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# Advancing Uncertainty analysis: Nano Fuzzy Frontier, Border and Exterior sets in Nano Fuzzy Topological spaces

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## Abstract

This research study focuses on the notions of nano fuzzy frontier, border and exterior sets, which play an important role in understanding the topological and algebraic aspects of fuzzy spaces. The Nano fuzzy J-open set is the new type of open set which was utilizing Nano fuzzy semi-interior and closure operators. In this study, we define and examine the Nano fuzzy J frontier, border, and exterior of the Nano fuzzy J open set, as well as the essential features and interactions between them.

**Keywords:** Nano fuzzy topolgy, Nano fuzzy J open set, Nano fuzzy J frontier, Nano fuzzy J border, Nano fuzzy J exterior

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## 1. Introduction

In 1965, fuzzy sets [12] were introduced by Lotfi A. Zadeh as a framework for modeling uncertainty, ambiguity, and imprecision. Fuzzy sets are mathematical structures that expand traditional sets by enabling items to have degrees of membership rather than binary membership (0 or 1, yes or no). Fuzzy topology [3] extends classical topology by including the idea of fuzziness or ambiguity within the definition of topological structures. Lellis Thivagar presented Nano topology [7] around 2013, an extension of rough set theory [9] that defines topological spaces based on approximations and boundaries on a subset in a universe utilising an equivalence relation.

Nano fuzzy topological spaces [8] are a mathematical notion that combines fuzzy topology with nano topology, two emerging areas of study. This fusion seeks to give a more complete framework for comprehending and analyzing complex systems, particularly at the nanoscale. Nano fuzzy topological spaces are valuable in disciplines including decision-making, data processing, artificial intelligence, and modeling of complex systems, when systems cannot be characterized with clear-cut boundaries and instead require a more subtle, fuzzy approach. Weaker notions of open sets have been investigated in nano fuzzy topological spaces [10, 11]. Additionally, we have introduced a new weaker notion of open set, called the nano fuzzy J open set [5], and studied its properties.

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Generally the concepts of frontier, border and exterior are partition the space and analyze the position and behavior of elements within and around a set. They are particularly helpful for investigating continuity, convergence, and separation features, and they are fundamental concepts in advanced mathematical research and real-world applications such as geographical analysis, data clustering, and decision-making models [1]. Provides a powerful analytical framework for applications in situations where decisions must be made under high uncertainty, incomplete information, or conflicting criteria. The concepts of frontier, border, and exterior of weaker notions of open sets have already been studied in various topological spaces [2, 4, 6]. In this paper, we define the frontier, border, and exterior of nano fuzzy J-open sets and discuss their properties.

## 2. Preliminaries

**Definition 2.1.** [12]Let $X$ be a universe of discourse and $x$ be any particular element of $X$. The **fuzzy set** $A$ defined on $X$ is a collection of ordered pairs,

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

where $\mu_A(x) : X \to [0, 1]$ is called the membership function.

**Definition 2.2.** [12]Suppose $I$ and $J$ are two fuzzy subsets (simply, $F_Y$ subset) in a set $Y$, then

(i) $I \leqslant J \iff \mu_I(q) \leqslant \mu_J(q), \quad \forall q \in Y.$
(ii) $I = J \iff I(q) = J(q), \quad \forall q \in Y.$
(iii) $(I \vee J)(q) = \max\{I(q), J(q)\}, \quad \forall q \in Y.$
(iv) $(I \wedge J)(q) = \min\{I(q), J(q)\}, \quad \forall q \in Y.$

**Definition 2.3.** [8]Let $R$ be an equivalence relation on a non-empty set $S$. With the membership function $\mu_F$, let $K$ be a $F_Y$ subset of $S$. The Nano fuzzy lower (upper) approximations and Nano fuzzy boundary of $K$ in the approximation $(S, R)$, denoted by $\underline{NF_z}(K)$, $\overline{NF_z}(K)$, and $NF_zB(K)$, are each defined as below:

$$\underline{NF_z}(K) = \left\{(k, \mu_{\underline{R}(A)}(k)) \mid y \in [k]_R,\ k \in S\right\},$$

$$\overline{NF_z}(K) = \left\{(k, \mu_{\overline{R}(A)}(k)) \mid y \in [k]_R,\ k \in S\right\},$$

$$NF_zB(K) = \overline{NF_z}(K) - \underline{NF_z}(K),$$

where

$$\mu_{\underline{R}(A)}(k) = \bigwedge_{y \in [k]_R} \mu_A(y), \quad \text{and} \quad \mu_{\overline{R}(A)}(k) = \bigvee_{y \in [k]_R} \mu_A(y).$$

The collection $\tau_R(K) = \left\{0_F,\ 1_F,\ \underline{NF_z}(K),\ \overline{NF_z}(K),\ NF_zB(K)\right\}$ produces a topology known as the Nano fuzzy topology, and $(S, \tau_R(K))$ is called a **Nano fuzzy topological space** (simply, $NF_YTS$). The members of $\tau_R(K)$ are known as **Nano fuzzy open sets** (simply, $NF_Yos$). Members of $\tau_{R^c}(K)$ are known as Nano fuzzy closed sets (simply, $NF_Ycs$).

