - 1$):

$$FV(X|r) = (r - k) \times [c_0(1 + r)^{T-1} + c_1(1 + r)^{T-2} + \dots + c_{T-2}(1 + r)^1 + c_{T-1}] \tag{9.9}$$

$$FV(X|r) = (r - k) \times FV(C|r) \tag{9.10}$$

By dividing the previous relationship on both sides by $\frac{1}{(1+r)^T}$, we have:

$$PV(X|r) = \frac{(r - k) \times FV(C|r)}{(1 + r)(1 + r)^{n-1}} \tag{9.11}$$

$$PV(X|r) = \frac{r - k}{1 + r} \times PV(C|r) \tag{9.12}$$

This is the end of proof. □

9.5. Result’s previous Theorem

$$\left\{ \begin{array}{l} \text{if } r > k \text{ and } C \in C^+ \Rightarrow X \in C^+, \\ \text{if } r > k \text{ and } C \in C^0 \Rightarrow X \in C^0, \\ \text{if } r > k \text{ and } C \in C^- \Rightarrow X \in C^-, \\ \text{if } r = k \Rightarrow X \in C^0, \\ \text{if } r < k \text{ and } C \in C^+ \Rightarrow X \in C^-, \\ \text{if } r < k \text{ and } C \in C^0 \Rightarrow X \in C^0, \\ \text{if } r < k \text{ and } C \in C^- \Rightarrow X \in C^+. \end{array} \right. \tag{9.13}$$

The aforementioned outcome was clearly visible in Figure 1. This figure actually shows that the NPV and the IRR of a smaller flow could be used in two levels rather than choosing based on the NPV in one.

