Math. & Comput. Sci., 7(1) 2026, 1–8
Online: ISSN 2717-2708
Mathematics and Computational Sciences
Journal Homepage: http://mcs.qut.ac.ir/mcs

Analytical and numerical investigation of Hopf bifurcation in an economic model

Javad Hadadi[a], Zohreh Eskandari[b,*]

[a] Department of Mathematical Sciences, Shahrekord University, Shahrekord, Iran
[b] Department of Mathematics, Faculty of Science, Fasa University, Fasa, Iran

Abstract

This study examines an economic model and explores the Hopf bifurcation by individually varying the indebtedness factor and the output-capital ratio parameter. Both analytical and numerical methods are used to determine the conditions and coefficients for the normal form of the Hopf bifurcation. The critical coefficient for this bifurcation is identified using the central manifold theory. Additionally, the phase portrait of the model near the critical values of the indebtedness factor and the output-capital ratio parameter is illustrated using the Matcont software package.

Keywords: Economic model, Hopf bifurcation, Critical coefficient, Matcont package

2020 MSC: 34C23, 37M20, 37N40, 37L10.

©2026 All rights reserved.

1. Introduction

Economy is one of the most important parameters in life, the absence of which can create destructive effects in life. With a better economy, we can live a better life and also focus on our interests and talents. On the other hand, macroeconomics can affect our lives. Therefore, by presenting models in the field of macroeconomics in which factors such as gross domestic product and foreign capital inflow play a role, we can achieve a better understanding of these phenomena and apply them on a smaller scale.

Economy, as one of the fundamental pillars of development and sustainability in human societies, plays a decisive role in improving the quality of life and enhancing public welfare. The absence of an efficient economic system can lead to profound and destructive consequences, including widespread unemployment, decreased living standards, and social instability. Conversely, a healthy and dynamic economy creates opportunities for job creation, increased productivity, and the flourishing of individual and collective talents. A thorough examination of economic structure and macroeconomic indicators enables policymakers to make more effective decisions that lead to sustainable growth [16].

Macroeconomics is the branch of economics that examines the processes and overall indicators of a countrys economy. These indicators include Gross Domestic Product (GDP), inflation rate, unemployment rate, and foreign capital inflows. GDP is one of the most important measures of a countrys economic

*Corresponding author
Email addresses: j1741366@gmail.com (Javad Hadadi), z-eskandari@fasau.ac.ir (Zohreh Eskandari)

doi: 10.30511/mcs.2025.2048171.1265

Received: 15 December 2024 Accepted: 27 October 2025

capacity, representing the total value of goods and services produced within a specific period [16]. Inflation and unemployment rates are also key indicators that reflect the economic health and the impacts of economic policies.

Foreign capital inflow is another vital macroeconomic indicator that plays a significant role in financing, technology transfer, and enhancing production capacity. The entry of foreign capital into a countrys economy can create employment opportunities, develop industries, and improve infrastructure. Moreover, foreign investment flows contribute to strengthening international economic relations and enhancing a countrys competitiveness [1]. However, the proper management of these investments and the implementation of supportive policies for their optimal utilization are of paramount importance.

Macroeconomic models serve as essential analytical tools that assist policymakers and researchers in understanding the structure and overall behavior of an economy, as well as predicting the impacts of economic policies. These models, in addition to aiding decision-making, enable the assessment of changes in indicators such as GDP and foreign capital inflows.

Until 1967, scientists believed that external factors cause economic fluctuations, but Goodwin [9] believed that internal factors cause economic fluctuations. After that, different linear, non-linear and coupled models were studied accordingly [2, 4, 5, 8, 6, 19]. These articles [12, 17, 18, 11, 10, 3, 13] also facilitate a deeper understanding of the study presented in this paper.