**Definition 2.4.** [8]Suppose $(S, \tau_R(K))$ is a $NF_Y$ topological space. Let $\gamma$ be a $F_Y$ subset of $S$. Then:

(i) The **Nano fuzzy interior** of $\gamma$ (simply, $NF_Y\operatorname{int}(\gamma)$) is described as $NF_Y\operatorname{int}(\gamma) = \bigvee\{\beta \mid \beta \leqslant \gamma \text{ and } \beta \text{ is a } NF_Yos\}$.
(ii) The **Nano fuzzy closure** of $\gamma$ (simply, $NF_Y\operatorname{cl}(\gamma)$) is described as $NF_Y\operatorname{cl}(\gamma) = \bigwedge\{\beta \mid \gamma \leqslant \beta \text{ and } \beta \text{ is a } NF_Ycs\}$.

**Definition 2.5.** [5]Suppose $(S, \tau_R(K))$ is a $NF_YTS$. Let $\gamma$ be a $F_Y$ subset of $S$. Then $\gamma$ is defined as

(i) A set $\gamma$ is called a **Nano fuzzy J open set** (simply, $NF_YJos$) if $\gamma \leqslant NF_Y\operatorname{scl}(NF_Y\operatorname{sint}(\gamma))$.

(ii) A set $\gamma$ is called a **Nano fuzzy J closed set** (simply, $NF_Y Jcs$) if $NF_Y \, sint\,(NF_Y \, scl(\gamma)) \leqslant \gamma$.

**Definition 2.6.** [5]

(i) The **Nano fuzzy J interior** of $\gamma$ (simply, $NF_Y \, Jint(\gamma)$) is described as the union of all $NF_Y Jos$'s contained in $\gamma$. That is, $NF_Y \, Jint(\gamma) = \bigvee\{\beta \mid \beta \leqslant \gamma$ and $\beta$ is a $NF_Y Jos\}$.

(ii) The **Nano fuzzy J closure** of $\gamma$ (simply, $NF_Y \, Jcl(\gamma)$) is described as the intersection of all $NF_Y Jcs$'s that contain $\gamma$. That is, $NF_Y \, Jcl(\gamma) = \bigwedge\{\beta \mid \gamma \leqslant \beta$ and $\beta$ is a $NF_Y Jcs\}$.

*Remark* 2.7. [5]For any $\alpha \in NF_Y Ts(S, \tau_R(K))$, we have $(NF_Y Jcl(\alpha))^c = NF_Y Jint(\alpha^c)$ and $(NF_Y Jint(\gamma))^c = NF_Y Jcl(\gamma^c)$.

**Proposition 2.8.** *[5]Let $(S, \tau_R(K))$ be a $NF_Y TS$ and $\alpha \in (S, \tau_R(K))$. Then, for any $F_Y$ subset $\beta \leqslant \alpha$, the following hold:*

- $NF_Y \, int(\beta) \leqslant NF_Y \, Jint(\beta) \leqslant NF_Y \, sint(\beta)$.
- $NF_Y \, cl(\beta) \geqslant NF_Y \, Jcl(\beta) \geqslant NF_Y \, scl(\beta)$.

**Theorem 2.9.** *[5]Let $\alpha$ be a $F_Y$ subset of a $NF_Y TS$ $(S, \tau_R(K))$. Then*

(i) *$\alpha$ is a $NF_Y Jos$ if and only if $\alpha = NF_Y \, Jint(\alpha)$.*
(ii) *$\alpha$ is a $NF_Y Jcs$ if and only if $\alpha = NF_Y \, Jcl(\alpha)$.*

## 3. Nano Fuzzy J Frontier

**Definition 3.1.** Let $(S, \tau_R(K))$ be a $NF_Y Ts$. Let $\mu$ be a $F_Y$ subset of $S$. Then the **Nano fuzzy J frontier** of $\mu$, denoted by $NF_Y \, JFr(\mu)$, is defined by:

$$NF_Y \, JFr(\mu) = NF_Y \, Jcl(\mu) \wedge NF_Y \, Jcl(\mu^c).$$

**Example 3.2.** Assume $S = \{q, r, s, t\}$ be the universal set, and the equivalence relation is given by: $S/R = \{\{q, s\}, \{r, t\}\}$. Let $K = \{q_{0.41}, \ r_{0.32}, \ s_{0.63}, \ t_{0.54}\}$ be a $F_Y$ subset of $S$.

