Math. & Comput. Sci., 7(1) 2026, 9–20
Online: ISSN 2717-2708
Mathematics and Computational Sciences
Journal Homepage: http://mcs.qut.ac.ir/mcs

Two-person games induced by Pentapartitioned Neutrosophic Payoffs

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Abstract

Two-person games have been extensively studied in classical game theory, but uncertainty and imprecision in real-world scenarios necessitate more advanced mathematical frameworks. Pentapartitioned Neutrosophic sets, provide a powerful tool for handling contradiction, ignorance, unknown, and inconsistent information. This paper explores two-person games in a Pentapartitioned Neutrosophic environment, where players strategies, payoffs, and outcomes are expressed using Pentapartitioned Neutrosophic numbers. We present fundamental definitions, solution concepts, and equilibrium conditions tailored for such games. The findings demonstrate that Pentapartitioned Neutrosophic game theory provides a more flexible and realistic approach to strategic interactions involving indeterminacy.

Keywords: Pentapartitioned Neutrosophic set, Pentapartitioned Neutrosophic game theory, Two person games, Pentapartitioned Neutrosophic two person games

2020 MSC: 91A44, 91A80, 91A86

1. Introduction

Two-person Game Theory is a framework that models strategic interactions between two decision makers, often referred to as players. The theory analyses how each player makes decisions based on their preferences, strategies, and the possible outcomes, aiming to maximize their payoff or minimize losses. Combining the concepts Two-person Game Theory and Pentaparitioned Neutrosophic Set theory, we obtain Two-person Game theories induced by Pentaparitioned Neutrosophic Sets aim to model and solve games where the uncertainty or imprecision in the information is expressed through Pentaparitioned Neutrosophic Sets. This type of game theory allows for more flexible and realistic modeling of real-world decision-making scenarios, where players may not have complete information or may face ambiguous or conflicting choices. The Pentaparitioned Neutrosophic approach helps capture the complexities of real-world situations more accurately than traditional crisp or fuzzy game theory. Using Atanassov's [2] Intuitionistic set theory (1986) and Zadeh's [16] Fuzzy set theory (1965), Florintin Marandache [12] introduced the Neutrosophic logic for the first time in 2005. Later, in 2020, Rama Mallick [8] and Surapati Pramanik established the idea of pentapartitioned neutrophilic sets. Neumann [10] and

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doi: 10.30511/mcs.2025.2075226.1518

Received: 20 October 2025 Accepted: 24 November 2025

Morgenstern introduced the theories of Games and Economical Behaviour in 1944. Afterthat Nash [9] described the Non Cooperative Games in 1951. Also the works on Game Theory of Binmore [4], Aliprantis [1], Ferguson [7], Cevikel [5] were must notable things. In recent days, Surpati Premanik [11] used Neutrosophic Game theoretical approach towards indo-pak conflict over jammu-kashmir in 2014. In 2019, Deli [6] discussed about Matrix Games using Simplified Neutrosphic Payoffs.

Recent literature provides several mathematical foundations that are useful for game-theoretic modeling and analysis. In 2025, Taati [15] introduced new classes of one-searchable graphs, whose structural properties can support strategic interactions represented through graph-based games, especially those involving search, pursuitevasion, and network-based strategies. In the context of decision making, Shureshjani [13] proposed a developed BestWorst method in fuzzy intuitionistic environments, offering a systematic framework to model players preferences when uncertainty or ambiguity affects strategic choices in 2021. Also in 2024 extending this direction, Shureshjani et al. [14] introduced a parametric distance measure for trapezoidal intuitionistic fuzzy numbers, which enhances the evaluation of strategies in multi-criteria group decision-making settings, an essential component in cooperative and non-cooperative game-theoretic problems where players must choose among multiple conflicting criteria.

2. Preliminary Results

Definition 2.1. Let X be a universal set. Then a **Pentapartitioned Neutrosophic Set** (PN set) A over X is an object of the form, A={(x, T_A(x), C_A(x), G_A(x), U_A(x), F_A(x))/x∈ X} where T, C, G, U, F : X[0,1], represents the degree of the truth membership, contradiction membership, ignorance membership, unknown membership and falsity membership functions respectively and $0 \leqslant T_A(x) + C_A(x) + G_A(x) + U_A(x) + F_A(x) \leqslant 5$ for each x∈ X. [8]

Proposition 2.2. *Let A, B be two Pentapartitioned Neutrosophic sets over* X. *Then,*