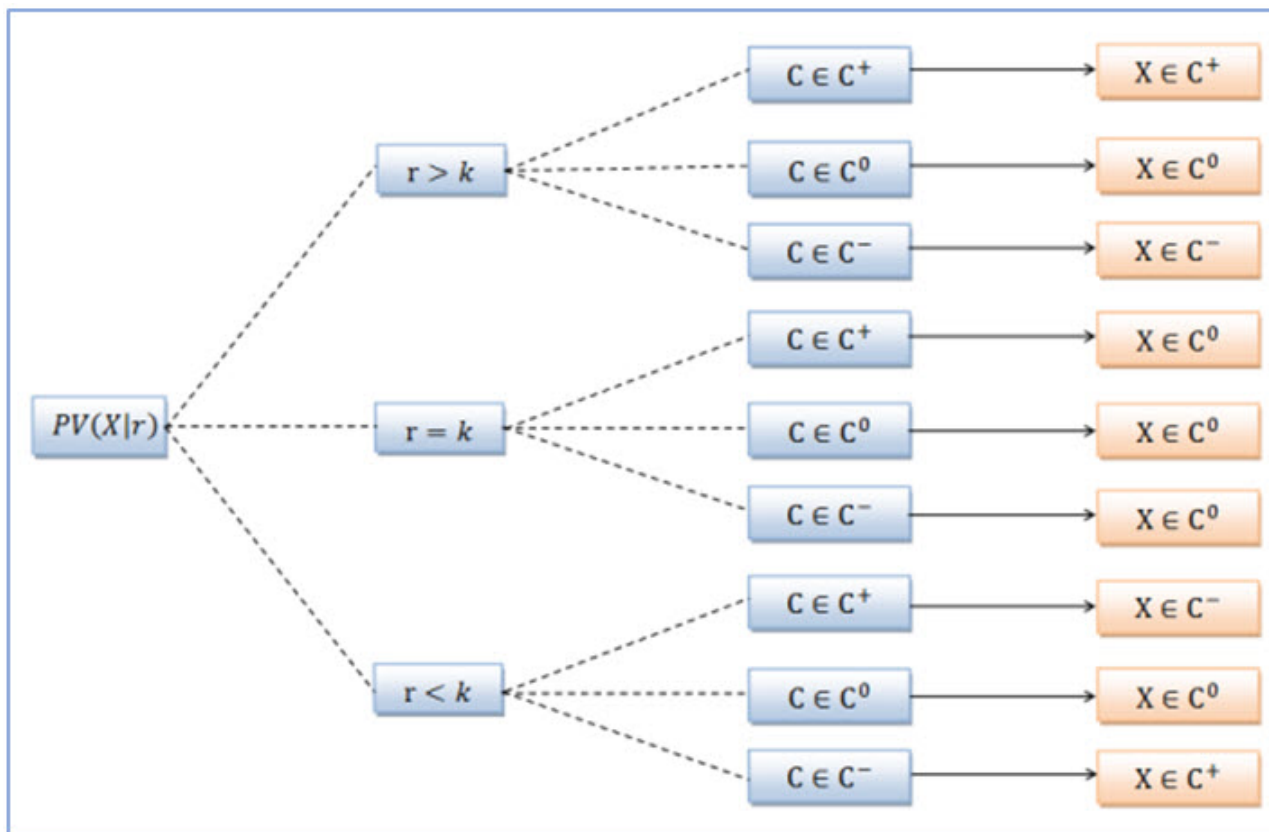


Figure 1: Net Present Value and Internal Rate of Return (Decision Tree)

Example 9.2. Consider a project with the cash flow of $X = (-1, 6, -11, 6)$ and a constant interest rate equal to $r = 10\%$. Three real internal rates of return are determined for this project ($k_1 = 0\%$, $k_2 = 100\%$, $k_3 = 200\%$). We now calculate the investment flow associated with the internal rate of return using Horner’s algorithm (k_2) (according to $x = 1 + k_2$). Table 8 incorporates the whole calculations.

Table 8: Part of the calculations

	-1	6	-11	6
2	0	-2	8	-6
0	-1	4	-3	0

The following step in this example involves presenting a final Table (Table 9) that includes all of the X -dependent cash flow streams (C^{k_i}) in order to decide whether to accept or reject the relevant project. Together with the associated sub-cash flows, the cash flow stream’s present value diagram is shown in Figure 2.

Table 9: The result of calculations

t	0	1	2	3	PV			
X	-1	6	-11	6	-0.128			
$C^{0\%}$	-1	5	-6		-1.413	(10% > 0%)	and $(C^{0\%} \in C^-)$	$X \in C^-$ reject
$C^{100\%}$	-1	4	-3		0.157	(10% < 100%)	and $(C^{100\%} \in C^+)$	$X \in C^-$ reject
$C^{200\%}$	-1	3	-2		0.074	(10% < 200%)	and $(C^{200\%} \in C^+)$	$X \in C^-$ reject