The Nano fuzzy lower and upper approximations of $K$, and the Nano fuzzy boundary are: $\underline{NF_z}(K) = \{q_{0.41}, \ r_{0.32}, \ s_{0.41}, \ t_{0.32}\}$, $\overline{NF_z}(K) = \{q_{0.63}, \ r_{0.54}, \ s_{0.63}, \ t_{0.54}\}$, $NF_z B(K) = \{q_{0.22}, \ r_{0.22}, \ s_{0.22}, \ t_{0.22}\}$. Thus, the collection of Nano fuzzy open sets is, $NF_Y os = \{0_F, \ 1_F, \ \underline{NF_z}(K), \ \overline{NF_z}(K), \ NF_z B(K)\}$. The collection of Nano fuzzy J closed sets is,

$$NF_Y \, Jcs = \{0_F, \ 1_F, \ \underline{NF_z}(K), \ \overline{NF_z}(K), \ \underline{NF_z}^c(K), \ \overline{NF_z}^c(K), \ NF_z B^c(K), \\ q_{(0.59-0.78)}, \ r_{(0.68-0.78)}, \ s_{(0.59-0.78)}, \ t_{(0.68-0.78)}\}.$$

Let $\mu = \{q_{0.32}, \ r_{0.22}, \ s_{0.38}, \ t_{0.26}\}$, $\mu^c = \{q_{0.68}, \ r_{0.78}, \ s_{0.62}, \ t_{0.74}\}$. Then $NF_Y \, Jcl(\mu) = \{q_{0.41}, \ r_{0.32}, \ s_{0.41}, \ t_{0.32}\}$, $NF_Y \, Jcl(\mu^c) = \{q_{0.68}, \ r_{0.78}, \ s_{0.62}, \ t_{0.74}\}$. Therefore, the Nano fuzzy J frontier of $\mu$ is $NF_Y JFr(\mu) = \{q_{0.41}, \ r_{0.32}, \ s_{0.41}, \ t_{0.32}\}$.

*Remark* 3.3. For a $F_Y$ subset $\mu$ of $S$, $NF_Y JFr(\mu)$ is $NF_Y Jcs$.

**Theorem 3.4.** *For a $F_Y$ subset $\mu \in NF_Y Ts$ $(S, \tau_R(K))$, we have $NF_Y JFr(\mu) = NF_Y JFr(\mu^c)$.*

*Proof.* Let $\mu$ be a $F_Y$ subset in $NF_Y Ts$ $(S, \tau_R(K))$. Then by Definition 3.1,

$$\begin{aligned} NF_Y JFr(\mu) &= NF_Y Jcl(\mu) \wedge NF_Y Jcl(\mu^c) \\ &= NF_Y Jcl(\mu^c) \wedge NF_Y Jcl(\mu) \\ &= NF_Y Jcl(\mu^c) \wedge NF_Y Jcl((\mu^c)^c) \\ &= NF_Y JFr(\mu^c) \quad \text{(by Definition 3.1)}. \end{aligned}$$

Hence, $NF_Y JFr(\mu) = NF_Y JFr(\mu^c)$. □

**Theorem 3.5.** *Let* $\mu$ *be a* $F_Y$ *subset in* $NF_YTs\ (S, \tau_R(K))$. *Then* $NF_YJFr(\mu) = NF_YJcl(\mu) - NF_YJint(\mu)$.

*Proof.* Let $\mu$ be a $NF_Y$os. Then $\mu$ is also a $NF_Y$Jos. By Definition 3.1,
$NF_YJFr(\mu) = NF_YJcl(\mu) \wedge NF_YJcl(\mu^c)$. By Remark 2.7, we have $NF_YJcl(\mu^c) = (NF_YJint(\mu))^c$.
Therefore,

$$\begin{aligned} NF_YJFr(\mu) &= NF_YJcl(\mu) \wedge (NF_YJint(\mu))^c \\ &= NF_YJcl(\mu) - NF_YJint(\mu) \quad (\text{using } H \wedge L^c = H - L). \end{aligned}$$

□

**Theorem 3.6.** *A* $F_Y$ *subset* $\mu_0$ *is a* $NF_YJcs$ *in* $S$ *if and only if* $NF_YJFr(\mu_0) \leqslant \mu_0$.

*Proof.* Let $\mu_0$ be a $NF_YJcs$ in the $NF_YTs(S, \tau_R(K))$. Then by Definition 3.1 and the theorem 3.5, $NF_YJFr(\mu_0) = NF_YJcl(\mu_0) \wedge NF_YJcl(\mu_0^c) = NF_YJcl(\mu_0) - NF_YJint(\mu_0)$. Since $\mu_0$ is a $NF_YJcs$, we have $NF_YJcl(\mu_0) = \mu_0$. Therefore, $NF_YJFr(\mu_0) = \mu_0 - NF_YJint(\mu_0) \leqslant \mu_0$. Hence, $NF_YJFr(\mu_0) \leqslant \mu_0$.