1. *A⊆B if and only if* $T_A(x) \leqslant T_B(x)$, $C_A(x) \leqslant C_B(x)$, $G_A(x) \geqslant G_B(x)$, $U_A(x) \geqslant U_B(x)$, $F_A(x) \geqslant F_B(x)$ *for each* $x \in X$. *[8]*
2. *A∪B=* {(x, max ($T_A(x)$, $T_B(x)$), max ($C_A(x)$, $C_B(x)$), min ($G_A(x)$, $G_B(x)$), min ($U_A(x)$, $U_B(x)$), min ($F_A(x)$, $F_B(x)$))/x∈X}. *[8]*
3. *A∩B=* {(x, min ($T_A(x)$, $T_B(x)$), min ($C_A(x)$, $C_B(x)$), max($G_A(x)$, $G_B(x)$), max ($U_A(x)$, $U_B(x)$), max ($F_A(x)$, $F_B(x)$))/x∈X}. *[8]*
4. *If A=* {(x, $T_A(x)$, $C_A(x)$, $G_A(x)$, $U_A(x)$, $F_A(x)$)/x∈X}, *then* A^C = {(x, $F_A(x)$, $U_A(x)$, 1-($G_A(x)$), $C_A(x)$, $T_A(x)$)/x∈X}. *[8]*
5. *A⊈B if atleast one of the following occurs* $T_A(x) \geqslant T_B(x)$, $C_A(x) \geqslant C_B(x)$, $G_A(x) \leqslant G_B(x)$, $U_A(x) \leqslant U_B(x)$, $F_A(x) \leqslant F_B(x)$ *for any* $x \in X$. *[8]*

Definition 2.3. In game theory, **Strategies** are essential for identifying the best choices for players in different scenarios. Two key categories of strategies are pure strategies and mixed strategies, both providing distinct methods for decision-making in games. [4]

1. In games of pure strategy, every player decides based on a clear, distinct strategy. A **Pure Strategy** consists of a collection of fixed actions that a player guarantees to execute. There is no randomness or uncertainty in the selection of actions, and players opt for a single strategy from their set of strategies. [6]
2. In mixed strategy games, players incorporate a factor of randomness or probability into their choices. In a **Mixed Strategy** setup, rather than committing to one pure strategy, players allocate probabilities to various strategies within their set. The choice to implement a specific strategy is determined randomly, according to the designated probabilities. [6]

Definition 2.4. A **Matrix Game** is a two-player game, denoted by $G = (\text{Player I}, \text{Player II}, S_1, S_2)$ such that: [5]

Strategy	s_1^2	s_2^2	$\dots$	s_n^2
s_1^1	(a_{11}, b_{11})	(a_{12}, b_{12})	$\dots$	(a_{1n}, b_{1n})
s_2^1	(a_{21}, b_{21})	(a_{22}, b_{22})	$\dots$	(a_{2n}, b_{2n})
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_m^1	(a_{m1}, b_{m1})	(a_{m2}, b_{m2})	$\dots$	(a_{mn}, b_{mn})

Table 1: Matrix of Games

1. Player I has a finite strategy set S_1 with m elements, [3]
2. Player II has a finite strategy set S_2 with n elements, and [3]
3. The payoffs of the players are functions $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$ of the outcomes $(s_1, s_2) \in S_1 \times S_2$.

The matrix game is played as follows: At a certain time Player I chooses a strategy $s_1 \in S_1$ and simultaneously Player II chooses a strategy $s_2 \in S_2$. Once this is done, each player i receives the payoff $u_i(s_1, s_2)$. If $S_1 = \{s_1^1, s_1^2, \dots, s_1^m\}, S_2 = \{s_2^1, s_2^2, \dots, s_2^n\}$ and we put $a_{ij} = u_1(s_1^i, s_2^j), \quad b_{ij} = u_2(s_1^i, s_2^j)$, [8] then the payoffs can be arranged in the form of the $m \times n$ matrix shown in Table 1. Here, $S_1 = \{s_1^1, s_1^2, \dots, s_1^m\}, \quad S_2 = \{s_2^1, s_2^2, \dots, s_2^n\}$ is called the set of all pure strategies available for Player I and Player II respectively.

3. Main Results

3.1. 2pPN Game Using Pure Strategies:

Definition 3.1. Consider S a collection of strategies in a 2pPN game. Suppose $K,\ L \subseteq S$ be the strategies of Player I and Player II respectively and since it is a two person Pentapartitioned Neutrosophic Game (briefly 2pPN game), the strategic form of the 2pPN game is defined as $(K,\ L,\ \mathcal{M})$ where $\mathcal{M}$ is a Pentapartitioned Neutrosohic set over $K \times L$ that is

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\} \tag{3.1}$$

The format stated in (3.1) represents Player I prefers the strategy $k \in K$ and simultaneously Player II prefers the strategy $l \in L$ without knowing the strategies of each other. Also

$$(T_M(k,l),\ C_M(k,l),\ G_M(k,l),\ U_M(k,l),\ F_M(k,l))$$

describes the gains of the Player I at the same time Player II gains the negation that of Player I at the strategy $(k,\ l)$.

