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Some mean ergodic theorems on locally compact hypergroups

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Abstract

In this paper, we study the mean ergodic theorems and the weighted ones on locally compact hypergroups. Among other obtained results, for the class of all commutative hypergroups $\mathcal{H}$ with a Plancherel measure $\tilde{\omega}$ that $\operatorname{supp}(\tilde{\omega}) = \widehat{\mathcal{H}}$, we prove that if $(k_j)_{j\in\mathbb{N}}$ is a subsequence of $\mathbb{N}$, $f \in L^2(\mathcal{H})$, and μ is a power bounded measure on $\mathcal{H}$ such that the sequence

$$\left(\frac{1}{m}\sum_{n=1}^{m}\underbrace{\mu * \ldots * \mu}_{k_n-\text{times}} * f\right)_{m\in\mathbb{N}}$$

weakly converges in $L^2(\mathcal{H})$, then the numerical sequence $\left(\frac{1}{m}\sum_{n=1}^{m}\alpha^{k_n}\right)_{m\in\mathbb{N}}$ is convergent too for all $\alpha \in \mathbb{C}$ with

$$\tilde{\omega}\left(\{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = \alpha\}\right) > 0.$$

Keywords: locally compact groups, locally compact hypergroups, mean ergodic theorem, power bounded measures, w*-convergence, Fourier transforms

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1. Introduction

In this paper, we intend to study and conclude some facts regarding locally compact hypergroups as an important extension of locally compact groups. The examples include also a large class of other locally compact spaces such as double coset spaces, polynomial hypergroups, and Sturm-Liouville hypergroups. Although there is no action necessarily between elements of a hypergroup, there exists a convolution algebra structure on the space of all regular measures of the hypergroup in some way which the abstract harmonic analysis can be obtained for it. Hence, in this structure, we are deprived of the group action, which is a very powerful tool in the theory of harmonic analysis on locally compact groups, and therefore working with hypergroups is much more complicated than on locally compact groups. Since the 1970's

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which locally compact hypergroups were introduced in [6, 12, 22, 24], extensive research has been done on this structure and harmonic analysis on them, and this branch of harmonic analysis has always remained active due to its many applications and attractions. See the book [3] and papers [1, 14, 15, 19, 26] as some recent research on this field. On the other hand, ergodic theory and more generally the theory of linear and nonlinear dynamical systems have occupied a large part of research in mathematical analysis during the last two decades. For example see [4, 5, 10] for some dynamical properties of bounded linear operators on Banach spaces.

In [27] and [21] some topics related to the mean ergodic theorems and ergodic sequences of probability measures on commutative hypergroups have been studied. L. Pavel in [21] as a main result proves that if $\mathcal{H}$ is a commutative hypergroup, then a sequence $(\mu_n)_n$ of probability measures on $\mathcal{H}$ is strongly ergodic (i.e. for each representation π of $\mathcal{H}$ on a Hilbert space H_π and for any $\xi \in H_\pi$ the sequence $(\pi_{\mu_n}\xi)_n$ converges in norm to some $\eta \in H_\pi$ such that $\pi_\mu \eta = \eta$ for all probability measure μ on $\mathcal{H}$) if and only if for any $1 \neq \gamma \in \widehat{\mathcal{H}}$, the dual of $\mathcal{H}$, we have $\lim_{n\to\infty} \widehat{\mu_n}(\gamma) = 0$. And equivalently, $w^* - \lim_{n\to\infty} \widehat{\mu_n}(\gamma) = m$, where m is the unique invariant mean on $B(\mathcal{H})$. Recently, in [17], for a locally compact group G and a power bounded regular measure μ on G, the existence of of the weak*-limit of the sequence $(\frac{1}{n}\sum_{j=1}^{n} a_j \mu^j)_n$ has been investigated, where $(a_n)_n$ is a good weight for the mean ergodic theorem. In this paper, we extend and improved some of the results of [16, 17] regarding mean ergodic theorems and the weighted ones for locally compact hypergroups, and in the meantime we have obtained results that are also novel for the group case. a subsequence of $\mathbb{N}$, $f \in L^2(\mathcal{H})$, and μ is a power As a main result we prove that if $\mathcal{H}$ is a commutative hypergroup with a Plancherel measure $\widetilde{\omega}$ that $\mathrm{supp}(\widetilde{\omega}) = \widehat{\mathcal{H}}$, $(k_j)_{j\in\mathbb{N}}$ is bounded measure on $\mathcal{H}$ such that the sequence

$$\left(\frac{1}{m}\sum_{n=1}^{m} \underbrace{\mu * \ldots * \mu}_{k_n-\text{times}} * f\right)_{m\in\mathbb{N}}$$

weakly converges in $L^2(\mathcal{H})$, then the numerical sequence $\left(\frac{1}{m}\sum_{n=1}^{m} \alpha^{k_n}\right)_{m\in\mathbb{N}}$ is convergent too for all $\alpha \in \mathbb{C}$ with $\tilde{\omega}\left(\{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = \alpha\}\right) > 0$. Also, we see that if $\mathcal{H}$ is a compact commutative hypergroup with a Haar measure ω, $(k_j)_{j\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$, and for every power bounded measures μ on $\mathcal{H}$, the sequence

$$\left(\frac{1}{n}\sum_{j=1}^{n} \mu^{k_j}\right)_{n\in\mathbb{N}}$$

is w^*-convergent in $\mathcal{M}(\mathcal{H})$, then the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \xi^{k_i}$ exists for all $\xi \in \{\gamma(x) : x \in \mathcal{H}, \gamma \in \widehat{\mathcal{H}}\}$.