Conversely, assume that $NF_YJFr(\mu_0) \leqslant \mu_0$. Then, $NF_YJcl(\mu_0) - NF_YJint(\mu_0) \leqslant \mu_0$. Since $NF_YJint(\mu_0) \leqslant \mu_0$, we conclude that $NF_YJcl(\mu_0) = \mu_0$. Hence, $\mu_0$ is a $NF_YJcs$. □

**Theorem 3.7.** *Let* $\mu$ *be a* $F_Y$ *subset in* $S$. *If* $\mu$ *is a* $NF_YJos$, *then* $NF_YJFr(\mu) \leqslant \mu^c$.

*Proof.* Let $\mu$ be a $NF_YJos$ in $NF_YTs\ (S, \tau_R(K))$. Then $\mu^c$ is a $NF_YJcs$. By theorem 3.6, $NF_YJFr(\mu^c) \leqslant \mu^c$. Also by Theorem 3.4, we get $NF_YJFr(\mu) \leqslant \mu^c$. □

*Remark* 3.8. The example below demonstrates that the converse of the above theorem need not be true

**Example 3.9.** From Example 3.2, let $\mu = \{q_{0.32}, r_{0.22}, s_{0.38}, t_{0.26}\}$, $\mu^c = \{q_{0.68}, r_{0.78}, s_{0.62}, t_{0.74}\}$, and $NF_YJFr(\mu) = \{q_{0.41}, r_{0.32}, s_{0.41}, t_{0.32}\}$. Here, $NF_YJFr(\mu) \leqslant \mu^c$, but $\mu$ is not a $NF_YJos$.

**Theorem 3.10.** *Let* $\alpha \leqslant \beta$ *and* $\beta$ *be any* $NF_YJcs$ *in* $S$. *Then* $NF_YJFr(\alpha) \leqslant \beta$.

*Proof.* We have $\alpha \leqslant \beta \Rightarrow NF_YJcl(\alpha) \leqslant NF_YJcl(\beta)$. By definition of $NF_YJFr$, $NF_YJFr(\alpha) = NF_YJcl(\alpha) \wedge NF_YJcl(\alpha^c) \leqslant NF_YJcl(\beta) \wedge NF_YJcl(\alpha^c) \leqslant NF_YJcl(\beta)$. Then by theorem 2.9, $NF_YJcl(\beta) = \beta$. Hence, $NF_YJFr(\alpha) \leqslant \beta$. □

**Theorem 3.11.** *Let* $\mu$ *be a* $NF_Y$*os in* $NF_YTs\ (S, \tau_R(K))$. *Then* $(NF_YJFr(\mu))^c = NF_YJint(\mu) \vee NF_YJint(\mu^c)$.

*Proof.* Let $\mu$ be a $NF_Y$os in $NF_YTs\ (S, \tau_R(K))$. Then by definition of $NF_YJFr$,
$(NF_YJFr(\mu))^c = (NF_YJcl(\mu) \wedge NF_YJcl(\mu^c))^c = (NF_YJcl(\mu))^c \vee (NF_YJcl(\mu^c))^c$.

By remark 2.7, we have $NF_YJcl(\mu^c) = (NF_YJint(\mu))^c$. Therefore, $(NF_YJFr(\mu))^c = NF_YJint(\mu^c) \vee NF_YJint(\mu)$. □

**Theorem 3.12.** *For a* $F_Y$ *subset* $\mu$ *in* $NF_YTs\ (S, \tau_R(K))$, *we have* $NF_YJFr(\mu) \leqslant NF_YFr(\mu)$.

*Proof.* Let $\mu$ be a $F_Y$ subset in $NF_YTs\ (S, \tau_R(K))$. Then by proposition 2.8, $NF_YJcl(\mu) \leqslant NF_Ycl(\mu)$ and $NF_YJcl(\mu^c) \leqslant NF_Ycl(\mu^c)$. By definition, $NF_YJFr(\mu) = NF_YJcl(\mu) \wedge NF_YJcl(\mu^c) \leqslant NF_Ycl(\mu) \wedge NF_Ycl(\mu^c) = NF_YFr(\mu)$. Hence, $NF_YJFr(\mu) \leqslant NF_YFr(\mu)$. □

**Theorem 3.13.** *For a* $F_Y$ *subset* $\mu$ *in* $NF_YTs\ (S, \tau_R(K))$, *we have* $NF_YJcl\,(NF_YJFr(\mu)) \leqslant NF_YJFr(\mu)$.