Definition 3.2. Consider a 2pPN game $(K,\ L,\ \mathcal{M})$ where $K,\ L$ be sets the strategies of Player I and Player II respectively and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Suppose $K = \{k_1,\ k_2, \dots,\ k_r\}$ and $L = \{l_1,\ l_2, \dots,\ l_s\}$ then the 2pPN game of order $r \times s$ is defined as

$$\mathcal{M} = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1s} \\ m_{21} & m_{22} & \dots & m_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ m_{r1} & m_{r2} & \dots & m_{rs} \end{pmatrix}$$

where

$$m_{ij} = (T_M(k_i, l_j),\ C_M(k_i, l_j),\ G_M(k_i, l_j),\ U_M(k_i, l_j),\ F_M(k_i, l_j))$$

where i varies from 1 to r and j varies from 1 to s.
The matrix $\mathcal{M}$ can also be represent in the tabulation form tabulated as follows:

Players	$\rightarrow$	Player II			
$\downarrow$	Strategies	l_1	l_2	$\dots$	l_s
Player I	k_1	m_{11}	m_{12}	$\dots$	m_{1s}
	k_2	m_{21}	m_{22}	$\dots$	m_{2s}
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	k_r	m_{r1}	m_{r2}	$\dots$	m_{rs}

Table 2: Construction of Payoff Matrix for Player I and Player II

Remark 3.3. The collection of all 2pPN game of order $r \times s$ is formulated as $(2pPN)_{r\times s}$. Also since $(T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l))$ describes the gains of the Player I at the same time Player II gains the negation that of Player I at the strategy (k, l), $(2pPN)_{r\times s}$ describes the gain of player I.

Definition 3.4. Consider a 2pPN game $(K, L, \mathcal{M})$ where K, L be sets the strategies of Player I and Player II respectively and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then (k^*, l^*) is said to be the saddle point of the 2pPN game if the following conditions hold:

1. $\max_{k_i \in K} \{(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j))\}$ $= (T_M(k^*, l^*), C_M(k^*, l^*), G_M(k^*, l^*), U_M(k^*, l^*), F_M(k^*, l^*))$
2. $\min_{l_i \in L} \{(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j))\}$ $= (T_M(k^*, l^*), C_M(k^*, l^*), G_M(k^*, l^*), U_M(k^*, l^*), F_M(k^*, l^*))$

Example 3.5. Consider the strategies $K = \{k_1, k_2, k_3\}$ of Player I and $L = \{l_1, l_2, l_3\}$ of Player II. Then the $(2pPN)_{r\times s}$ is formulated as;

		Player II		
		l_1	l_2	l_3
Player I	k_1	(0.9, 0.8, 0.1, 0.1, 0.1)	(0.7, 0.6, 0.2, 0.5, 0.6)	(0.4, 0.4, 0.3, 0.7, 0.7)
	k_2	(0.8, 0.7, 0.2, 0.3, 0.3)	(0.9, 0.7, 0.1, 0.2, 0.2)	(0.8, 0.6, 0.3, 0.3, 0.4)
	k_3	(0.2, 0.2, 0.8, 0.7, 0.9)	(0.3, 0.2, 0.9, 0.8, 0.9)	(0.4, 0.3, 0.6, 0.7, 0.8)

Table 3: Payoff Matrix of Player I and Player II

Remark 3.6. The saddle point (k^*, l^*) of the 2pPN game represents the least winning strategy of Player I is k^* and the most losing strategy of Player II is l^*. The saddle point of the 2pPN game is often called as the optimal strategy of the 2pPN game.

Example 3.7. Consider the 2pPN game stated in Example 3.5.
Then
Here (0.8, 0.6, 0.3, 0.3, 0.4) is the $\min_{l_i \in L}$ of the second row as well as the $\max_{k_i \in K}$ of the third column.
Thus (k_2, l_3) is the saddle point of the given 2pPN game.

Remark 3.8. Every 2pPN game need not be having Pentapartitioned Neutrosophic saddle point.

Definition 3.9. Consider a 2pPN game $(K, L, \mathcal{M})$ where K, L be sets the strategies of Player I and Player II respectively and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then the Pentapartitioned Neutrosophic Upper Value of 2pPN game is defined as

$$\min_{l_j \in L} \left[\max_{k_i \in K} \{(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j))\} \right]$$

		Player II			
		l_1	l_2	l_3	Row Minimum $(\min_{l_i \in L})$
Player I	k_1	(0.9, 0.8, 0.1, 0.1, 0.1)	(0.7,0.6, 0.2, 0.5, 0.6)	(0.4, 0.4, 0.3, 0.7, 0.7)	(0.4, 0.4, 0.3, 0.7, 0.7)
	k_2	(0.8, 0.7, 0.2, 0.3, 0.3)	(0.9,0.7, 0.1, 0.2, 0.2)	(0.8, 0.6, 0.3, 0.3, 0.4)	(0.8, 0.6, 0.3, 0.3, 0.4)
	k_3	(0.2, 0.2, 0.8, 0.7, 0.9)	(0.3,0.2, 0.9, 0.8, 0.9)	(0.4, 0.3, 0.6, 0.7, 0.8)	(0.2, 0.2, 0.8, 0.7, 0.9)
	Column Maximum $(\max_{k_i \in K})$	(0.9, 0.8, 0.1, 0.1, 0.1)	(0.9,0.7, 0.1, 0.2, 0.2)	(0.8, 0.6, 0.3, 0.3, 0.4)	

Table 4: Payoff Matrix for Player I and Player II with Row Minimum and Column Maximum

and is denoted by V_{up}.
It is also known as the gain floor for the winning of Player I of the 2pPN game.