Today, different models of the economy have been proposed. Here we are talking about one of these models that was proposed by Bouali [4] in 1999 and recently this model has been modified. The model we discuss here is a modified Van der Pol oscillator model, which is as follows:

$$\begin{aligned} \dot{x} &= \mu y + \rho x(\delta - y^2), \\ \dot{y} &= \nu y + \omega x + \sigma z, \\ \dot{z} &= \gamma x - \kappa y, \end{aligned} \tag{1.1}$$

where x is savings, y is gross domestic product and z is foreign capital inflow. The parameters μ, ρ, δ, ν, ω, σ, γ, and κ are all positive, with their definitions provided in reference [5]. Specifically, μ represents the marginal propensity to save, ρ denotes the fraction of capitalized profit, δ corresponds to GDP potential, and ν indicates the marginal propensity to consume. Additionally, ω signifies the proportion of savings, σ represents the output-capital ratio, γ captures the inflow-saving ratio, and κ accounts for the indebtedness factor.

The system (1.1) has three equilibrium points as $X_0 = (0, 0, 0)$, $X_1 = (x^*, y^*, z^*)$ and $X_2 = (-x^*, -y^*, -z^*)$ that $x^* = \frac{1}{\gamma}\sqrt{\frac{\mu\kappa\gamma+\delta\rho\gamma^2}{\rho}}$, $y^* = \frac{1}{\kappa}\sqrt{\frac{\mu\kappa\gamma+\delta\rho\gamma^2}{\rho}}$ and $z^* = -\dfrac{\kappa\omega+\gamma\nu}{\gamma\sigma\kappa}\sqrt{\frac{\mu\kappa\gamma+\delta\rho\gamma^2}{\rho}}$. We examine the stability and bifurcations in each of the equilibrium points in other sections.

2. Stability and bifurcation of equilibrium point X_0

The Jacobian matrix of the system (1.1) is as follows:

$$\tilde{J} = \begin{pmatrix} \rho(\delta - y^2) & -2\rho x y + \mu & 0 \\ \omega & \nu & \sigma \\ \gamma & -\kappa & 0 \end{pmatrix},$$

by setting the values at the point X_0 we have:

$$J = \tilde{J}|_{(0,0,0)} = \begin{pmatrix} \rho\delta & \mu & 0 \\ \omega & \nu & \sigma \\ \gamma & -\kappa & 0 \end{pmatrix}, \tag{2.1}$$

therefore, the characteristic equation of the relation (2.1) is obtained as follows:

$$\begin{aligned} f(\lambda) &= \lambda^3 - (\rho\delta+\nu)\lambda^2 + (\rho\delta\nu + \sigma\kappa - \mu\omega)\lambda - \sigma(\rho\kappa\delta + \mu\gamma), \\ &= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \end{aligned}$$

It is clear that the system (1.1) has a positive eigenvalue at the equilibrium point X_0 because $f(0) < 0$. Also, it does not have a zero eigenvalue, but it can have a complex eigenvalue with zero real part. According to the Routh-Hurwitz criterion, if $\frac{-a_3}{a_1} + a_2 < 0$, the other two eigenvalues have a negative real part, and the equilibrium point X_0 is saddle, if $\frac{-a_3}{a_1} + a_2 > 0$, the other two eigenvalues have positive real parts and the equilibrium point X_0 is source, and if $\frac{-a_3}{a_1} + a_2 = 0$, then there are two complex eigenvalues with zero real part as $\lambda = \pm i\sqrt{\frac{a_3}{a_1}} = \pm i\theta^*$, so the equilibrium point X_0 is non-hyperbolic and there is a possibility of Hopf bifurcation. Therefore, we keep all the parameters constant except one such as κ. We find the critical value of this parameter for which we have a pair of complex eigenvalues:

$$\kappa^* = \frac{-\delta^2\rho^2\nu + \delta\mu\rho\omega - \delta\rho\nu^2 + \sigma\mu\gamma + \mu\nu\omega}{\sigma\nu}.$$

Then we check the Hopf bifurcation condition. First, we check the transversality condition, so we have:

$$\frac{d\Re(\lambda)}{d\kappa}|_{\kappa=\kappa^*} = \frac{b_1b_3 + b_2b_4}{b_3^2 + b_4^2}$$

where

$$\begin{aligned} &b_1 = -\nu\sigma\rho\delta, \qquad b_2 = \nu\sigma\theta^*, \qquad b_4 = 2\theta^*\nu^2 + 2\delta\rho\theta^*\nu, \\ &b_3 = -\mu(\delta\rho\omega + \sigma\gamma) + 3{\theta^*}^2\nu + \delta^2\rho^2\nu. \end{aligned}$$