*Proof.* Let $\mu$ be a $F_Y$ subset in $NF_YTs\ (S, \tau_R(K))$. Then by definition,

$$\begin{aligned} NF_YJcl\,(NF_YJFr(\mu)) &= NF_YJcl\,(NF_YJcl(\mu) \wedge NF_YJcl(\mu^c)) \\ &\leqslant NF_YJcl\,(NF_YJcl(\mu)) \wedge NF_YJcl\,(NF_YJcl(\mu^c)) \\ &\leqslant NF_YJcl(\mu) \wedge NF_YJcl(\mu^c) \\ &= NF_YJFr(\mu) \quad (\text{by definition of } NF_YJcl). \end{aligned}$$

Hence, $NF_YJcl(NF_YJFr(\mu)) \leqslant NF_YJFr(\mu)$. □

**Theorem 3.14.** *For a* $F_Y$ *subset* $\mu$ *in* $NF_YTs\ (S, \tau_R(K))$, *we have* $NF_YJFr\,(NF_YJint(\mu)) \leqslant NF_YJFr(\mu)$.

*Proof.* Let $\mu$ be a $F_Y$ subset in $NF_YTs\ (S, \tau_R(K))$. Then by definition of $NF_YJFr$,

$$\begin{aligned} NF_YJFr\,(NF_YJint(\mu)) &= NF_YJcl\,(NF_YJint(\mu)) \wedge NF_YJcl\,((NF_YJint(\mu))^c) \\ &= NF_YJcl\,(NF_YJint(\mu)) \wedge NF_YJcl\,(NF_YJcl(\mu^c)) \\ &\leqslant NF_YJcl(\mu) \wedge NF_YJcl(\mu^c) \\ &= NF_YJFr(\mu). \quad (\because NF_YJcl(NF_YJint(\mu)) \leqslant NF_YJcl(\mu)) \end{aligned}$$

Hence, $NF_YJFr(NF_YJint(\mu)) \leqslant NF_YJFr(\mu)$. □

**Theorem 3.15.** *For a* $F_Y$ *subset* $\mu$ *in* $NF_YTs\ (S, \tau_R(K))$, *we have:* $NF_YJFr(NF_YJcl(\mu)) \leqslant NF_YJFr(\mu)$.

*Proof.* Let $\mu$ be a $F_Y$ subset in $NF_YTs\ (S, \tau_R(K))$. Then by definition of $NF_YJFr$

$$\begin{aligned} NF_YJFr(NF_YJcl(\mu)) &= NF_YJcl(NF_YJcl(\mu)) \wedge NF_YJcl((NF_YJcl(\mu))^c) \\ &= NF_YJcl(\mu) \wedge NF_YJcl(NF_YJint(\mu^c)) \\ &\leqslant NF_YJcl(\mu) \wedge NF_YJcl(\mu^c) \\ &= NF_YJFr(\mu) \quad (\because NF_YJcl(NF_YJint(\mu)) \leqslant NF_YJcl(\mu)). \end{aligned}$$

Hence, $NF_YJFr(NF_YJcl(\mu)) \leqslant NF_YJFr(\mu)$. □

**Theorem 3.16.** *If* $\mu$ *is both a* $NF_YJos$ *and a* $NF_YJcs$, *then:* $NF_YJFr(\mu) = \mu \wedge \mu^c$.

*Proof.* Let $\mu$ be both a $NF_YJos$ and a $NF_YJcs$. Then $\mu^c$ is also both a $NF_YJos$ and a $NF_YKJcs$. Therefore, $NF_YJcl(\mu) = \mu, \quad NF_YJcl(\mu^c) = \mu^c$. Hence, by definition of $NF_YJFr$ $NF_YJFr(\mu) = NF_YJcl(\mu) \wedge NF_YJcl(\mu^c) = \mu \wedge \mu^c$. □

**Theorem 3.17.** *Let* $\mu$ *be a* $F_Y$ *subset in the* $NF_YTs\ (S, \tau_R(K))$. *Then:* $NF_YJint(\mu) \leqslant \mu - NF_YJFr(\mu)$.

*Proof.* Let $\mu$ be a $F_Y$ subset in the $NF_YTs\ (S, \tau_R(K))$. Now, by definition of $NF_YJFr$

$$\begin{aligned} \mu - NF_YJFr(\mu) &= \mu \wedge (NF_YJFr(\mu))^c \\ &= \mu \wedge [NF_YJcl(\mu) \wedge NF_YJcl(\mu^c)]^c \\ &= \mu \wedge [NF_YJint(\mu^c) \vee NF_YJint(\mu)] \\ &= [\mu \wedge NF_YJint(\mu^c)] \vee [\mu \wedge NF_YJint(\mu)] \\ &= [\mu \wedge NF_YJint(\mu^c)] \vee NF_YJint(\mu) \\ &\geqslant NF_YJint(\mu). \end{aligned}$$