Definition 3.10. Consider a 2pPN game $(K,\ L,\ \mathcal{M})$ where $K,\ L$ be sets the strategies of Player I and Player II respectively and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then the Pentapartitioned Neutrosophic Lower Value of 2pPN game is defined as

$$\max_{k_i \in K}\left[\min_{l_j \in L}\left\{\left(T_M(k_i,l_j),\ C_M(k_i,l_j),\ G_M(k_i,l_j),\ U_M(k_i,l_j),\ F_M(k_i,l_j)\right)\right\}\right]$$

and is denoted by V_{low}.
It is also known as the loss ceiling for the Player II of the 2pPN game.

Definition 3.11. Consider a 2pPN game $(K,\ L,\ \mathcal{M})$ where $K,\ L$ be sets the strategies of Player I and Player II respectively and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

If

$$\min_{l_j \in L}\left[\max_{k_i \in K}\left\{\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right\}\right]$$
$$= \max_{k_i \in K}\left[\min_{l_j \in L}\left\{\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right\}\right]$$

that is $V_{up} = V_{low}$ then V is called the Pentapartitioned Neutrosophic Value of 2pPN game where $V_{up} = V_{low} = V$.

Example 3.12. Consider the 2pPN game stated in Example 3.5.
From the Table 5 we have $V_{up} = V_{low} = (0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4)$, Hence, (k_2, l_3) is the best strategy for winning of the player I.
Hence the value of the game $V = (0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4)$.

Theorem 3.13. *Consider a 2pPN game* $(K,\ L,\ \mathcal{M})$ *where* $K,\ L$ *be sets the strategies of Player I and Player II respectively and*

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

		Player II				
		l_1	l_2	l_3	Row Minimum $(\min_{l_i \in L})$	$max_{k_i \in K}$ $(\min_{l_i \in L})$
Player I	k_1	(0.9, 0.8, 0.1, 0.1, 0.1)	(0.7,0.6, 0.2, 0.5, 0.6)	(0.4, 0.4, 0.3, 0.7, 0.7)	(0.4, 0.4, 0.3, 0.7, 0.7)	(0.8, 0.6, 0.3, 0.3, 0.4)
	k_2	(0.8, 0.7, 0.2, 0.3, 0.3)	(0.9,0.7, 0.1, 0.2, 0.2)	(0.8, 0.6, 0.3, 0.3, 0.4)	(0.8, 0.6, 0.3, 0.3, 0.4)	
	k_3	(0.2, 0.2, 0.8, 0.7, 0.9)	(0.3,0.2, 0.9, 0.8, 0.9)	(0.4, 0.3, 0.6, 0.7, 0.8)	(0.2, 0.2, 0.8, 0.7, 0.9)	
	Column Maximum $(max_{k_i \in K})$	(0.9, 0.8, 0.1, 0.1, 0.1)	(0.9,0.7, 0.1, 0.2, 0.2)	(0.8, 0.6, 0.3, 0.3, 0.4)		
	$\min_{l_i \in L}(max_{k_i \in K})$	(0.8, 0.6, 0.3, 0.3, 0.4)				

Table 5: Minimax and Maximin

Then $V_{low} \subseteq V_{up}$

Proof: *Consider* V_{low} *and* V_{up} *be the Pentapartitioned Neutrosophic Upper and Lower Value of the 2pPN game* $(K,\ L,\ \mathcal{M})$ *where* $K,\ L$ *be sets the strategies of Player I and Player II respectively and*

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then we have,

$$V_{low} = \max_{k_i \in K}\left[\min_{l_j \in L}\left\{\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right\}\right]$$

Choose any $k_i^{\#} \in K$, *so*

$$\max_{k_i \in K}\left[\min_{l_j \in L}\left\{\left(T_M(k_i,l_j),\ C_M(k_i,l_j),\ G_M(k_i,l_j),\ U_M(k_i,l_j),\ F_M(k_i,l_j)\right)\right\}\right]$$

$$\subseteq \min_{l_j \in L}\left\{\left(T_M(k_i^{\#},\ l_j),\ C_M(k_i^{\#},\ l_j),\ G_M(k_i^{\#},\ l_j),\ U_M(k_i^{\#},\ l_j),\ F_M(k_i^{\#},\ l_j)\right)\right\}$$

Thus,

$$V_{low} \subseteq \min_{l_j \in L}\left\{\left(T_M(k_i^{\#},\ l_j),\ C_M(k_i^{\#},\ l_j),\ G_M(k_i^{\#},\ l_j),\ U_M(k_i^{\#},\ l_j),\ F_M(k_i^{\#},\ l_j)\right)\right\}$$