First, for convenience of readers, we recall some basics regarding hypergroups and for more details we refer to the monograph [3], and the basic paper [12].

Next, ϵ_x is a Dirac measure at the point x, and $\mathrm{supp}(\mu)$ denotes the support of a measure μ.

A locally compact hypergroup is a locally compact Hausdorff topological space $\mathcal{H}$ equipped with a product $*$ on $M(\mathcal{H})$, the space of all complex Radon measures on $\mathcal{H}$, which is called a *convolution* and makes it a Banach algebra, and also with an involutive homeomorphism $x \mapsto x^-$ from $\mathcal{H}$ onto $\mathcal{H}$ such that for every $a, b \in \mathcal{H}$ the following condition hold:

1. $\epsilon_a * \epsilon_b$ is a probability measure and $\mathrm{supp}(\epsilon_a * \epsilon_b)$ is compact.
2. $(x, y) \mapsto \epsilon_x * \epsilon_y$ from $\mathcal{H} \times \mathcal{H}$ into $\mathcal{M}^+(\mathcal{H})$ is continuous.
3. $(x, y) \mapsto \mathrm{supp}(\epsilon_x * \epsilon_y)$ from $\mathcal{H} \times \mathcal{H}$ to the family of non-empty compact subsets of $\mathcal{H}$ is continuous.
4. $(\epsilon_a * \epsilon_b)^- = \epsilon_{b^-} * \epsilon_{a^-}$.
5. There exists an (identity) element $e \in \mathcal{H}$ that for each $x \in \mathcal{H}$, $\epsilon_x * \epsilon_e = \epsilon_e * \epsilon_x = \epsilon_x$. Also, $e \in \mathrm{supp}(\epsilon_a * \epsilon_b)$ if and only if $b = a^-$.

In the sequel, always $\mathcal{H}$ is a locally compact hypergroup. $\mathcal{H}$ is called *commutative* whenever $\epsilon_x * \epsilon_y = \epsilon_y * \epsilon_x$ for all $x, y \in \mathcal{H}$. A non-zero non-negative regular measure ω on $\mathcal{H}$ is called a *left Haar measure* for $\mathcal{H}$ if for every $x \in \mathcal{H}$, $\epsilon_x * \omega = \omega$. Although it has been just a conjecture in the general case, it was proved that any commutative or compact or discrete in addition to so many other classes of hypergroups admit a left Haar measure; see [3, 12, 25]. For example, the set $\mathcal{H} := \{0, 1, 2\}$ equipped with the discrete topology, and the identity mapping as involution, and the following convolution is a commutative hypergroup:

$$\epsilon_0 * \epsilon_0 := \epsilon_0, \quad \epsilon_0 * \epsilon_1 := \epsilon_1, \quad \epsilon_0 * \epsilon_2 := \epsilon_2,$$

$$\epsilon_1 * \epsilon_1 := 0.25\epsilon_0 + 0.05\epsilon_1 + 0.70\epsilon_2,$$

$$\epsilon_2 * \epsilon_2 := 0.25\epsilon_0 + 0.30\epsilon_1 + 0.45\epsilon_2,$$

$$\epsilon_1 * \epsilon_2 := 0.70\epsilon_1 + 0.30\epsilon_2.$$

Always, in this paper any Lebesgue spaces on $\mathcal{H}$ is considered regarding the left Haar measure ω.
For every measurable function $f : \mathcal{H} \to \mathbb{C}$ and $a, b \in \mathcal{H}$ we define

$$f(a * b) := \int_{\mathcal{H}} f \, d(\epsilon_a * \epsilon_b),$$

while this integral exists.

Assume that $\mathcal{H}$ is a commutative hypergroup. Then, the set of all continuous bounded functions $\gamma : \mathcal{H} \to \mathbb{C}$ with $\gamma(x^-) = \overline{\gamma(x)}$ for all $x \in \mathcal{H}$, and

$$\gamma(x)\gamma(y) = \int_{\mathcal{H}} \gamma(t) \, d(\epsilon_x * \epsilon_y)(t)$$

for all $x, y \in \mathcal{H}$, is denoted by $\widehat{\mathcal{H}}$ and is called the *dual* of $\mathcal{H}$. Then, Fourier-Stieltjes transform of a measure $\mu \in \mathcal{M}(\mathcal{H})$ is defined as

$$\hat{\mu} : \widehat{\mathcal{H}} \to \mathbb{C}, \qquad \hat{\mu}(\gamma) = \int_{\mathcal{H}} \overline{\gamma(x)} \, d\mu(x).$$

Also, the Fourier transform of every $f \in L^1(\mathcal{H})$ is defined as

$$\hat{f} : \widehat{\mathcal{H}} \to \mathbb{C}, \qquad \hat{f}(\gamma) = \int_{\mathcal{H}} \overline{\gamma(x)} f(x) \, d\omega(x).$$