Therefore, $NF_YJint(\mu) \leqslant \mu - NF_YJFr(\mu)$. □

## 4. Nano Fuzzy J Border and Nano Fuzzy J Exterior

**Definition 4.1.** Let $(S, \tau_R(K))$ be a $NF_YTs$. Let $\mu$ be a $F_Y$ subset of $S$. Then the **Nano fuzzy J-border** of $\mu$, denoted by $NF_YJBr(\mu)$, is defined as

$$NF_YJBr(\mu) = \mu - NF_YJint(\mu).$$

**Example 4.2.** Assume $S = \{q, r, s, t\}$ be the universal set and the equivalence relation be $S/R = \{\{q, r\}, \{s, t\}\}$. Let $K = \{q_{0.41}, r_{0.56}, s_{0.36}, t_{0.61}\}$ be a $F_Y$ subset of $S$. Then, $(\underline{NF_Z})(K) = \{q_{0.41}, r_{0.41}, s_{0.36}, t_{0.36}\}, \quad (\overline{NF_Z})(K) =$

$\{q_{0.56}, r_{0.56}, s_{0.61}, t_{0.61}\}$, $NF_Z B(K) = \{q_{0.15}, r_{0.15}, s_{0.25}, t_{0.25}\}$.
Thus, $NF_Y os = \{0_F, 1_F, (\underline{F_Z N})(K), (\overline{F_Z N})(K), BF_Z N(K)\}$ and

$$NF_Y Jos = \{0_F,\ 1_F,\ \underline{NF_z}(K),\ \overline{NF_z}(K),\ \underline{NF_z}^c(K),\ \overline{NF_z}^c(K),\ q_{(0.15-0.41)},\ r_{(0.15-0.41)},\ s_{(0.25-0.36)},\ t_{(0.25-0.36)}\}.$$

Let $\mu = \{q_{0.52}, r_{0.46}, s_{0.38}, t_{0.36}\}$. Then, $NF_Y Jint(\mu) = \{q_{0.41}, r_{0.41}, s_{0.36}, t_{0.36}\}$.
$NF_Y JBr(\mu) = \{q_{0.11}, r_{0.05}, s_{0.02}, t_0\}$.

**Theorem 4.3.** *If a $F_Y$ subset $\mu$ of $NF_Y Ts$ $(S, \tau_R(K))$ is $NF_Y Jcs$, then $NF_Y JBr(\mu) = NF_Y JFr(\mu)$.*

*Proof.* Let $\mu$ be a $NF_Y Jcs$ subset of S. Then by theorem, $NF_Y Jcl(\mu) = \mu$. Now,
$NF_Y JFr(\mu) = NF_Y Jcl(\mu) - NF_Y Jint(\mu) = \mu - NF_Y Jint(\mu) = NF_Y JBr(\mu)$. Hence proved. □

**Theorem 4.4.** *For a $F_Y$ subset $\mu$ of $NF_Y Ts(S, \tau_R(K))$, $\mu$ is $NF_Y Jos$ if and only if $NF_Y JBr(\mu) = 0_F$.*

*Proof. Necessity:* Let $\mu$ be a $NF_Y Jos$ of S. Then by theorem, $NF_Y Jint(\mu) = \mu$. Now, $NF_Y JBr(\mu) = \mu - NF_Y Jint(\mu) = \mu - \mu = 0_F$.

*Sufficiency:* Suppose $NF_Y JBr(\mu) = 0_F$. This implies $\mu - NF_Y Jint(\mu) = 0_F$. Therefore $NF_Y Jint(\mu) = \mu$, and hence $\mu$ is $NF_Y Jos$. □

**Corollary 4.5.** *In a $NF_Y Ts$, $NF_Y JBr(0_F) = 0_F$ and $NF_Y JBr(1_F) = 0_F$.*

*Proof.* Since $0_F$ and $1_F$ are $NF_Y Jos$, by Theorem 4.4, $NF_Y JBr(0_F) = 0_F$ and $NF_Y JBr(1_F) = 0_F$. □

**Theorem 4.6.** *For a $F_Y$ subset $\mu$ of $NF_Y Ts$ $(S, \tau_R(K))$, $NF_Y JBr(NF_Y Jint(\mu)) = 0_F$.*

*Proof.* By the definition of $NF_Y JBr$, $NF_Y JBr(NF_Y Jint(\mu)) = NF_Y Jint(\mu) - NF_Y Jint(NF_Y Jint(\mu))$. We know that $NF_Y Jint(NF_Y Jint(\mu)) = NF_Y Jint(\mu)$, and hence $NF_Y JBr(NF_Y Jint(\mu)) = 0_F$. □

**Theorem 4.7.** *For a $F_Y$ subset $\mu$ of $NF_Y Ts$ $(S, \tau_R(K))$, $NF_Y JBr(\mu) \leqslant NF_Y JFr(\mu)$.*

*Proof.* Since $\mu \leqslant NF_Y Jcl(\mu)$, we have $\mu - NF_Y Jint(\mu) \leqslant NF_Y Jcl(\mu) - NF_Y Jint(\mu)$. This implies, $NF_Y JBr(\mu) \leqslant NF_Y JFr(\mu)$. □