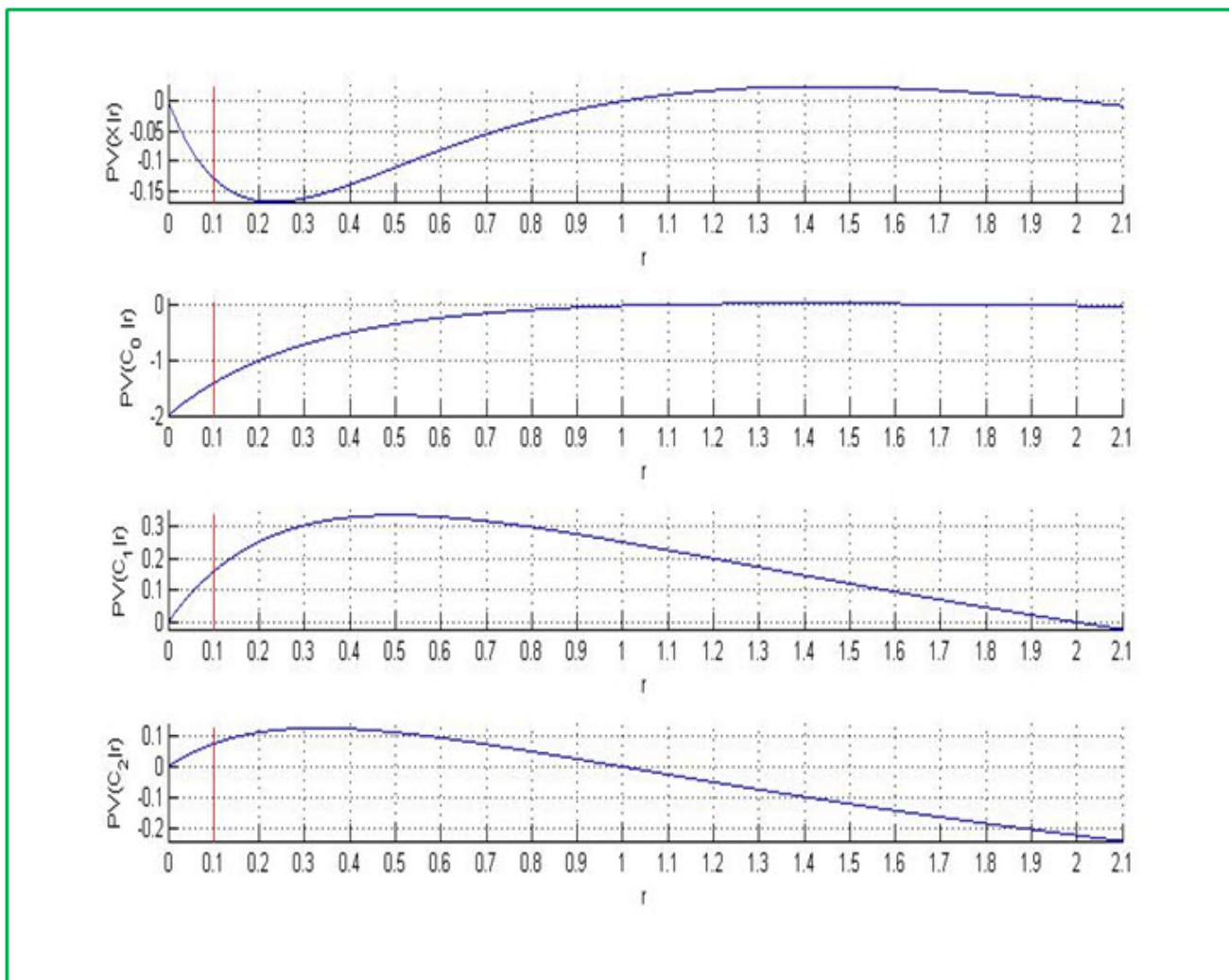


Figure 2: Cash flow chart

Example 9.3. Consider a project with the cash flow sequence $X = (5, 4, -5, -7)$. The corresponding cash flow diagram is presented in Figure 3.

The existing challenge is to determine whether investing in this project would be profitable, considering an interest rate of 15%.

This cash flow has only one internal rate of return ($k = 0.14$). If the project is evaluated using the classical internal rate of return (IRR) method, given that $k < r$, it can be concluded that the investment in this project would result in a loss.

However, if the proposed approach is applied to assess the project, we obtain:

$$C = (5.0000, 9.7198, 6.1191) \quad \text{and} \quad PV(C | r = 0.15) = 18.07 > 0, \quad \text{and since} \quad C \in C^+.$$

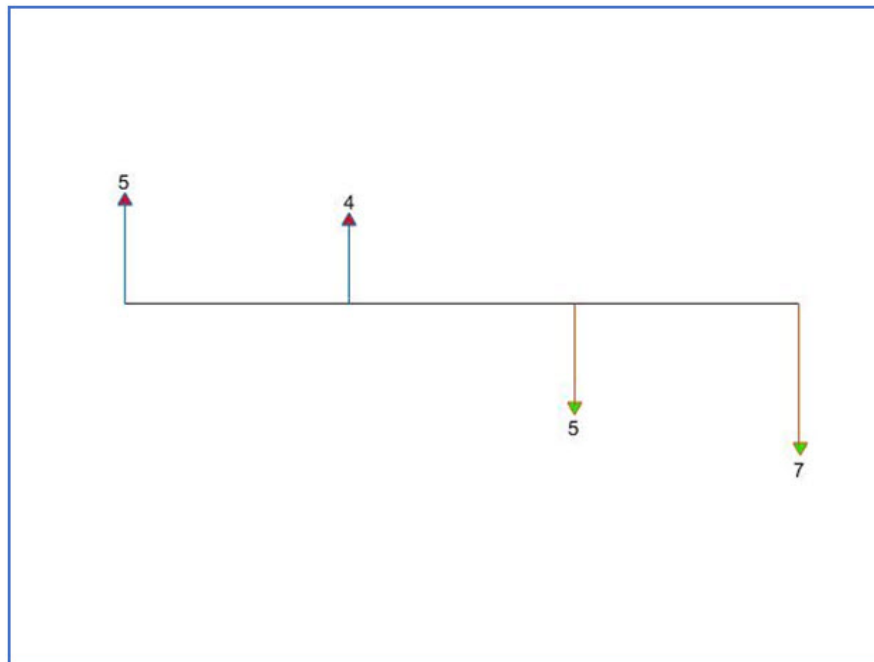


Figure 3: Cash Flow diagram of the project under consideration

Thus, based on the decision tree in Figure 1, the investment in this project is associated with a loss ($X \in C^+$).