Choose any $l_j^{\#} \in L$. *Then,*

$$\min_{l_j \in L}\left\{\left(T_M(k_i^{\#},\ l_j),\ C_M(k_i^{\#},\ l_j),\ G_M(k_i^{\#},\ l_j),\ U_M(k_i^{\#},\ l_j),\ F_M(k_i^{\#},\ l_j)\right)\right\}$$

$$\subseteq \left(T_M(k_i^{\#},\ l_j^{\#}),\ C_M(k_i^{\#},\ l_j^{\#}),\ G_M(k_i^{\#},\ l_j^{\#}),\ U_M(k_i^{\#},\ l_j^{\#}),\ F_M(k_i^{\#},\ l_j^{\#})\right)$$

Thus

$$V_{low} \subseteq \left(T_M(k_i^{\#},\ l_j^{\#}),\ C_M(k_i^{\#},\ l_j^{\#}),\ G_M(k_i^{\#},\ l_j^{\#}),\ U_M(k_i^{\#},\ l_j^{\#}),\ F_M(k_i^{\#},\ l_j^{\#})\right)$$

Also,

$$V_{low} \subseteq \max_{k_i \in K}\left[\left(T_M(k_i,\ l_j^{\#}),\ C_M(k_i,\ l_j^{\#}),\ G_M(k_i,\ l_j^{\#}),\ U_M(k_i,\ l_j^{\#}),\ F_M(k_i,\ l_j^{\#})\right)\right]$$

$$V_{low} \subseteq \min_{l_j \in L}\left[\max_{k_i \in K}\left\{\left(T_M(k_i,l_j),\ C_M(k_i,l_j),\ G_M(k_i,l_j),\ U_M(k_i,l_j),\ F_M(k_i,l_j)\right)\right\}\right]$$

Hence $V_{low} \subseteq V_{up}$.

Remark 3.14. From Theorem 3.13 we conclude that in a 2pPN game, V_{low} is not always equal to V_{up}.

Theorem 3.15. *Consider a 2pPN game* $(K, L, \mathcal{M})$ *where* K, L *be sets the strategies of Player I and Player II respectively and*

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then

1. $V_{low} \subseteq \left((k^*,l^*), T_M(k^*,l^*), C_M(k^*,l^*), G_M(k^*,l^*), U_M(k^*,l^*), F_M(k^*,l^*)\right)$
2. $\left((k^*,l^*), T_M(k^*,l^*), C_M(k^*,l^*), G_M(k^*,l^*), U_M(k^*,l^*), F_M(k^*,l^*)\right) \subseteq V_{up}$

where (k^*, l^*) *is the saddle point of the 2pPN game.*

Proof: *Consider* V_{low} *and* V_{up} *be the Pentapartitioned Neutrosophic Upper and Lower Value of the 2pPN game* $(K, L, \mathcal{M})$ *where* K, L *be sets the strategies of Player I and Player II respectively and*

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

And Since (k^*, l^*) *is the saddle point of the 2pPN game.*
Then

$$\begin{aligned}&\max_{k_i\in K}\left\{\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right\}\\ &\quad = \left(T_M(k^*,l^*), C_M(k^*,l^*), G_M(k^*,l^*), U_M(k^*,l^*), F_M(k^*,l^*)\right)\end{aligned} \tag{3.2}$$

And

$$\begin{aligned}&\min_{l_j\in L}\left\{\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right\}\\ &\quad = \left(T_M(k^*,l^*), C_M(k^*,l^*), G_M(k^*,l^*), U_M(k^*,l^*), F_M(k^*,l^*)\right)\end{aligned} \tag{3.3}$$

Now,

1. V_{low}

$$\begin{aligned}&= \max_{k_i\in K}\left[\min_{l_j\in L}\left\{\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right\}\right]\\ &\subseteq \max_{k_i\in K}\left[\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right]\end{aligned}$$

Then by (3.2)

$$= (T_M(k^*, l^*), C_M(k^*, l^*), G_M(k^*, l^*), U_M(k^*, l^*), F_M(k^*, l^*))$$

Hence
$V_{low} \subseteq ((k^*,l^*), T_M(k^*,l^*), C_M(k^*,l^*), G_M(k^*,l^*), U_M(k^*,l^*), F_M(k^*,l^*))$

2. V_{up}

$$\begin{aligned}&= \min_{l_j\in L}\left[\max_{k_i\in K}\left\{\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right\}\right]\\ &\supseteq \min_{l_j\in L}\left[\left(T_M(k_i,l_j), C_M(k_i,l_j), G_M(k_i,l_j), U_M(k_i,l_j), F_M(k_i,l_j)\right)\right]\end{aligned}$$

Then by (3.3)

$$= (T_M(k^*, l^*), C_M(k^*, l^*), G_M(k^*, l^*), U_M(k^*, l^*), F_M(k^*, l^*))$$

Hence,
$\left((k^*,l^*), T_M(k^*,l^*), C_M(k^*,l^*), G_M(k^*,l^*), U_M(k^*,l^*), F_M(k^*,l^*)\right) \subseteq V_{up}$