**Theorem 4.8.** *Let $\mu$ be a $F_Y$ subset of $NF_Y Ts$ $(S, \tau_R(K))$. Then, $NF_Y JBr(\mu) = \mu \wedge NF_Y Jcl(\mu^c)$.*

*Proof.* We know that $(NF_Y Jint(\mu))^c = NF_Y Jcl(\mu^c)$. Therefore, $\mu \wedge NF_Y Jcl(\mu^c) = \mu \wedge (NF_Y Jint(\mu))^c$. By using the property $H \wedge L^c = H - L$, we get $\mu \wedge (NF_Y Jint(\mu))^c = \mu - NF_Y Jint(\mu) = NF_Y JBr(\mu)$. Hence, $NF_Y JBr(\mu) = \mu \wedge NF_Y Jcl(\mu^c)$. □

**Theorem 4.9.** *For a $F_Y$ subset $\mu$ of $NF_Y Ts$ $(S, \tau_R(K))$,*

*(i)* $NF_Y Jint(\mu) \vee NF_Y JBr(\mu) = \mu$
*(ii)* $NF_Y Jint(\mu) \wedge NF_Y JBr(\mu) = 0_F$

*Proof.* (i) Let $\mu_0 \in \mu$. If $\mu_0 \in NF_Y Jint(\mu)$, then the result is obvious. If $\mu_0 \notin NF_Y Jint(\mu)$, then by the definition of $NF_Y JBr(\mu)$, $\mu_0 \in NF_Y JBr(\mu)$. Hence $\mu_0 \in NF_Y Jint(\mu) \vee NF_Y JBr(\mu)$ and so $\mu \leqslant NF_Y Jint(\mu) \vee NF_Y JBr(\mu)$.

On the other hand, since $NF_Y Jint(\mu) \leqslant \mu$ and $NF_Y JBr(\mu) \leqslant \mu$, we have $NF_Y Jint(\mu) \vee NF_Y JBr(\mu) \leqslant \mu$. Hence,$NF_Y Jint(\mu) \vee NF_Y JBr(\mu) = \mu$.

(ii) Suppose $NF_Y Jint(\mu) \wedge NF_Y JBr(\mu) \neq 0_F$. Let $\mu_0 \in NF_Y Jint(\mu) \wedge NF_Y JBr(\mu)$. Then $\mu_0 \in NF_Y Jint(\mu)$ and $\mu_0 \in NF_Y JBr(\mu)$. Since $NF_Y JBr(\mu) = \mu - NF_Y Jint(\mu)$, we have $\mu_0 \in \mu$. But $\mu_0 \in NF_Y Jint(\mu)$, which leads to a contradiction. Hence, $NF_Y Jint(\mu) \wedge NF_Y JBr(\mu) = 0_F$. □

**Definition 4.10.** Let $(S, \tau_R(K))$ be a $NF_YTs$. Let $\mu$ be a $F_Y$ subset of $S$. Then the *Nano fuzzy J-exterior* of $\mu$, denoted by $NF_YJExt(\mu)$, is defined as the Nano fuzzy interior of $\mu^c$. That is,

$$NF_YJExt(\mu) = NF_YJint(\mu^c).$$

*Remark* 4.11. For a $F_Y$ subset $\mu$ of $NF_YTs$ $(S, \tau_R(K))$, $NF_YJExt(1_F) = 0_F$ and $NF_YJExt(0_F) = 1_F$.

**Example 4.12.** In Example 4.2, let $\mu = \{q_{0.52}, r_{0.46}, s_{0.38}, t_{0.36}\}$. Then $\mu^c = \{q_{0.48}, r_{0.54}, s_{0.62}, t_{0.64}\}$.
Hence, $NF_YJExt(\mu) = NF_YJint(\{q_{0.48}, r_{0.54}, s_{0.62}, t_{0.64}\}) = \{q_{0.41}, r_{0.41}, s_{0.36}, t_{0.36}\}$.

**Theorem 4.13.** *For a $F_Y$ subset $\mu$ of $NF_YTs$ $(S, \tau_R(K))$, $NF_YJExt(\mu) = (NF_YJcl(\mu))^c$.*

*Proof.* We know that $NF_YJint(\mu^c) = (NF_YJcl(\mu))^c$. Hence, $NF_YJExt(\mu) = NF_YJint(\mu^c) = (NF_YJcl(\mu))^c$. □

**Theorem 4.14.** *For a $F_Y$ subset $\mu$ of $NF_YTs$ $(S, \tau_R(K))$, $\mu$ is $NF_YJcs$ if and only if $NF_YJExt(\mu) = \mu^c$.*

*Proof. Necessity:* Let $\mu$ be a $NF_YJcs$ of $S$. Then $\mu^c$ is $NF_YJos$. By theorem 2.9, $NF_YJint(\mu^c) = \mu^c$. Now, $NF_YJExt(\mu) = NF_YJint(\mu^c) = \mu^c$.