To validate both the proposed and classical approaches, we use the Present Value (PV) method, which yields $PV(X | r = 0.15) = 0.095 > 0$. Consequently, investing in this project (X) would be profitable. Therefore, it is evident that the proposed approach, without any deficiencies, leads to an accurate decision, and it is recommended that the investment be made.

Example 9.4. A company active in technical and engineering services (book printing industry) intends to purchase a laser printer. The cash flow associated with this project is: $X = (-500$, 50$, 50$, 50$, 50$, 50$, 50$, 50$, 50$, 50$, 50$, 50$, 50$)$. According to reports from the Central Bank, the monthly interest rate is 5%. The financial process of this project is illustrated in Figure 4. The management team of the company seeks to determine, using the IRR method, whether the investment in this project would be profitable.

The IRR for this project is $k = 0.03 < r = 0.05$, and the auxiliary cash flow based on the proposed approach is: $C = (-500$, -464.61$, -428.19$, -390.71$, -352.13$, -312.42$, -271.55$, -229.49$, -186.2$, -141.64$, -95.78$, -48.58$)$.

Given that $PV(Cr = 0.05) = -2873$ and $C \in C^-$, according to the decision tree in Figure 1, we conclude $X \in C^-$, and therefore, the investment in this project would not only fail to be profitable but would also result in financial losses for the company. Thus, the management team is advised not to proceed with the purchase of the printer.

10. Discussion

Mathematics has always offered help in resolving problems that arise around us and largely contributes to dealing with them. However, we should constantly look for more optimal ways of performing and experimenting with mathematical theories rather than just relying on previous findings. As noted, the proposed demonstration technique is simpler and more concise than the Hazen technique, providing a simpler way to analyze single- and multi-rate projects. As suggested in the previous sections, it is sufficient to just use the designed decision tree for this end. While the proposed evaluation approach will partly meet financial projects, one should pay attention that this problem may entail other challenges such as the estimation of Internal Rates of Return, which makes the mentioned problem more difficult. In