Corollary 3.16. *Consider a 2pPN game* $(K,\ L,\ \mathcal{M})$ *where* $K,\ L$ *be sets the strategies of Player I and Player II respectively and*

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

And suppose $V_{up} = V_{low} = V$ *then*

$$V = ((k^*,l^*), T_M(k^*,l^*), C_M(k^*,l^*), G_M(k^*,l^*), U_M(k^*,l^*), F_M(k^*,l^*))$$

where (k^*,l^*) *is the saddle point of the 2pPN game.*

Definition 3.17. Consider a 2pPN game $(K,\ L,\ \mathcal{M})$ where $K,\ L$ be sets the strategies of Player I and Player II respectively and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then the domination on strategies are defined as follows:
On strategies of Player I: If maximum of

$$\left(T_M\left(k_x,\ l_j\right),\ C_M\left(k_x,\ l_j\right),\ G_M\left(k_x,\ l_j\right),\ U_M\left(k_x,\ l_j\right),\ F_M\left(k_x,\ l_j\right)\right)$$

and

$$\left(T_M\left(k_y,\ l_j\right),\ C_M\left(k_y,\ l_j\right),\ G_M\left(k_y,\ l_j\right),\ U_M\left(k_y,\ l_j\right),\ F_M\left(k_y,\ l_j\right)\right)$$

is equal to

$$\left(T_M\left(k_x,\ l_j\right),\ C_M\left(k_x,\ l_j\right),\ G_M\left(k_x,\ l_j\right),\ U_M\left(k_x,\ l_j\right),\ F_M\left(k_x,\ l_j\right)\right)$$

for all strategy $l_j \in L$ then we defined that the strategy k_x is **dominates** the strategy k_y.
On strategies of Player II: If minimum of

$$\left(T_M\left(k_i,\ l_x\right),\ C_M\left(k_i,\ l_x\right),\ G_M\left(k_i,\ l_x\right),\ U_M\left(k_i,\ l_x\right),\ F_M\left(k_i,\ l_x\right)\right)$$

and

$$\left(T_M\left(k_i,\ l_y\right),\ C_M\left(k_i,\ l_y\right),\ G_M\left(k_i,\ l_y\right),\ U_M\left(k_i,\ l_y\right),\ F_M\left(k_i,\ l_y\right)\right)$$

is equal to

$$\left(T_M\left(k_i,\ l_x\right),\ C_M\left(k_i,\ l_x\right),\ G_M\left(k_i,\ l_x\right),\ U_M\left(k_i,\ l_x\right),\ F_M\left(k_i,\ l_x\right)\right)$$

for all strategy $k_i \in K$ then we defined that the strategy l_x is **dominates** the strategy l_y.

Example 3.18. Illustrative example of solving a 2pPN Game using the domination of strategies:
In this process, we can reduce a 2pPN game to get the optimal strategy by removing the dominated strategies. This elimination process of rows and columns makes way to the best strategy of Player I in a 2pPN game.
Consider the 2pPN game stated in Example 3.5
We can write Table 3 as,

$$\mathcal{M} = \begin{pmatrix} (0.9,\ 0.8,\ 0.1,\ 0.1,\ 0.1) & (0.7,\ 0.6,\ 0.2,\ 0.5,\ 0.6) & (0.4,\ 0.4,\ 0.3,\ 0.7,\ 0.7) \\ (0.8,\ 0.7,\ 0.2,\ 0.3,\ 0.3) & (0.9,\ 0.7,\ 0.1,\ 0.2,\ 0.2) & (0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4) \\ (0.2,\ 0.2,\ 0.8,\ 0.7,\ 0.9) & (0.3,\ 0.2,\ 0.9,\ 0.8,\ 0.9) & (0.4,\ 0.3,\ 0.6,\ 0.7,\ 0.8) \end{pmatrix}$$

Since the row for strategy K_1 is dominates the row for strategy K_3. Thus we eliminate row 3 in $\mathcal{M}$. Then

$$\mathcal{M} = \begin{pmatrix} (0.9,\ 0.8,\ 0.1,\ 0.1,\ 0.1) & (0.7,\ 0.6,\ 0.2,\ 0.5,\ 0.6) & (0.4,\ 0.4,\ 0.3,\ 0.7,\ 0.7) \\ (0.8,\ 0.7,\ 0.2,\ 0.3,\ 0.3) & (0.9,\ 0.7,\ 0.1,\ 0.2,\ 0.2) & (0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4) \end{pmatrix}$$

Since the column for strategy l_3 is dominates the column for strategy l_1. Hence we eliminate column 1 in $\mathcal{M}$. Then

$$\mathcal{M} = \begin{pmatrix} (0.7,\ 0.6,\ 0.2,\ 0.5,\ 0.6) & (0.4,\ 0.4,\ 0.3,\ 0.7,\ 0.7) \\ (0.9,\ 0.7,\ 0.1,\ 0.2,\ 0.2) & (0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4) \end{pmatrix}$$