By [12, Theorem 7.3I], this mapping can be extend from $L^1(\mathcal{H}) \cap L^2(\mathcal{H})$ to a bijection isometry $\mathcal{F} : L^2(\mathcal{H}) \to L^2(\widehat{\mathcal{H}})$, where $L^2(\widehat{\mathcal{H}})$ is equipped with a measure $\tilde{\omega}$, which is called the *Plancherel measure* corresponding with ω. For more details on this topic we refer to the book [3]. By [3, Proposition 2.2.26], for every $\mu \in \mathcal{M}(\mathcal{H})$ and $f \in L^2(\mathcal{H})$, we have

$$\mathcal{F}(\mu * f)(\gamma) = \hat{\mu}(\gamma) \cdot \hat{f}(\gamma) \quad \text{ on } \operatorname{supp}(\widetilde{\omega}).$$

2. w*-limit theorems on hypergroups

In this section, we study weighted mean ergodic theorems regarding power bounded measures on locally compact hypergroups. First, we recall the following definitions.

Definition 2.1. A measure $\mu \in \mathcal{M}(\mathcal{H})$ is called *power bounded* whenever there is some constant $C > 0$ such that for each $n \in \mathbb{N}_0$ we have $\|\mu^n\| \leqslant C$, where $\mu^n := \mu * \ldots * \mu$ (n-times). The set of all power bounded measures on $\mathcal{H}$ is denoted by $\mathcal{M}_{pb}(\mathcal{H})$.

Trivially, any probability measure on a compact hypergroup is power bounded. For more details regarding power bounded measures on locally compact groups we refer to [23].

Definition 2.2. Assume that X is a Banach space, and $T : X \to X$ is linear. Then, T is called *power bounded* whenever there is some $C > 0$ such that $\|T^n\| \leqslant C$ for all $n \in \mathbb{N}_0$, where $T^n := T \circ \ldots \circ T$ (n-times).

Assume that $\mathcal{H}$ is a locally compact hypergroup with a left Haar measure ω. For every $\mu \in \mathcal{M}(\mathcal{H})$ and measurable $f, g : \mathcal{H} \to \mathbb{C}$ define

$$(\mu * f)(s) := \int_{\mathcal{H}} f(x^- * s)\, d\mu(x), \qquad (f * g)(s) := \int_{\mathcal{H}} f(s * x) g(x^-)\, d\omega(x)$$

for all $s \in \mathcal{H}$, while the integrals exist. Let $1 < p < \infty$ and $q := \frac{p}{p-1}$. Then, by [12, Theorem 6.2B] we have

$$\|\mu * f\|_p \leqslant \|\mu\|\, \|f\|_p.$$

This implies that the mapping $T_\mu^{(p)} : L^p(\mathcal{H}) \to L^p(\mathcal{H})$ defined by

$$T_\mu^{(p)}(f) := \mu * f$$

for all $f \in L^p(\mathcal{H})$, is bounded with $\|T_\mu^{(p)}\| \leqslant \|\mu\|$. In fact, for every $n \in \mathbb{N}_0$ we have $\|T_{\mu^n}^{(p)}\| \leqslant \|\mu^n\|$, and this implies that if the measure μ is power bounded, then the operator $T_\mu^{(p)}$ is power bounded too.

As usual, for every $f \in C_0(\mathcal{H})$ we denote

$$\langle \mu, f \rangle := \int_{\mathcal{H}} f\, d\mu.$$

Thanks to [12, Theorem 6.2F and Theorem 5.5P], for every $f \in L^p(\mathcal{H})$ and $g \in L^q(\mathcal{H})$ we have $f * g, f * g^- \in C_0(\mathcal{H})$. For each measurable $f : \mathcal{H} \to \mathbb{C}$ define $f^-(x) := f(x^-)$ for all $x \in \mathcal{H}$. Let $(e_\alpha)_\alpha$ be a net in $C_c^+(\mathcal{H})$ with $\|e_\alpha\|_1 = 1$ and $\mathrm{supp}(e_\alpha) \to \{e\}$. Then, by [12, Lemma 5.1B] for every $f \in C_c^+(\mathcal{H})$ we have

$$\lim_\alpha \|(f\omega) * e_\alpha^- - f\|_{\sup} = 0.$$

This implies that the set $\mathrm{span}\,(\{f * g^- : f, g \in C_c(\mathcal{H})\})$ is dense in $C_0(\mathcal{H})$.

Note that thanks to Fubini's Theorem and [12, Theorem 5.1D], for every $f, g \in L^2(\mathcal{H})$,

$$\begin{aligned}
\langle \mu, f * g^- \rangle &= \int_{\mathcal{H}} (f * g^-)(x)\, d\mu(x) \\
&= \int_{\mathcal{H}} \int_{\mathcal{H}} f(x * y) g^-(y^-)\, d\omega(y)\, d\mu(x) \\
&= \int_{\mathcal{H}} \int_{\mathcal{H}} f(y) g^-(y^- * x)\, d\omega(y)\, d\mu(x) \\
&= \int_{\mathcal{H}} \left(\int_{\mathcal{H}} g^-(y^- * x)\, d\mu(x) \right) f(y)\, d\omega(y) \\
&= \int_{\mathcal{H}} \left(\int_{\mathcal{H}} g(x^- * y)\, d\mu(x) \right) f(y)\, d\omega(y) \\
&= \int_{\mathcal{H}} (\mu * g)(y)\, f(y)\, d\omega(y) \\
&= \langle \mu * g, \bar{f} \rangle_{L^2},
\end{aligned}$$

where $\bar{f}(x) := \overline{f(x)}$ for all $x \in \mathcal{H}$, so,

$$\langle \mu, f * g^- \rangle = \langle \mu * g, \bar{f} \rangle_{L^2} \tag{2.1}$$

for all $f, g \in L^2(\mathcal{H})$ and $\mu \in \mathcal{M}(\mathcal{H})$.