If $b_1b_3 + b_2b_4 \neq 0$, the trasversality condition is satisfied, then we check the non-degeneracy condition. To obtain the first Lyapunov coefficient, we first calculate the left and right eigenvectors corresponding to the eigenvalue $\lambda = i\theta^*$, which are as follows:

$$\begin{aligned} &Jq = \lambda q, \qquad J^Tp = \bar{\lambda}p \qquad \langle q, q\rangle = 1, \qquad \langle p, q\rangle = 1, \\ &q_1 = (1, \frac{i\theta^* - \rho\delta}{\mu}, \frac{-\theta^{*2} - i\delta\rho\theta^* - i\theta^*\,\nu + \delta\rho\nu - \mu\omega}{\mu\sigma})^T, \\ &p_1 = (\frac{i\theta^*\,\omega - \gamma\sigma}{\sigma\,(i\theta^* + \rho\delta)}, \frac{-i\theta^*}{\sigma}, 1)^T, \\ &q = \frac{q_1}{\sqrt{\langle q_1, q_1\rangle}}, \qquad p = \frac{p_1}{\langle p_1, q\rangle}. \end{aligned}$$

The first Lyapunov coefficient is calculated from the following equation [14]:

$$L = \frac{1}{2\theta^*}\Re(\langle p, C(q, q, \bar{q})\rangle) = \frac{1}{2\theta^*}\Re(\frac{2\,(i\theta^* + 3\,\rho\delta)\,\rho\,(i\theta^*\,\omega + \gamma\sigma)}{\mu^2\sigma\langle p_1, q_1\rangle\langle q_1, q_1\rangle}).$$

If the first coefficient of Lyapunov becomes non-zero, the system (1.1) at equilibrium point X_0 for $\kappa = \kappa^*$ has a Hopf bifurcation, which we will talk about in the following [15].

Theorem 2.1. *The system (1.1) at the equilibrium point X_0 for $\kappa = \kappa^*$ with the condition $L \neq 0$ and $\frac{d\Re(\lambda)}{d\kappa}|_{\kappa=\kappa^*} \neq 0$, has Hopf bifurcation. If the first Lyapunov coefficient L becomes negative, a stable cycle bifurcate from the desired equilibrium point, and if the first lyapunov coefficient becomes positive, an unstable cycle is created.*

3. Stability and bifurcation of equilibrium points X_1 and X_2

Similar to the previous case, by valuing the Jacobian matrix at the equilibrium point X_1, we have:

$$\hat{J} = \begin{pmatrix} \rho(\delta - y^{*2}) & -2\rho x^* y^* + \mu & 0 \\ \omega & \nu & \sigma \\ \gamma & -\kappa & 0 \end{pmatrix}, \tag{3.1}$$

The characteristic equation of relation (3.1) is as follows:

$$\begin{aligned} g(\lambda) &= \lambda^3 + (\frac{\gamma\mu}{\kappa} - \nu)\lambda^2 + (\frac{2\,\kappa\delta\rho\omega}{\gamma} + \sigma\kappa + \mu\omega - \frac{\gamma\mu\nu}{\kappa})\lambda + 2\,\kappa\sigma\delta\rho + 2\,\gamma\sigma\mu, \\ &= \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3. \end{aligned}$$

Now, using the Routh-Hurwitz criterion, we check the stability of equilibrium point X_1. What has been said about the stability of equilibrium point X_1 is true for stability of the equilibrium point X_2. It is clear that X_1 has at least one negative real eigenvalue because $g(0) > 0$.

-If $c_1 > 0$ and $\frac{-c_3}{c_1} + c_2 > 0$, it has three eigenvalues with negative real part, which in this case, the equilibrium point X_1 is sink.

-If we have $c_1 > 0$ and $\frac{-c_3}{c_1} + c_2 < 0$, or $c_1 < 0$ and $\frac{-c_3}{c_1} + c_2 > 0$, or $c_1 < 0$ and $\frac{-c_3}{c_1} + c_2 < 0$, or $c_1 = 0$, or $c_1 < 0$ and $\frac{-c_3}{c_1} + c_2 = 0$, then X_1 has three eigenvalues with a positive real part and two negative real part eigenvalue, which in this case, the equilibrium point X_1 is saddle.