Since the row for strategy K_2 is dominates the row for strategy K_1. Thus we eliminate row 1 in $\mathcal{M}$. Then

$$\mathcal{M} = \begin{pmatrix} (0.9,\ 0.7,\ 0.1,\ 0.2,\ 0.2) & (0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4) \end{pmatrix}$$

Since the column for strategy l_3 is dominates the column for strategy l_1. Hence we eliminate column 1 in $\mathcal{M}$. Thus

$$\mathcal{M} = \begin{pmatrix} (0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4) \end{pmatrix}$$

Hence we conclude that $(0.8,\ 0.6,\ 0.3,\ 0.3,\ 0.4)$ is the optimal value of the 2pPN game for the Players I and II choosing the strategies k_2 and l_3 respectively.

3.2. 2pPN Game Using Mixed Strategies:

Definition 3.19. Consider a 2pPN game $(K,\ L,\ \mathcal{M})$, where K be the set of mixed strategies of Player I and L be the set of mixed strategies of Player II and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Suppose $K = \{k_1,\ k_2, \ldots,\ k_r\}$ and $L = \{l_1,\ l_2, \ldots,\ l_s\}$ then the 2pPN game of order $r \times s$ is defined as $\mathcal{M} = \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1s} \\ m_{21} & m_{22} & \ldots & m_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ m_{r1} & m_{r2} & \ldots & m_{rs} \end{pmatrix}$ such that

$$m_{ij} = \left(T_M(k_i, l_j),\ C_M(k_i, l_j),\ G_M(k_i, l_j),\ U_M(k_i, l_j),\ F_M(k_i, l_j)\right)$$

where i varies from 1 to r and j varies from 1 to s.
Also the expected payoff values of the Players is calculated by, $E(k,l) = k^t \mathcal{M} l$.
That is

$$\begin{aligned} E(k,l) = & \Bigg[1 - \prod_{l_j \in L} \prod_{k_i \in K} \left(1 - T_M(k_i, l_j)^{k_i l_j}\right), 1 - \prod_{l_j \in L} \prod_{k_i \in K} \left(1 - C_M(k_i, l_j)^{k_i l_j}\right), \\ & \prod_{l_j \in L} \prod_{k_i \in K} \left(G_M(k_i, l_j)^{k_i l_j}\right), \prod_{l_j \in L} \prod_{k_i \in K} \left(U_M(k_i, l_j)^{k_i l_j}\right), \prod_{l_j \in L} \prod_{k_i \in K} \left(F_M(k_i, l_j)^{k_i l_j}\right)\Bigg] \end{aligned}$$

Definition 3.20. Consider a 2pPN game $(K,\ L,\ \mathcal{M})$, where K be the set of mixed strategies of Player I and L be the set of mixed strategies of Player II and

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then the saddle point of the mixed strategies $K,\ L$ is denoted as $(x_*,\ y_*)$ and the expected value of the $(x_*,\ y_*)$ is defined as; $E(x_*,\ y_*) = \max_{k \in K} [\min_{l \in L} E(k,l)] = \min_{l \in L} [\max_{k \in K} E(k,l)]$.

Remark 3.21. Literally $E(x_*,\ y_*)$ describes that Player I will maximize his/her expectation by choosing a considerable k and Player II will minimize the maximized expected value of Player I by choosing a considerable l.

Proposition 3.22. ***Derivation of Maxi-min strategy of Player I***
Consider a 2pPN game $(K, L, \mathcal{M})$*, where* K *be the set of mixed strategies of Player I and* L *be the set of pure strategies of Player II and*

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

Then

$$E(k_i, l_j) = \Big[1 - \prod_{k_i \in K}\left(1 - T_M(k_i,l_j)^{k_i}\right), 1 - \prod_{k_i \in K}\left(1 - C_M(k_i,l_j)^{k_i}\right), \prod_{k_i \in K}\left(G_M(k_i,l_j)^{k_i}\right), \prod_{k_i \in K}\left(U_M(k_i,l_j)^{k_i}\right), \prod_{k_i \in K}\left(F_M(k_i,l_j)^{k_i}\right)\Big]$$

Now a minimization is made by Player II on the Expectation $E(k_i, l_j)$*, and is denoted by* E_1*. Then*

$$E_1 = \Big[\min_{l_i \in L}\left(1 - \prod_{k_i \in K}\left(1 - T_M(k_i,l_j)^{k_i}\right)\right), \min_{l_j \in L}\left(1 - \prod_{k_i \in K}\left(1 - C_M(k_i,l_j)^{k_i}\right)\right), \max_{l_j \in L}\left(\prod_{k_i \in K}\left(G_M(k_i,l_j)^{k_i}\right)\right), \max_{l_j \in L}\left(\prod_{k_i \in K}\left(F_M(k_i,l_j)^{k_i}\right)\right), \max_{l_j \in L}\left(\prod_{k_i \in K}\left(F_M(k_i,l_j)^{k_i}\right)\right)\Big]$$