The following fact plays a key role in some proofs.

Theorem 2.3. *Assume that* $(k_j)_{j=1}^{\infty}$ *is a strictly increasing sequence of natural numbers. Then, the followings are equivalent.*

1. *For each linear operator* T *on a Hilbert space* H *with* $\|T\| \leqslant 1$, *the sequence*
$$\left(\frac{1}{n}\sum_{j=1}^{n} T^{k_j}(x)\right)_{n\in\mathbb{N}} \tag{2.2}$$
converges in H *with norm topology, for all* $x \in H$.
2. *For each linear isometry* T *on a Hilbert space* H *the sequence* (2.2) *converges in* H *with norm topology, for all* $x \in H$.
3. *For each linear isometry* T *on a Hilbert space* H *the sequence* (2.2) *weakly converges in* H *for all* $x \in H$.
4. *For every* $\xi \in \mathbb{T}$, *the sequence*
$$\left(\frac{1}{n}\sum_{j=1}^{n} \xi^{k_j}\right)_{n\in\mathbb{N}}$$
is convergent.

Proof. See [8, Theorem 21.14]. □

The following theorem is known as Sz.-Nagy dilation Theorem. See [20].

Theorem 2.4. *Let* T *be a power-bounded operator on a Hilbert space* $\mathcal{H}$. *Then, there exists an invertible bounded linear operator* $S : \mathcal{H} \to \mathcal{H}$ *such that the operator* $\|S^{-1}TS\| \leqslant 1$.

Recall that an operator T on a Banach space X is called a *contraction* whenever $\|T\| \leqslant 1$.

Theorem 2.5. *Let* $\mathcal{H}$ *be a locally compact hypergroup with a left Haar measure* ω. *Assume that* $(k_j)_{j\in\mathbb{N}}$ *is a strictly increasing sequence in* $\mathbb{N}$ *such that for every* $\xi \in \mathbb{T}$, *the sequence*
$$\left(\frac{1}{n}\sum_{j=1}^{n} \xi^{k_j}\right)_{n\in\mathbb{N}}$$
is convergent. Then, for every $\mu \in \mathcal{M}_{pb}(\mathcal{H})$, *the sequence*
$$\left(\frac{1}{n}\sum_{j=1}^{n} \mu^{k_j}\right)_{n\in\mathbb{N}}$$
is w^**-convergent in* $\mathcal{M}(\mathcal{H})$.

Proof. Assume that $(k_j)_{j\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that for every $\xi \in \mathbb{T}$, the sequence
$$\left(\frac{1}{n}\sum_{j=1}^{n} \xi^{k_j}\right)_{n\in\mathbb{N}}$$
is convergent, and let $\mu \in \mathcal{M}_{pb}(\mathcal{H})$. Hence, the operator $T_\mu^{(2)}$ is power bounded on the Hilbert space $L^2(\mathcal{H})$. By the Sz.-Nagy dilation Theorem 2.4, there exists an invertible bounded linear operator $S : L^2(\mathcal{H}) \to L^2(\mathcal{H})$ such that $\|S^{-1}T_\mu^{(2)}S\| \leqslant 1$. Let $f, g \in L^2(\mathcal{H})$, and put $f' := S^{-1}(f)$ and $g' := S^*(g)$. Then, by Theorem 2.3, there is $h \in L^2(\mathcal{H})$ with
$$h = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} C^{k_i} f',$$

where the convergence is in $\|\cdot\|_2$-norm, and $C := S^{-1}T_\mu^{(2)}S$. Then, thanks to the relation

$$\left\langle SC^{k_i}S^{-1}f, g\right\rangle = \left\langle C^{k_i}(S^{-1}f), (S^*g)\right\rangle,$$

we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left\langle (T_\mu^{(2)})^{k_i}f, g\right\rangle &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left\langle (SCS^{-1})^{k_i}f, g\right\rangle\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left\langle C^{k_i}f', g'\right\rangle\\ &= \left\langle \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}C^{k_i}f', g'\right\rangle = \left\langle h, g'\right\rangle.\end{aligned}$$

Hence, the limit $\lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=1}^{n}(T_\mu^{(2)})^{k_i}f, g\right\rangle$ exists for all $f, g \in L^2(\mathcal{H})$. By the relation (2.1), for every $f, g \in L^2(\mathcal{H})$ we have

$$\lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=1}^{n}\mu^{k_i}, f * g^-\right\rangle = \lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=1}^{n}\mu^{k_i} * g, \bar{f}\right\rangle_{L^2},$$

and thus, for each $f, g \in L^2(\mathcal{H})$, the limit

$$\lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=1}^{n}\mu^{k_i}, f * g^-\right\rangle$$

exists. Since the linear span of $\left\{f * g^- : f, g \in L^2(\mathcal{H})\right\}$ is dense in $C_0(\mathcal{H})$, the limit $w^* - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^{k_i}$ exists. □