-If $c_1 > 0$ and $\frac{-c_3}{c_1} + c_2 = 0$, it has two pure imaginary eigenvalues and one negative real eigenvalue, in which case the pure imaginary eigenvalues are $\lambda = \pm i\sqrt{\frac{c_3}{c_1}} = \pm i\theta_*$, and there is a probability of Hopf bifurcation.

Now, keeping all the parameters constant except for one such as σ, we find the critical value of this parameter, for which we have a pair of complex eigenvalues, so we have:

$$\sigma_* = \frac{\left(2\,\kappa^2\delta\rho\omega + \mu\omega\gamma\kappa - \gamma^2\mu\nu\right)(\mu\gamma - \kappa\nu)}{\kappa^2\gamma\,(2\,\delta\rho\kappa + \mu\gamma + \kappa\nu)},$$

then we check the Hopf bifurcation condition. To check the transversality condition, we have:

$$\frac{d\Re(\lambda)}{d\kappa}|_{\kappa=\kappa^*} = \frac{d_1 d_3 + d_2 d_4}{d_3^2 + d_4^2}$$

where

$$\begin{aligned} &d_1 = 2\kappa^2\gamma\rho\delta + 2\mu\gamma^2\kappa, \qquad d_2 = \kappa^2\gamma\theta_*, \qquad d_4 = -2\theta_*\nu\gamma\kappa + 2\theta_*\gamma^2\mu, \\ &d_3 = \kappa^2(2\rho\delta\omega + \sigma\gamma) - \nu\gamma^2\mu + \gamma\kappa(\mu\omega - 3\theta_*^2). \end{aligned}$$

If $d_1 d_3 + d_2 d_4 \neq 0$, the trasversality condition is satisfied, then we check the non-degeneracy condition. To obtain the first Lyapunov coefficient, we first calculate the left and right eigenvectors corresponding to the eigenvalue $\lambda = i\theta_*$, which are as follows:

$$\begin{aligned} &\hat{J}q = \lambda q, \qquad \hat{J}^T p = \bar{\lambda} p, \qquad \langle q, q\rangle = 1, \qquad \langle p, q\rangle = 1, \\ &q_1 = (-\frac{\kappa\,(2\,\kappa\delta\rho + \mu\gamma)}{\gamma\,(i\kappa\theta_* + \mu\gamma)}, 1, \frac{i\kappa\,(2\,\kappa\delta\rho + i\kappa\theta_* + 2\,\mu\gamma)}{(i\kappa\theta_* + \mu\gamma)\,\theta_*})^T, \\ &p_1 = (1, -\frac{i\kappa\theta_* - 2\,\kappa\delta\rho - 2\,\mu\gamma}{i\gamma\theta_* + \kappa\omega + \gamma\nu}, \frac{i\mu\gamma^2\theta_* - i\kappa\gamma\theta_*\,\nu - 2\,\kappa^2\delta\rho\omega - \mu\omega\gamma\kappa + \gamma^2\mu\nu + \kappa\gamma\theta_*^2}{\gamma\kappa\,(i\gamma\theta_* + \kappa\omega + \gamma\nu)})^T, \\ &q = \frac{q_1}{\sqrt{\langle q_1, q_1\rangle}}, \qquad p = \frac{p_1}{\langle p_1, q\rangle}, \end{aligned}$$

The first Lyapunov coefficient is calculated from the following equation [14]:

$$\hat{L} = \frac{1}{2\theta_*}\Re(\langle p, C(q,q,\bar{q})\rangle - 2\langle p, B(q, \hat{J}^{-1}B(q,\bar{q}) + \langle p, B(\bar{q},(2\theta_* iI_3 - \hat{J})^{-1}B(q,q))\rangle),$$

so, we have:

$$\begin{aligned}\hat{L} = \frac{1}{2\theta_*}\Re\Bigg(&\frac{1}{\langle q_1,q_1\rangle\langle p_1,q_1\rangle}\Bigg(\frac{-2\rho\kappa(2\,\kappa\delta\rho+\gamma\mu)(i\kappa\theta_*-3\,\gamma\mu)}{\gamma(\gamma^2\mu^2+\kappa^2\theta_*^2)}\\
&+\frac{-4\,\rho\kappa(4\,\delta\mu\rho\kappa\gamma+\mu^2\gamma^2-\kappa^2{\theta_*}^2)(2\,i\kappa\theta_*-2\,\kappa\delta\rho+\mu\gamma)}{\gamma(i\kappa\theta_*+\mu\gamma)(\mu^2\gamma^2+\kappa^2{\theta_*}^2)}\\
&+\frac{1}{N}\Big(-4\,(\kappa\delta\rho+\mu\gamma)(\rho\kappa((\theta_*\,i-4\,\delta\rho)\kappa-\mu\gamma)\\
&\times((\sigma(\theta_*\,i+\delta\rho)\gamma+2\,(i\delta\rho-\frac{\theta_*}{2})\omega\theta_*)\kappa^2\\
&+(-\frac{1}{2}\gamma^2\sigma\mu+(-2\,i{\theta_*}^3+{\theta_*}^2\nu)\gamma)\kappa+\mu\gamma^2(\nu i+2\,\theta_*)\theta_*))\Big)\Bigg)\Bigg),\end{aligned}$$

where

$$\begin{aligned}N =&\Bigg(\Big((\theta_*\,i+\rho\delta)\sigma\gamma+2\,i\delta\theta_*\,\rho\omega\Big)\kappa^2+\gamma(\gamma\sigma\mu+(\mu\omega i-4\,i{\theta_*}^2+2\,\theta_*\,\nu)\theta_*)\kappa-(\nu i+2\,\theta_*)\gamma^2\mu\theta_*\Bigg)\\
&\times\gamma(\mu^2\gamma^2+\kappa^2{\theta_*}^2).\end{aligned}$$

If the first coefficient of Lyapunov becomes non-zero, the system (1.1) at equilibrium point X_1 for $\sigma = \sigma_*$ has a Hopf bifurcation.

Theorem 3.1. *The system (1.1) at the equilibrium point X_1 for $\sigma = \sigma_*$ with the condition $\hat{L} \neq 0$ and $\frac{d\Re(\lambda)}{d\sigma}|_{\sigma=\sigma_*} \neq 0$, has Hopf bifurcation. If the first Lyapunov coefficient $\hat{L}$ becomes negative, a stable cycle bifurcate from the desired equilibrium point, and if the first lyapunov coefficient becomes positive, an unstable cycle is created.*

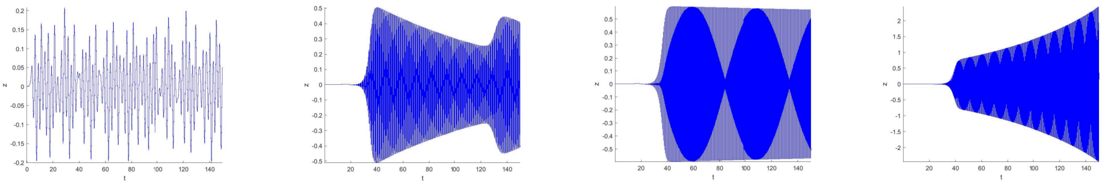
4. Numerical simulation

Now, using Matcont package in MATLAB [7], we numerically check the Hopf bifurcation at the origin. We keep all parameters constant except the parameter κ. Then we assign a specific numerical value to each parameter, $\mu = 0.02, \rho = 0.4, \delta = 1, \sigma = 50, \gamma = 10, \nu = 0.05$ and $\omega = 0.1$. Then, by releasing the parameter κ, at $\kappa = \kappa^* = 3.996759$, we see the Hopf bifurcation, whose first Lyapunov coefficient is equal to $L = -7.45 \times 10^{-5}$. The first Lyapunov coefficient is negative, so we see a stable cycle that is bifurcated from the origin.

Then we examine the Hopf bifurcation in X_1 and X_2. For this purpose, we give a certain value to all the parameters $\mu = 0.1, \rho = 0.9, \delta = 0.4, \gamma = 50, \nu = 0.001, \omega = 2, \kappa = 0.5$ and we keep the parameter σ free. At $\sigma = \sigma_* = 0.38126958$, Hopf bifurcation occurs, whose first Lyapunov coefficient is equal to $\hat{L} = 1.5203 \times 10^{-3}$.