Then the Player I make the maximization by using the mixed strategy on E_1 *and is denoted by* E_1^**. Thus*

$$E_1^* = \Big[\max_{k_i \in K}\left(\min_{l_i \in L}\left(1 - \prod_{k_i \in K}\left(1 - T_M(k_i,l_j)^{k_i}\right)\right)\right), \max_{k_i \in K}\left(\min_{l_i \in L}\left(1 - \prod_{k_i \in K}\left(1 - C_M(k_i,l_j)^{k_i}\right)\right)\right), \min_{l_i \in L}\left(\max_{l_j \in L}\left(\prod_{k_i \in K}\left(G_M(k_i,l_j)^{k_i}\right)\right)\right), \min_{l_i \in L}\left(\max_{l_j \in L}\left(\prod_{k_i \in K}\left(U_M(k_i,l_j)^{k_i}\right)\right)\right), \min_{l_i \in L}\left(\max_{l_j \in L}\left(\prod_{k_i \in K}\left(F_M(k_i,l_j)^{k_i}\right)\right)\right)\Big]$$

Hence the mixed strategy E_1^* *is known as the Maxi-min Strategy of Player I.*

Proposition 3.23. ***Derivation of Mini-max strategy of Player II***
Consider a 2pPN game $(K, L, \mathcal{M})$*, where* K *be the set of pure strategies of Player I and* L *be the set of mixed strategies of Player II and*

$$\mathcal{M} = \{((k,l), T_M(k,l), C_M(k,l), G_M(k,l), U_M(k,l), F_M(k,l)) / (k,l) \in K \times L\}$$

. Then

$$E(k_i, l_j) = \Big[1 - \prod_{l_j \in L}\left(1 - T_M(k_i,l_j)^{l_j}\right), 1 - \prod_{l_j \in L}\left(1 - C_M(k_i,l_j)^{l_j}\right), \prod_{l_j \in L}\left(G_M(k_i,l_j)^{l_j}\right), \prod_{l_j \in L}\left(U_M(k_i,l_j)^{l_j}\right), \prod_{l_j \in L}\left(F_M(k_i,l_j)^{l_j}\right)\Big]$$

Now a maximization is made by Player I by using a mixed strategy on the Expectation $E(k_i, l_j)$, *and is denoted by* E_2. *Then*

$$E_2 = \Bigg[\max_{k_i \in K}\left(1 - \prod_{l_j \in L}\left(1 - T_M(k_i, l_j)^{l_j}\right)\right), \max_{k_i \in K}\left(1 - \prod_{l_j \in L}\left(1 - C_M(k_i, l_j)^{l_j}\right)\right),$$
$$\min_{k_i \in K}\left(\prod_{l_j \in L}\left(G_M(k_i, l_j)^{l_j}\right)\right), \min_{k_i \in K}\left(\prod_{l_j \in L}\left(U_M(k_i, l_j)^{l_j}\right)\right),$$
$$\min_{k_i \in K}\left(\prod_{l_j \in L}\left(F_M(k_i, l_j)^{l_j}\right)\right)\Bigg]$$

Then the Player II make the minimization by using the pure strategy on E_1 *and is denoted by* E_1^*. *Thus*

$$E_2^* = \Bigg[\min_{l_i \in L}\left(\max_{k_i \in K}\left(1 - \prod_{l_j \in L}\left(1 - T_M(k_i, l_j)^{l_j}\right)\right)\right),$$
$$\min_{l_i \in L}\left(\max_{k_i \in K}\left(1 - \prod_{l_j \in L}\left(1 - C_M(k_i, l_j)^{l_j}\right)\right)\right), \max_{l_j \in L}\left(\min_{k_i \in K}\left(\prod_{l_j \in L}\left(G_M(k_i, l_j)^{l_j}\right)\right)\right),$$
$$\max_{l_j \in L}\left(\min_{k_i \in K}\left(\prod_{l_j \in L}\left(U_M(k_i, l_j)^{l_j}\right)\right)\right), \max_{l_j \in L}\left(\min_{k_i \in K}\left(\prod_{l_j \in L}\left(F_M(k_i, l_j)^{l_j}\right)\right)\right)\Bigg]$$

Hence the mixed strategy E_2^* *is known as the Mini-max Strategy of Player II.*

Remark 3.24. In a 2pPN game, we have $E_1 \subseteq E_2$.

4. Conclusion

In this research article we dealt with the Game Theoritical Approach of Pentapartitioned Neutrosophic Sets through the Two-Person Pentapartitioned Neutrosophic Games. The analysis of two-person PN games reveals that the inclusion of PN parameters enhances the flexibility of equilibrium determination, particularly when crisp payoffs are insufficient to describe the player's strategic behaviours. Further research can foucs on multi-person PN games, dynamic PN interactions and algorithmic methods for computing PN euqilibria under varying decision environments. Such extensions would further strengthen the applicability of PN game theory in economics, management and decision making theories.

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