Remark 2.6. Let $\mathcal{H}$ be a commutative hypergroup and $\gamma \in \widehat{\mathcal{H}}$. Then, for every $x, y, z \in \mathcal{H}$,

$$\begin{aligned}\gamma(x * y * z) &= \int_{\mathcal{H}}\gamma(t)\, d((\epsilon_x * \epsilon_y) * \epsilon_z)(t)\\ &= \int_{\mathcal{H}}\int_{\mathcal{H}}\gamma(s * t)\, d(\epsilon_x * \epsilon_y)(s)\, d\epsilon_z(t)\\ &= \int_{\mathcal{H}}\gamma(s * z)\, d(\epsilon_x * \epsilon_y)(s)\\ &= \gamma(z)\int_{\mathcal{H}}\gamma(s)\, d(\epsilon_x * \epsilon_y)(s)\\ &= \gamma(z)\,\gamma(x * y)\\ &= \gamma(x)\,\gamma(y)\,\gamma(z).\end{aligned}$$

In this way, for every $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathcal{H}$,

$$\gamma(x_1 * x_2 * \ldots * x_n) = \prod_{j=1}^{n}\gamma(x_j). \tag{2.3}$$

Theorem 2.7. *Assume that $\mathcal{H}$ is a compact commutative hypergroup with a Haar measure ω. Let $(k_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$, and for every $\mu \in \mathcal{M}_{pb}(\mathcal{H})$, the sequence*

$$\left(\frac{1}{n}\sum_{j=1}^{n}\mu^{k_j}\right)_{n\in\mathbb{N}}$$

be w^*-convergent in $\mathcal{M}(\mathcal{H})$. Then, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \xi^{k_i}$$

exists for all $\xi \in \{\gamma(x) : x \in \mathcal{H}, \gamma \in \widehat{\mathcal{H}}\}$.

Proof. Let for every $\mu \in \mathcal{M}_{pb}(\mathcal{H})$, the sequence

$$\left(\frac{1}{n} \sum_{j=1}^{n} \mu^{k_j} \right)_{n\in\mathbb{N}}$$

be w^*-convergent in $\mathcal{M}(\mathcal{H})$. Let $\xi \in \{\gamma(x) : x \in \mathcal{H}, \gamma \in \widehat{\mathcal{H}}\}$. Then, there is some $x \in \mathcal{H}$ and $\gamma \in \widehat{\mathcal{H}}$ with $\xi = \gamma(x)$. Note that for every $n \in \mathbb{N}$ we have

$$\|\epsilon_x^n\| = \|\epsilon_x * \ldots * \epsilon_x\| \leqslant \|\epsilon_x\| \ldots \|\epsilon_x\| = 1.$$

This implies that the measure ϵ_x is power bounded on $\mathcal{H}$. Hence, by the hypothesis, the sequence

$$\left(\frac{1}{n} \sum_{j=1}^{n} \epsilon_x^{k_j} \right)_{n\in\mathbb{N}}$$

is w^*-convergent in $\mathcal{M}(\mathcal{H})$. Since $\mathcal{H}$ is compact, $\widehat{\mathcal{H}} \subseteq C_0(\mathcal{H})$. Hence, $\gamma \in C_0(\mathcal{H})$, and so

$$\left(\frac{1}{n} \sum_{j=1}^{n} \epsilon_x^{k_j}(\gamma) \right)_{n\in\mathbb{N}}$$

converges. But, by the relation (2.3),

$$\sum_{j=1}^{n} \epsilon_x^{k_j}(\gamma) = \sum_{j=1}^{n} \gamma(x * \ldots * x) = \sum_{j=1}^{n} \gamma(x)^{k_j} = \sum_{j=1}^{n} \xi^{k_j},$$

and this completes the proof. □

In spite of the group case, the support of the Plancherel measure is not the whole dual of the hypergroup. Nevertheless, there are several important classes of hypergroups such as strong hypergroups which this condition holds for them; see [3, Theorem 2.4.3]. Recall that a hypergroup is called *strong* if its dual is a hypergroup with the conjugate as involution, and a convolution $*$ satisfying

$$\chi(x)\eta(x) = \int_{\mathcal{H}} \gamma(x)\, d(\epsilon_\chi * \epsilon_\eta)(\gamma), \qquad (x \in \mathcal{H}, \chi, \eta \in \widehat{\mathcal{H}}).$$

For instance, see the paper [7] in which C.F. Dunkl and C.E. Ramirez introduced an important class of strong compact countable hypergroups $\mathcal{H}_a$, where $0 < a \leqslant \frac{1}{2}$. The dual of this hypergroup $\widehat{\mathcal{H}_a}$ identifies with $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ equipped with the identity mapping as involution, 0 as the identity element, and the following convolution:

$$\epsilon_n * \epsilon_n := \frac{a^n}{1-a}\epsilon_0 + \sum_{k=1}^{n-1} a^{n-k}\epsilon_k + \frac{1-2a}{1-a}\epsilon_n,$$

where $n \in \mathbb{N}$, and also for every distinct $m, n \in \mathbb{N}$, $\epsilon_m * \epsilon_n := \epsilon_{\max\{m,n\}}$.