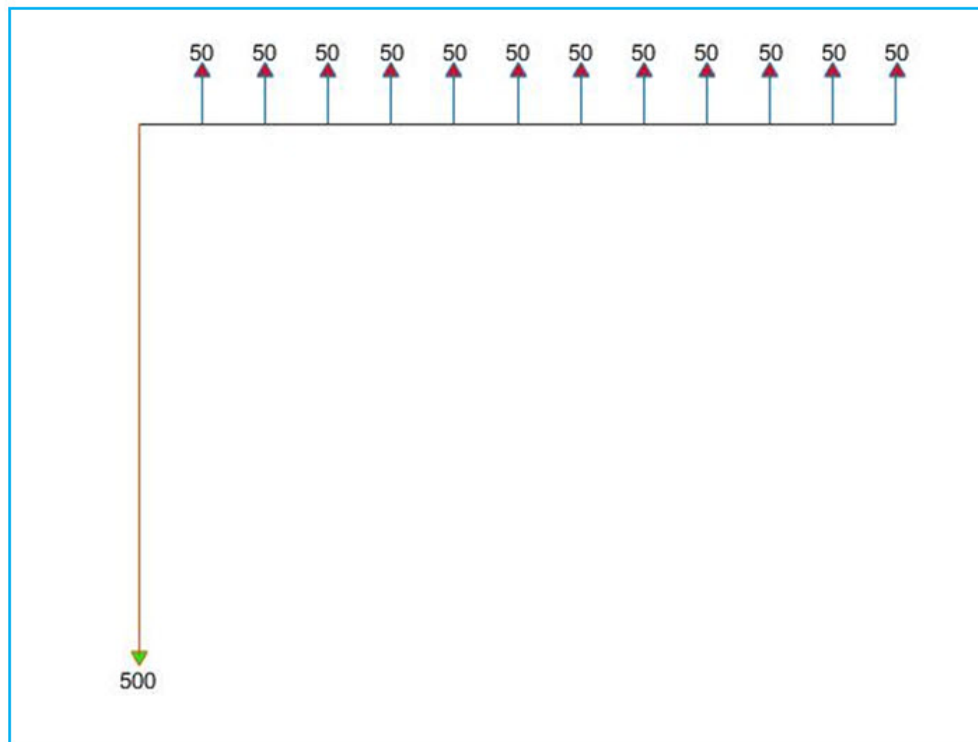


Figure 4: Financial flow chart of the laser printer purchase project

this study, MATLAB software was used to estimate rates of return. One should also pay attention that the novel technique has failed to compare several financial projects and identify the most lucrative of them, while simply evaluating the desirability of a project separately. This [problem] is also expected to be dealt with by later researchers. As stated earlier, the key objective of this study was to demonstrate how Horners Theorem can be practically applied in economics and in evaluating financial projects and how it may help develop and improve econometric methods. Findings showed that the proposed approach could also largely contribute to developing and improving Hazens approach. The novel approach makes use of Project Classification (PC), which is missing in Hazen’s technique

11. Conclusion

In the opinion of some critics of econometrics. Econometrics highly depends on the interpretation of raw data without having a relationship with proven economic theories or searching for ordinary procedures. The important point is to describe the findings revealed in the data acceptably through a theory, even if it means that you should develop your theories for the main process. In this lieu, new techniques and methods for the assessment of financial projects can contribute to the decision-making process of financial managers. IRR is among these techniques always considered beside the Present Value (PV) method. However, it should be mentioned that the IRR method may lead to wrong decisions in some cases and sometimes may confuse managers. The Hazen Approach is one of the novel methods in this case, which broadly has addressed the multi-rate problems of financial projects. In this work, we have tried to encourage students of engineering and economics to use the internal rate of return method more frequently by examining it from a different perspective, although Hazen’s approach provides a new perspective for demystifying the ambiguities and contradictions within the method offered. This method has not been used in many practical works and is only cited theoretically by scientists. In addition, this method proposes a two-criteria decision-making process for evaluating projects and plans with different tariffs. Return is not recognized as a sufficient criterion for making informed decisions.

One of the main limitations of this study is neglecting issues related to the IRR method. In fact, the scarcity of published articles in this area has limited our access to both theoretical and practical findings. Nevertheless, an effort has been made to address the challenges associated with the IRR method by utilizing mathematical techniques and tools. All stages of the work have been presented and validated through algebraic methods to avoid subjective or experiential opinions.

11.1. Further Research

- It should be noted that with this new method, we could only judge the quality of projects based on their actual return; Therefore, mixed returns and project rankings were completely ignored in this study.
- A second recommendation for future research is to extend this method to the classification of projects and plans and to generalize it to the economic evaluation of projects using mixed returns.
- The study suggests developing new algorithms for calculating both real and mixed rates of return using mathematical methods and computer programs in future research. Another area of exploration is comparing fossil fuels, such as diesel, and natural gas in power plants. The IRR method could be applied to economically evaluate these two types of fuel to determine which is more economically viable and environmentally friendly. Furthermore, researchers are encouraged to calculate the upper and lower bounds of real rates of return using algebraic methods. Calculating rates of return for fuzzy and gray cash flows presents a new challenge, and it is recommended to develop a straightforward and comprehensible method for this purpose to evaluate uncertain financial projects based on rates of return.
- In some instances, there may be one or more negative rates of return. It is suggested that solutions be provided for interpreting these negative rates during financial project evaluations.
- Additionally, in many cases involving the IRR method, the rates of return may be expressed as complex numbers. It is recommended to establish solutions for effectively utilizing these complex rates to evaluate financial projects.