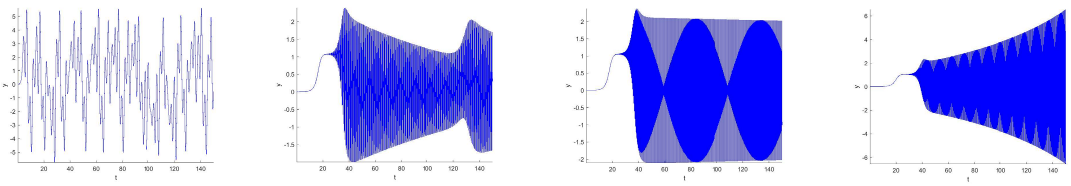
5. Conclusion

In this paper, we considered an economic model and by changing the parameter κ, which was called the indebtedness factor, the Hopf bifurcation occurred. These findings indicate that the equilibrium point X_0 undergoes a stability transition as κ crosses κ^*. Beyond this threshold, the system experiences a shift



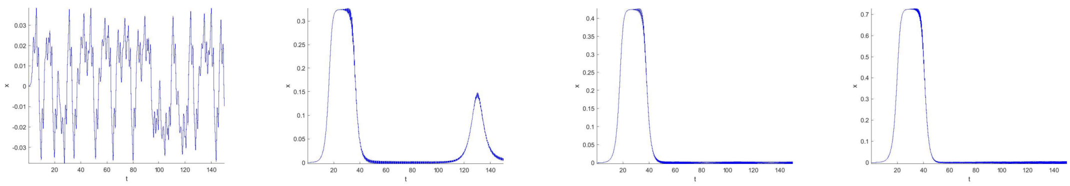
(a) (b) (c) (d)

Figure 1: Changes in foreign capital inflow with the initial point $(0.0001, 0.0001, 0.0001)$ for (a) $\kappa = 0.1$ (b) $\kappa = 3$ (c) $\kappa = \kappa^* = 3.996759$ and (d) $\kappa = 7$.



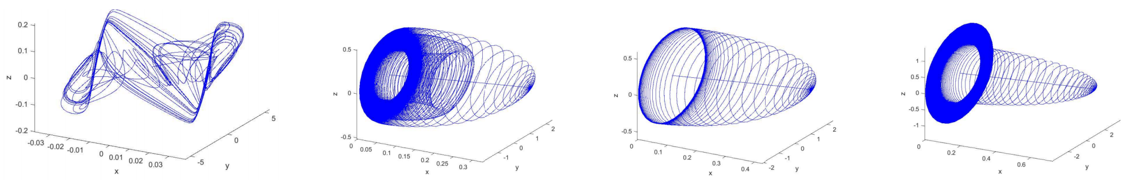
(e) (f) (g) (h)

Figure 2: Changes in gross domestic product with the initial point $(0.0001, 0.0001, 0.0001)$ for (e) $\kappa = 0.1$ (f) $\kappa = 3$ (g) $\kappa = \kappa^* = 3.996759$ and (h) $\kappa = 7$.



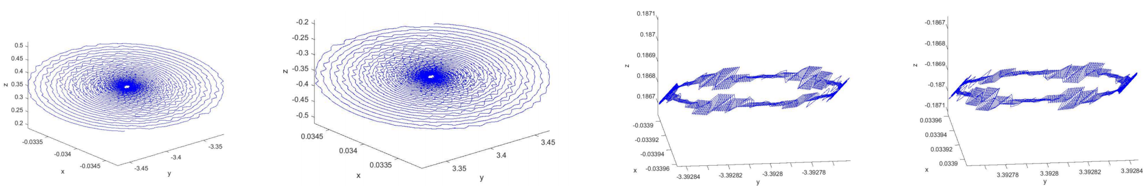
(i) (j) (k) (l)

Figure 3: Changes in savings with the initial point $(0.0001, 0.0001, 0.0001)$ for (i) $\kappa = 0.1$ (j) $\kappa = 3$ (k) $\kappa = \kappa^* = 3.996759$ and (l) $\kappa = 7$.