The measure $\widetilde{\omega}$ given by

$$\widetilde{\omega}(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ \frac{1-a}{a^k}, & \text{if } k \geqslant 1 \end{cases}$$

would be the Plancherel measure of $\mathcal{H}_a$ which clearly, $\operatorname{supp}(\widehat{\omega}) = \mathbb{N}_0 = \widehat{\mathcal{H}_a}$.

Theorem 2.8. *Let $(k_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. Assume that $\mathcal{H}$ is a commutative hypergroup with $\operatorname{supp}(\widetilde{\omega}) = \widehat{\mathcal{H}}$, $\mu \in \mathcal{M}_{pb}(\mathcal{H})$ and $\alpha \in \mathbb{C}$ such that*

$$\tilde{\omega}\left(\{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = \alpha\}\right) > 0.$$

Then, if $\left(\frac{1}{m}\sum_{n=1}^{m}(T_{\mu}^{(2)})^{k_n} f\right)_{m\in\mathbb{N}}$ is weakly convergent, then $\left(\frac{1}{m}\sum_{n=1}^{m}\alpha^{k_n}\right)_{m\in\mathbb{N}}$ converges.

Proof. Since

$$\tilde{\omega}\left(\{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = \alpha\}\right) > 0$$

and $\tilde{\omega}$ is a regular measure on $\widehat{\mathcal{H}}$ with $\operatorname{supp}(\widetilde{\omega}) = \widehat{\mathcal{H}}$, one can find a non-empty open set with compact support

$$F \subseteq \{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = \alpha\}.$$

Then, $0 \neq \chi_F \in L^2(\widehat{\mathcal{H}})$. Since the Fourier transform $\mathcal{F}$ is onto, there is some non-zero $f \in L^2(\mathcal{H})$ such that $\mathcal{F}(f) = \chi_F$. Then,

$$\mathcal{F}(\mu * f)(\xi) = \widehat{\mu}(\xi)\,\hat{f}(\xi) = \widehat{\mu}(\xi)\,\chi_F(\xi) = \alpha\,\chi_F(\xi) = \mathcal{F}(\alpha f),$$

so $\mu * f = \alpha f$ because $\mathcal{F}$ is one to one. Setting

$$\tau_m(\alpha) := \frac{1}{m}\sum_{n=1}^{m}\alpha^{k_n}, \qquad (m \in \mathbb{N})$$

we obtain

$$\frac{1}{m}\sum_{n=1}^{m}(T_{\mu}^{(2)})^{k_n} f = \frac{1}{m}\sum_{n=1}^{m}(\alpha^{k_n} f) = \left(\frac{1}{m}\sum_{n=1}^{m}\alpha^{k_n}\right) f = \tau_m(\alpha) f,$$

thus,

$$\left\langle \frac{1}{m}\sum_{n=1}^{m}(T_{\mu}^{(2)})^{k_n} f, f\right\rangle = \langle \tau_m(\alpha) f, f\rangle = \tau_m(\alpha)\langle f, f\rangle = \tau_m(\alpha)\|f\|_2^2.$$

Therefore, the sequence $(\tau_m(\alpha))_{m\in\mathbb{N}}$ is convergent. □

Let $\mathcal{H}$ be a Pontryagin commutative hypergroup, and $a \in \mathcal{H}$. Then, $\mu := \delta_a$ is power bounded, and note that $\widehat{\delta_a}(\xi) = \overline{\xi(a)}$, so

$$\{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = \alpha\} = \{\xi \in \widehat{\mathcal{H}} : \xi(a) = \bar{\alpha}\}.$$

Remark 2.9. Note that thanks to the above proof, the condition

$$\tilde{\omega}\left(\{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = \alpha\}\right) > 0$$

in the Theorem 2.8 is equivalent to that α to be an eigenvalue of the operator $T_{\mu}^{(2)}$.

Corollary 2.10. *Let $(k_j)_{j\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. Assume that $\mathcal{H}$ is a commutative hypergroup with* $\operatorname{supp}(\widetilde{\omega}) = \widehat{\mathcal{H}}$*, and $\mu \in \mathcal{M}_{pb}(\mathcal{H})$ with*

$$\tilde{\omega}\left(\{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = 1\}\right) > 0.$$

Then, if $\alpha \in \mathbb{T}$ and $\left(\frac{1}{m}\sum_{n=1}^{m} \alpha^{k_n} \underbrace{\mu * \ldots * \mu}_{k_n-\text{times}} * f\right)_{m\in\mathbb{N}}$ is weakly convergent, then $\left(\frac{1}{m}\sum_{n=1}^{m} \alpha^{k_n}\right)_{m\in\mathbb{N}}$ converges.

Proof. Set $\nu := \alpha\mu$. Then, for every $n \in \mathbb{N}$, $\|\nu^n\| = \|\alpha^n \mu^n\| = |\alpha|^n \|\mu^n\| = \|\mu^n\|$, so, $\nu \in \mathcal{M}_{pb}(\mathcal{H})$. Also, $\hat{\nu} = \alpha\,\hat{\mu}$, thus

$$\{\xi \in \widehat{\mathcal{H}} : \hat{\nu}(\xi) = \alpha\} = \{\xi \in \widehat{\mathcal{H}} : \hat{\mu}(\xi) = 1\}.$$

Finally, note that

$$(T_\nu^{(2)})^{k_n} f = \alpha^{k_n} \underbrace{\mu * \ldots * \mu}_{k_n-\text{times}} * f.$$

Now, applying Theorem 2.8, the proof is complete. □

Next fact would be an extension of [9, Theorem 3.4] and [16, Proposition 2.5] to locally compact hypergroups.