*Sufficiency:* Suppose $NF_YJExt(\mu) = \mu^c$. This implies $NF_YJint(\mu^c) = \mu^c$. Therefore, $\mu^c$ is $NF_YJos$, and hence $\mu$ is $NF_YJcs$. □

**Theorem 4.15.** *For $F_Y$ subsets $\mu_1$ and $\mu_2$ of $NF_YTs$ $(S, \tau_R(K))$, the following statements are valid:*

*(i)* $NF_YJExt(\mu_1 \vee \mu_2) \leqslant NF_YJExt(\mu_1) \wedge NF_YJExt(\mu_2)$
*(ii)* $NF_YJExt(\mu_1 \wedge \mu_2) \leqslant NF_YJExt(\mu_1) \vee NF_YJExt(\mu_2)$

*Proof.* (i)

$$\begin{aligned} NF_YJExt(\mu_1 \vee \mu_2) &= NF_YJint((\mu_1 \vee \mu_2)^c) \\ &= NF_YJint(\mu_1^c \wedge \mu_2^c) \\ &\leqslant NF_YJint(\mu_1^c) \wedge NF_YJint(\mu_2^c) \\ &= NF_YJExt(\mu_1) \wedge NF_YJExt(\mu_2). \end{aligned}$$

(ii)

$$\begin{aligned} NF_YJExt(\mu_1 \wedge \mu_2) &= NF_YJint((\mu_1 \wedge \mu_2)^c) \\ &= NF_YJint(\mu_1^c \vee \mu_2^c) \\ &\geqslant NF_YJint(\mu_1^c) \vee NF_YJint(\mu_2^c) \\ &= NF_YJExt(\mu_1) \vee NF_YJExt(\mu_2). \end{aligned}$$

□

## 5. Conclusion

This paper introduces the concepts of Nano fuzzy J-frontier, border, and exterior for Nano fuzzy J-open sets, along with an exploration of their properties. These developments contribute to a deeper understanding of the topological and algebraic characteristics of fuzzy spaces. The concepts of nano fuzzy frontier, boundary, and exterior offer an effective analytical framework for applications in scenarios that include high uncertainty, incomplete information, or conflicting criteria. In the future, these concepts can be applied in areas such as decision-making processes, medical diagnosis, and GIS modeling.

## Conflict of interest

The author declare that they do not have any conflict of intrest regarding the publication of this paper.

## References


[1] Abbasi Shureshjani, R.,(2021). A developed Best-Worst method to solve multi-criteria decision-making problems under intuitionistic fuzzy environments, Mathematics and Computational Sciences, 2(3), 43-55. 1

[2] Ayyasamy,& Vadivel, & Kalaiyarasan, V., & S.Tamilselvan, Prof, & Sundar, C. John., (2025). Some New Operators Using Fuzzy Nano M-Open Sets, Proyecciones (Antofagasta), 44, 294-309. 1

[3] Changi, C.L., (1968). Fuzzy topological spaces, J. Math. Anal. Appl., 24, 182-190. 1

[4] Girija, S & S, Selvanayaki & Ilango, Gnanambal., (2018). Study on frontier, border and exterior sets in inuitionistic topological spaces. Journal of Physics: Conference Series. 1

[5] Jackson, S., & Gnanaselvam Jude, I., (2025). Advancing Nano Fuzzy Topology: New open sets and Decision - Making Apllications, Proceedings of International Conference on Current Advances in Pure and Applied Mathematics by TDMNS College, ISBN:978-81-994323-5-2 , 46-57. 1, 2.5, 2.6, 2.7, 2.8, 2.9

[6] Jayakumar, P. & Kumar, M Sathish. (2021). A new type of exterior and frontier in Topological spaces, Journal of Physics: Conference Series. 1850. 012006. 10.1088/1742-6596/1850/1/012006. 1

[7] Lellis Thivagar, M., & Richard, C., (2013). On nano forms of weakly open set, Inter. J. Math. Stat. Invent., 1, 31-37. 1

[8] Navalakhe, R., & Rajwade, P., (2019). On Nano fuzzy topological spaces, International Review of Fuzzy Mathematics, 14, 127-136. 1, 2.3, 2.4

[9] Pawlak, Z., (1982). Rough sets, Inter. J. Comput. Inform. Sci., 11, 341-356. 1

[10] Rajwade, P., & Navalakhe, R., & Jain, V., (2022). Nano Generalied Fuzzy Semi Open Sets and Nano Fuzzy β-Open Sets in Nano Fuzzy Topological Spaces, GANITA, Bhartiya Ganita Parishad, 72, 285-289. 1

[11] Rajwade, P., & Navalakhe, R., (2024). Nano Fuzzy Z - Open Sets and its Application in Nano Fuzzy Topological Space, Journal of Scientific Research, 16(1), 201-211. 1

[12] Zadeh, L.A., (1965). Fuzzy sets, Information and Control., 8, 338-353. 1, 2.1, 2.2