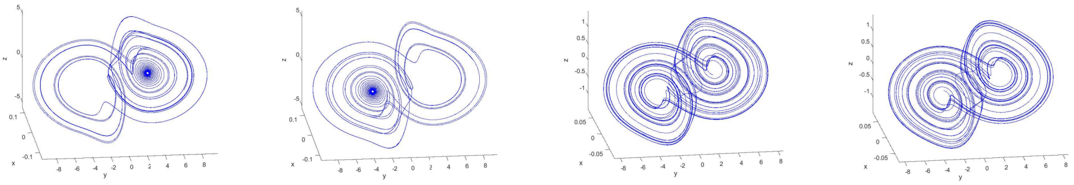
(m) (n) (p) (q)

Figure 4: Phase portrait of the system (1.1) with the initial point $(0.0001, 0.0001, 0.0001)$ for (m) $\kappa = 0.1$ (n) $\kappa = 3$ (p) $\kappa = \kappa^* = 3.996759$ and (q) $\kappa = 7$.



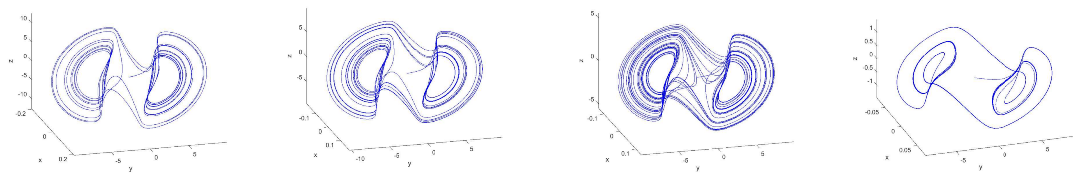
(r) (s) (t) (u)

Figure 5: Phase portrait of the system (1.1) for (r) $\sigma = 0.2$ with the initial point $(-0.0339, -3.3928, 0.1868)$. (s) $\sigma = 0.2$ with the initial point $(0.0339, 3.3928, -0.1868)$. (t) $\sigma = \sigma_* = 0.38126958$ with the initial point $(-0.0339, -3.3928, 0.1868)$. (u) $\sigma = \sigma_* = 0.38126958$ with the initial point $(0.0339, 3.3928, -0.1868)$.



(w) (x) (y) (z)

Figure 6: Phase portrait of the system (1.1) for (w) $\sigma = 1$ with the initial point $(-0.0339, -3.3928, 0.1868)$. (x) $\sigma = 1$ with the initial point $(0.0339, 3.3928, -0.1868)$. (y) $\sigma = 10$ with the initial point $(-0.0339, -3.3928, 0.1868)$. (z) $\sigma = 10$ with the initial point $(0.0339, 3.3928, -0.1868)$.



(I) (II) (III) (IV)

Figure 7: Phase portrait of the system (1.1) with the initial point around the origin equilibrium point $(0.0001, 0.0001, 0.0001)$ for (I) $\sigma = 0.2$ (II) $\sigma = \sigma_* = 0.38126958$ (III) $\sigma = 1$. (IV) $\sigma = 10$.

in stability, resulting in the emergence and continuous expansion of a limit cycle. This expansion progressively moves the system away from the equilibrium at the origin, which corresponds to an economic recession. See Figures 1, 2, 3, and 4.

Also we saw that stability and similar bifurcation occur at the equilibrium point X_1 and X_2. By changing the parameter σ, which is the output-capital ratio, Hopf bifurcation occurred in σ_*. This shows that before reaching the critical value of the parameter σ_*, X_1 and X_2 are stable and they lose their stability with the Hopf bifurcation. By selecting initial points around the equilibrium states, it is observed that the equilibrium points X_1 and X_2 remain stable for $\sigma < \sigma_*$ before reaching the critical threshold. However, at $\sigma = \sigma_*$, a limit cycle is observed near them. For $\sigma > \sigma_*$, the equilibrium points become unstable. In this scenario, the equilibrium point at the origin also becomes unstable, preventing the system from moving towards an economic recession. See Figures 5, 6, and 7.