Theorem 2.11. *If $\mu \in \mathcal{M}_{pb}(\mathcal{H})$, then there exists some measure $\theta_\mu \in M_i(\mathcal{H})$ such that*

$$w^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu^i = \theta_\mu.$$

Proof. Assume that $\mu \in \mathcal{M}_{pb}(\mathcal{H})$. Then, there exists some constant $C > 0$ such that for every $n \in \mathbb{N}_0$, $\|\mu^n\| \leqslant C$. Hence, for every $n \in \mathbb{N}_0$,

$$\left\|\frac{1}{n}\sum_{i=1}^{n} \mu^i\right\| \leqslant \frac{1}{n}\sum_{i=1}^{n} \|\mu^i\| \leqslant C.$$

Hence, thanks to Banach-Alaoglu Theorem, there exist $\theta \in \mathcal{M}(\mathcal{H})$ and also some subnet $\{n_j\}_j$ of the natural numbers such that

$$\lim_j \left\langle \frac{1}{n_j}\sum_{i=1}^{n_j} \mu^i, f\right\rangle = \langle \theta, f\rangle$$

for all $f \in C_0(\mathcal{H})$. Note that for each j we have

$$\mu * \left(\frac{1}{n_j}\sum_{i=1}^{n_j} \mu^i\right) = \frac{1}{n_j}\sum_{i=2}^{n_j+1} \mu^i = \frac{1}{n_j}\sum_{i=1}^{n_j} \mu^i + \frac{1}{n_j}(\mu^{n_j+1} - \mu).$$

Hence, since $\|\mu^{n_j+1} - \mu\| \leqslant 2C$ for all j, we can conclude that $\mu * \theta = \theta$ because the convolution product is separately w^*-continuous. This implies that $\mu^{n_j} * \theta = \theta$ for all j, and so $\theta * \theta = \theta$. Similarly, if $\theta' \in \mathcal{M}(\mathcal{H})$ is another closure point for the set

$$\left\{\frac{1}{n}\sum_{i=1}^{n} \mu^i : n \in \mathbb{N}_0\right\},$$

then we have $\theta' = \theta * \theta' = \theta$, and therefore, the proof is complete. □

Definition 2.12. The measure θ in Theorem 2.11 is called the *limit measure associated with* μ and is denoted by θ_μ.

Clearly, any power bounded operator T is Cesáro bounded and the condition $\lim_{n\to\infty} \frac{T^{n-1}(x)}{n} = 0$ automatically holds for every x. We will apply the following theorem in the proof of Theorem 2.15.

Theorem 2.13. *Assume that X is a Banach space, and T is a Cesáro bounded linear operator on X. Assume that $x \in X$ with $\lim_{n\to\infty} \frac{T^{n-1}(x)}{n} = 0$, and let $y \in X$. Then, the followings are equivalent:*

1. $y = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} T^i(x)$.
2. $y = w - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} T^i(x)$.

Proof. See Chapter 2 of the book [13]. □

Definition 2.14. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is called a *good weight for the mean ergodic theorem* if for every Hilbert space H and every linear contraction T on H, the sequence

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_n T^n(f)\right)_{n\in\mathbb{N}}$$

converges for all $f \in H$.

Now, we give one of the main results of this paper regarding hypergroups.

Theorem 2.15. *Assume that $\mathcal{H}$ is a second countable locally compact hypergroup and let $\mu \in \mathcal{M}_{pb}(\mathcal{H})$. If $(a_n)_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem, then setting*

$$a_\xi := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k \xi^k,$$

we have

$$w^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i \mu^i = \sum_{\xi\in\sigma_p\left(T_\mu^{(2)}\right)\cap\mathbb{T}} a_\xi \theta_{\xi\mu}.$$

Proof. Thanks to [2, Proposition 2.3], since the topology on $\mathcal{H}$ is second countable, Hausdorff and locally compact and the Haar measure ω is regular, we conclude that $L^2(\mathcal{H})$ is separable. Since $\mu \in \mathcal{M}_{pb}(\mathcal{H})$, the operator $T_\mu^{(2)}$ is power bounded on $L^2(\mathcal{H})$. Then, by [11], $T_\mu^{(2)}$ cannot have an uncountable number of eigenvalues of modulus 1. Assume that $\xi \in \sigma(T_\mu^{(2)}) \cap \mathbb{T}$. We claim that for every $f, g \in L^2(\mathcal{H})$,

$$\lim_{n\to\infty} \left\langle \frac{1}{n}\sum_{i=1}^{n} (T_{\xi\mu}^{(2)})^i fg \right\rangle = \langle \theta_\mu * f, g\rangle.$$