References

- [1] Aalaei, M. (2022). Pricing life settlements in the secondary market using fuzzy internal rate of return. *Journal of Mathematics and Modeling in Finance*, 2(2):53–62. 6
- [2] Abbasian, E. and Ahmadi, M. (2024). The quantitative and qualitative assessments of investment risk in government development projects (case study: Joint oil fields of iran and iraq). *Iranian Economic Review*, 28(1):250–272. 1
- [3] Abouzari, P., Khanlari, K., and Esmaeelabadi, R. (2024). Comparison and evaluation of moment frame frequency and response accuracy under characteristic matrices based on classical and modified shape functions. *Environmental Engineering and Management Journal*, 23(12):2631–2642. 5
- [4] Ajide, F. and Dada, J. (2024). Economic complexity and shadow economy in africa. *Iranian Economic Review*, 28(3):788–813. 1
- [5] Alamian, V. and Ayeshgar, E. (2024). The effect of problem posing on learning geometric concepts of students based on apos theory. *Journal of New Researches in Mathematics*, 9(6):23–31. 5
- [6] Ashrafganjooe, R., Akbari Fard, H., Jalaee Esfandabadi, S., and Mashinchi, M. (2024). Estimating the impact of uncertainty of macroeconomic and foreign trade variables on income distribution among income deciles. *Quarterly Journal of Quantitative Economics*, 21(2):75–125. 1
- [7] Asquth, D. and Bethel, J. (1995). Using heuristics to evaluate projects: The case of ranking projects by irr. *The Engineering Economist*, 40(3):287–294. 2
- [8] Babaei, R. and Mohammaditabar, D. (2024). An inventory lot sizing model of deteriorating items with time and price dependent demand, by considering the time value of money. *Journal of Optimization in Industrial Engineering*, 1(17):77–86. 6
- [9] Babaie, B. and Jasbi, A. (2021). Calculation of the profitability of fuzzy investment cash flows using the reliable economic rate of return measure. *Research in Production and Operations Management*, 12(1):77–101. 8

- [10] Berggren, J. (1990). Innovation and tradition in sharaf al-din al-tusi's muadalat. *Journal of the American Oriental Society*, 110(2):304–309. [9.2](#)
- [11] Brounen, D., De Jong, A., and Koedijk, K. (2004). Corporate finance in europe: Confronting theory with practice. *Financial Management*, 33(4):99–106. [2](#)
- [12] Burden, R. and Faires, J. (2011). *Numerical Analysis*. Richard Stratton, Canada. [2](#)
- [13] Cajori, F. (1911). Horner's method of approximation anticipated by ruffini. *Bulletin of the American Mathematical Society*, 17(8):409–414. [9.2](#)
- [14] Chisini, O. (1929). Sul concetto di media [on the concept of mean]. *periodic di matematica*, 4:106–116. [2](#)
- [15] Danielson, M. (2024). A robust model to estimate a firm's average economic return. *Journal of Contemporary Accounting & Economics*, 20(1):100–112. [8](#)
- [16] Desai, P. (2015). On the internal rate of return criterion: A note. *The Engineering Economist*, 21(2):141–144. [2](#)
- [17] Erdik, C. (2019). Investigation of mathematics teachers' opinions about problem posing. *Journal on Mathematics Education*, 10(1):1–20. [1](#), [5](#)
- [18] Frisch, R. (1933). Editor's note. *Econometrica*, 1:1–4. [1](#)
- [19] Graham, J. and Harvey, C. (2001). The theory and practice of corporate finance: Evidence from the field. *Journal of Financial Economics*, 60:44–51. [2](#)
- [20] Gronchi, S. (1986). On investment criteria based on the internal rate of return. *Oxford Economic Papers*, 1:174–180. [2](#)
- [21] Hadian, R., Hashemi, S., and Samadi, S. (2017). Evaluation of the effect of investment on future performance considering the level of firm financial constraint. *Journal of Asset Management and Financing*, 5(3):117–132. [3](#)
- [22] Hanafizadeh, P. and Salmani, H. (2021). Robust net present value with infinite lifetime. *Journal of Mathematics and Modeling in Finance*, 1(1):13–34. [6](#)