Readers of this paper can explore various bifurcations within the model by adjusting different parameters. These include codimension-one bifurcations such as saddle-node bifurcation and Hopf bifurcation, as well as codimension-two bifurcations like the Bogdanov-Takens bifurcation. Additionally, transforming this system into a fractional-order model and analyzing the resulting bifurcations may provide valuable insights.

References

[1] Alfaro, L. & Chauvin, J., (2020). Foreign direct investment, finance, and economic development, Faculty & Research 1 ,231-258. 1

[2] Assous, M. & Carret, V., (2022). Modeling economic instability: A history of early macroeconomics, Springer . 1

[3] Ayoade, A.A. & Agboola, S.O. & Thota, S., (2025). Bifurcation for a disease model with the effect of mass media, Mathematics and Computational Sciences ,e729327. 1

[4] Bouali, S., (1999). Feedback loop in extended Van der Pol's equation applied to an economic model of cycles, International Journal of Bifurcation and Chaos 9(4) ,745-756. 1

[5] Camargo, V.E. & Amaral, A.S. & Crepaldi, A.F. & Ferreira, F.F., (2022). Synchronization and bifurcation in an economic model, Chaos: An Interdisciplinary Journal of Nonlinear Science 32(10) ,103112. 1, 1

[6] De Cesare, L. & Sportelli, M., (2005). A dynamic IS-LM model with delayed taxation revenues, Chaos, Solitons & Fractals 25(1) ,229-238. 1

[7] Dhooge, A. & Govaerts, W. & Kuznetsov, Y.A. & Meijer, H.G.E. & Sautois, B., (2008). New features of the software MatCont for bifurcation analysis of dynamical systems, Mathematical and Computer Modelling of Dynamical Systems 14(2) ,147-175. 4

[8] Gardini, L. & Gori, L. & Guerrini, L. & Sodini, M., (2018). Introduction to the focus issue nonlinear economic dynamics, Chaos: An Interdisciplinary Journal of Nonlinear Science 28(5) ,055801. 1

[9] Goodwin, R.M., (1967). A Growth Cycle: Socialism, Capitalism and Economic Growth, 1967, ED. CH Feinstein. In Essays in economic dynamics ,165-170. 1

[10] Hadadi, J. & Ghaziani, R.K. & Alidousti, J. & Eskandari, Z., (2024). Dynamics and bifurcations of a discrete-time neural network model with a single delay, Journal of Mathematical Modeling (JMM) 12(3) . 1

[11] Hadadi, J. & Alidousti, J. & Khoshsiar Ghaziani, R. & Eskandari, Z., (2025). Bifurcations and complex dynamics of a two dimensional neural network model with delayed discrete time, Mathematical Methods in the Applied Sciences 48(7), 8091-8102. 1

[12] Hamou, A.A. & Rasul, R.R. & Hammouch, Z. & Özdemir, N., (2022). Analysis and dynamics of a mathematical model to predict unreported cases of COVID-19 epidemic in Morocco, Computational and Applied Mathematics 41(6) ,289. 1

[13] Ibrahim, M., (2022). Numerical simulation and optimal fishing effort for a fishery with total allowable catch (TAC), Mathematics and Computational Sciences 3(4) ,1-15. 1

[14] Kuznetsov, Y.A., (1998). Elements of applied bifurcation theory, Springer . 2, 3

[15] Lorenz, H.W., (1993). Nonlinear dynamical economics and chaotic motion (Vol. 334), Springer . 2

[16] Mankiw, N.G., (1998). Principles of microeconomics (Vol. 1), Elsevier. 1

[17] Ouakka, A. & Elazzouzi, A. & Hammouch, Z., (2025). An SVIQR model with vaccination-age, general nonlinear incidence rate and relapse: Dynamics and simulations, International Journal of Biomathematics 18(02) ,2350092. 1

[18] Singh, H. & Srivastava, H.M. & Hammouch, Z. & Nisar, K.S., (2021). Numerical simulation and stability analysis for the fractional-order dynamics of COVID-19, Results in Physics 20 . 1

[19] Volos, C.K. & Kyprianidis, I.M. & Stouboulos, I.N., (2012). Synchronization phenomena in coupled nonlinear systems applied in economic cycles, WSEAS Trans. Systems 11(12) ,681-690. 1