For each $n \in \mathbb{N}$ denote $\mu_n := \frac{1}{n}\sum_{i=1}^{n}(\xi\mu)^i$. Recall $w^* - \lim_{n\to\infty} \mu_n = \theta_{\xi\mu}$. As we mentioned at the beginning of this section, $\bar{g} * f^- \in C_0(\mathcal{H})$, and we have

$$\begin{aligned}\lim_{n\to\infty} \langle \mu_n * f, g\rangle &= \lim_{n\to\infty} \left\langle \mu_n, \bar{g} * f^- \right\rangle \\ &= \left\langle \theta_{\xi\mu}, \bar{g} * f^- \right\rangle \\ &= \left\langle \theta_{\xi\mu} * f, g \right\rangle.\end{aligned}$$

This means that

$$\frac{1}{n}\sum_{i=1}^{n}(T^{(2)}_{\xi,\mu})^{i}f \to \theta_{\xi\mu} * f$$

weakly in $L^2(\mathcal{H})$, as $n \to \infty$. Hence, by Theorem 2.13, for every $f \in L^2(\mathcal{H})$ and $\xi \in \mathbb{T}$ we have

$$\frac{1}{n}\sum_{i=1}^{n}\xi^{i}(T^{(2)}_{\mu})^{i}f \to \theta_{\xi\mu} * f$$

with $\|\cdot\|_2$-norm.

Since μ is power bounded, the corresponding operator $T^{(2)}_{\mu}$ is power bounded on $L^2(\mathcal{H})$. So, thanks to Theorem 2.4, there is some invertible operator S on $L^2(\mathcal{H})$ that $\|S^{-1}T^{(2)}_{\mu}S\| \leqslant 1$. Note that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}a_i(T^{(2)}_{\mu})^{i}f &= \frac{1}{n}\sum_{i=1}^{n}a_i(S(S^{-1}T^{(2)}_{\mu}S)S^{-1})^{i}f\\ &= S\left(\frac{1}{n}\sum_{i=1}^{n}a_i(S^{-1}T^{(2)}_{\mu}S)^{i}\right)S^{-1}f\end{aligned}$$

Thanks to [8, Theorem 21.2] and since $(a_n)_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem, we conclude that for every $g \in L^2(\mathcal{H})$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i(S^{-1}T^{(2)}_{\mu}S)^{i}g = \sum_{\xi\in\sigma_p(S^{-1}T^{(2)}_{\mu}S)\cap\mathbb{T}} a(\xi)P'_{\xi}g$$

exists with $\|\cdot\|_2$-norm where P'_{ξ} is the orthogonal projection onto $\ker(S^{-1}T^{(2)}_{\mu}S - \xi I)$. Now, setting $g := S^{-1}f$, the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i(S^{-1}T^{(2)}_{\mu}S)^{i}S^{-1}f$$

exists with $\|\cdot\|_2$-norm. Since S is continuous,

$$\lim_{n\to\infty}S\left(\frac{1}{n}\sum_{i=1}^{n}a_i(S^{-1}T^{(2)}_{\mu}S)^{i}S^{-1}f\right) = S\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i(S^{-1}T^{(2)}_{\mu}S)^{i}S^{-1}f\right)$$

also exists with $\|\cdot\|_2$-norm, and hence there is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i(T^{(2)}_{\mu})^{i}f$$

with $\|\cdot\|_2$-norm for all $f \in L^2(\mathcal{H})$. Now, just note that

$$\sigma_p(S^{-1}T^{(2)}_{\mu}S)\cap\mathbb{T} = \sigma_p(T^{(2)}_{\mu})\cap\mathbb{T},$$

and

$$\ker(S^{-1}T^{(2)}_{\mu}S - \xi I) = \ker(T^{(2)}_{\mu} - \xi I).$$

In fact,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i(T^{(2)}_{\mu})^{i}f = \sum_{\xi\in\sigma_p\left(T^{(2)}_{\mu}\right)\cap\mathbb{T}} a_{\xi}P_{\xi}f$$

with $\|\cdot\|_2$-norm, where P_ξ is the orthogonal projection onto $\ker\left(T_\mu^{(2)} - \xi I\right)$. Since $P_\xi f = \theta_{\xi\mu} * f$, for every $f, g \in L^2(\mathcal{H})$,

$$\begin{aligned}\lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=1}^{n} a_i\mu^i, f * g^-\right\rangle &= \lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=1}^{n} a_i (T_\mu^{(2)})^i g, \bar{f}\right\rangle \\ &= \left\langle \sum_{\xi\in\sigma_p\left(T_\mu^{(2)}\right)\cap\mathbb{T}} a_\xi P_\xi g, \bar{f}\right\rangle \\ &= \left\langle \sum_{\xi\in\sigma_p\left(T_\mu^{(2)}\right)\cap\mathbb{T}} a_\xi \theta_{\xi\mu} * g, \bar{f}\right\rangle \\ &= \left\langle \sum_{\xi\in\sigma_p\left(T_\mu^{(2)}\right)\cap\mathbb{T}} a_\xi \theta_{\xi\mu}, f * g^-\right\rangle.\end{aligned}$$

So, since $\operatorname{span}\left(\{f * g^- : f, g \in L^2(\mathcal{H})\}\right)$ is dense in $C_0(\mathcal{H})$,

$$w^* - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_i\mu^i = \sum_{\xi\in\sigma_p\left(T_\mu^{(2)}\right)\cap\mathbb{T}} a_\xi\theta_{\xi\mu}.$$

□

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