- [23] Hazen, G. (2003). A new perspective on multiple internal rates of return. *The Engineering Economist*, 48(1):31–51. [7](#), [7](#), [7.1](#), [7.2](#), [7.3](#), [7.4](#), [8](#)
- [24] Hazen, G. (2009). An extension of the internal rate of return to stochastic cash flows. *Management Science*, 55(6):1030–1034. [2](#)
- [25] Horner, W. (1819). A new method of solving numerical equations of all orders, by continuous approximation. *Philosophical Transactions of the Royal Society of London*, 109:308–335. [9.3](#)
- [26] ILLÉS, M. (2012). Transforming the net present value for a comparable one. *Club of Economics in Miskolc TMP*, 1(8):24–32. [2](#)
- [27] Jafari, H. (2020). Development of average of internal rate of return method. *Mathematics in Natural Science*, 6(1):20–26. [6](#)
- [28] Jafari, H. and Ehsanifar, M. (2024). Using constrained optimization to find real roots of polynomial(rrp). *Theory of Approximation and Applications*, 17(1):1–17. [6](#)
- [29] Jafari, H., Ehsanifar, M., and Sheykhan, A. (2020a). Finding optimum facility's layout by developed simulated annealing algorithm. *International Journal of Research in Industrial Engineering*, 9(2):172–182. [5](#)
- [30] Jafari, H., Etebari, A., and Salehi, K. (2022). Improvement of periodic rates in the average internal rate of return method. *Mathematics and Computational Sciences*, 3(4):23–32. [1](#), [6](#)
- [31] Jafari, H., Faraji, M., and Khaleghi Ardehali, P. (2020b). A heuristic method to calculate the real internal rate of return (rirr). *International Journal of Applied Operational Research*, 10(1):31–40. [6](#), [2](#)
- [32] Javaheri, B., Ghaderi, S., Borzoui, N., and Amani, R. (2024). Value added in different economic sectors, foreign direct investment and economic freedom in iran. *Journal of Econometric Modelling*, 9(3):9–36. [1](#)
- [33] Kester, G., Chang, R., Echanis, E., Haikal, S., Isa, M., and Skully, M. (1999). Capital budgeting practices in the asia-pacific region: Australia, hong kong, indonesia, malaysia, philippines and singapore. *Financial Practice and Education*, 9(1):57–63. [2](#)
- [34] Khalilzade, D. and Ahrabi, A. (2024). Designing optimal predictive control model for boost converter in solar inverters with the help of meta-engineering algorithms. *Transactions on Machine Intelligence*, 7(2):147–160. [5](#)
- [35] Lalkhezri, H. and Ashena, M. (2024). Spillover effects of selected macroeconomic variables on exchange market pressure: a var-dcc-garch approach. *Journal of Econometric Modelling*, 9(2):9–34. [1](#)
- [36] Lerskullawat, A. (2018). Financial development, financial constraint, and firm investment: Evidence from thailand. *Kasetsart Journal of Social Sciences*, 40(1):55–66. [1](#)
- [37] Liljebloom, E. and Vaihekoski, M. (2004). Investment evaluation methods and required rate of return in finnish publicly listed companies. *Finnish Journal of Business Economics*, 53(1):122–129. [2](#)
- [38] Liu, W., Zhou, R., and Wang, S. (2014). The model of cash flow valuation based on the svensson model. *Systems Engineering Theory & Practice*, 34(1):61–66. [2](#)
- [39] Lohmann, J. (1988). The irr, npv and the fallacy of their investment rate assumptions. *The Engineering Economist*, 33(4):303–330. [8](#)
- [40] Magni, C. (2010). Average internal rate of return and investment decisions: a new perspective. *The Engineering Economist*, 55(2):150–180. [1](#), [2](#)
- [41] Mohammad Talebi, B., Mohammad Talebi, M., Jahangiri, M., and Eshghiaraghi, M. (2018). Investigating the effect of internal rate of return on cash re-cycling on the abnormal returns of companies accepted in tehran stock exchange. *Advances in Mathematical Finance and Applications*, 5(3):1–10. [3](#)

- [42] Momeni, M., Soleimani, H., and Mirzapour Al-e Hashem, S. (2022). A nonlinear mathematical model for autonomous vehicle routing problem by considering traffic congestion for each route. *Journal of Quality Engineering and Production Optimization*, 7(2):14–33. 5
- [43] Moradi, B., Shakeri, H., and NamdarZangeneh, S. (2012). Solving the paradox of multiple irrs in engineering economic problems by choosing an optimal α -cut. *International Journal of Industrial Engineering & Production Research*, 1(23):45–52. 1, 2
- [44] Mujahed, M. and Elshareif, E. (2017). Internal rate of return (irr): A new proposed approach. In Benlamri, R. and Sparer, M., editors, *Leadership, Innovation and Entrepreneurship as Driving Forces of the Global Economy*, pages 761–768. Springer Proceedings in Business and Economics. 8
- [45] Norstrom, C. (1972). A sufficient condition for a unique non-negative internal rate of return. *Journal of Financial and Quantitative Analysis*, 7(3):1835–1839. 2
- [46] Nuugulu, S., Gideon, F., and Patidar, K. (2024). An efficient numerical scheme for a time-fractional blackscholes partial differential equation derived from the fractal market hypothesis. *Fractal and Fractional*, 8(8):461–473. 1
- [47] Osborne, I. (2010). A resolution to the npvirr debate? *The Quarterly Review of Economics and Finance*, 50(2):234–239. 1
- [48] Ouyang, H., Li, F., and Liu, Y. (2025). Environmental risk assessment and risk management for green finance decision-making. *Environmental Engineering and Management Journal*, 24(1):117–130. 1
- [49] Pahlavan, S., Khanmohammadi, M., and Yazdani, S. (2025). Development of credit rating scales in investment industry companies. *International Journal of Finance & Managerial Accounting*, 10(38):111–126. 1
- [50] Payne, J., Carrington, H., and Gale, L. (1999). Comparative financial practice in the us and canada: Capital budgeting and risk assessment techniques. *Financial Practice and Education*, 9(1):32–39. 2
- [51] Pressacco, F., Magni, C., and Stucchi, P. (2014). A quasi-irr for a project without irr. *Engineering Economist*, 2(11):1–25. 2
- [52] Rahimibadr, B. (2008). An econometric model for wheat price forecasting in iran. *Journal of European Economy*, 7:12–19. 1
- [53] Russell, M. and Rickard, J. (1982). Uniqueness of non-negative internal rate of return. *Journal of the Institute of Actuaries*, 109:435–445. 2
- [54] Ryan, P. and Ryan, G. (2002). Capital budgeting practices of the fortune 1000: How have things changed? *Journal of Business and Management*, 8(4):45–52. 2
- [55] Saboori, A., Rabiei, M., Zandi, F., and Shakuri, B. (2024). Presenting the optimal monetary policy effect model on inflation and exchange rate control. *International Journal of Nonlinear Analysis and Applications*, 15(8):215–224. 6
- [56] Samiee, M., Noorolahzadeh, N., and Saraf, F. (2024). Presenting a model for identifying the risk aversion of shareholders according to the influence and role of the governments monetary and fiscal policies and the mass behavior of shareholders. *International Journal of Nonlinear Analysis and Applications*, 15(11):333–344. 6
- [57] Samsami, H., Hasanpour, H., and Nowjavan, M. (2023). The impact of removing fictitious assets from bank balance sheet on money supply and other macroeconomic variables. *Iranian Journal of Economic Studies*, 12(1):249–270. 1
- [58] Sanchez, S., Liao, T., Ray, T., and Traintaphyllou, E. (1999). An example for dealing with impreciseness of future cash flows during the selection of economic alternatives. *International Journal of Industrial Engineering*, 6(1):38–47. 2
- [59] Sangpech, P. and Ueasangkomsate, P. (2022). Supply chain management and the circular economy: A review of current research and future trends. In *2022 Joint International Conference on Digital Arts, Media and Technology with ECTI Northern Section Conference on Electrical, Electronics, Computer and Telecommunications Engineering (ECTI DAMT & NCON)*, pages 24–28, Chiang Rai, Thailand. 1
- [60] Sedaghat Galeshklami, I., Jabbari, H., Panahian, H., Arabzadeh, M., and Mazroie Nasrabadi, I. (2025). Investigate the possible effects of the implementation of value-added tax on the amount of government budget deficit, cost of domestic products, demand pressure, inflationary expectations and inflation in iran. *International Journal of Nonlinear Analysis and Applications*, 16(4):311–324. 6
- [61] Sokolov, M. (2024). Npv, irr, pi, pp, and dpp: a unified view. *Journal of Mathematical Economics*, 114:23–29. 4
- [62] Soper, C. (1959). The marginal efficiency of capital: A future note. *The Economic Journal*, 69:174–178. 2
- [63] Spanos, A. (1986). *Statistical foundations of econometric modeling*. Cambridge University Press. 1
- [64] Velnicu, A., Bujoreanu, C., and Boaca, G. (2024). Numerical analysis of vehicle loads using matlab. *Environmental Engineering and Management Journal*, 23(11):2379–2389. 5
- [65] Vukeyi, M. and Vujadinovi, V. (2008). Boost quality of engineering project solutions through economic analysis and comparing alternatives. *International Journal for Quality Research*, 2(4):285–293. 2
- [66] Weber, T. (2013). On the (non-)equivalence of irr and npv. Technical report, *Economics and Strategy*, Ecole Polytechnique Fédérale de Lausanne, Switzerland. 8