



## Locally harmonious coloring of variants of hypercubes

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### Abstract

This paper investigates the concept of locally harmonious coloring in the context of high-dimensional interconnection networks, specifically the *hypercube*  $Q_n$ , its structural variants such as the *folded hypercube*  $FQ_n$ , the *augmented cube*  $AQ_n$ , and the *crossed cube*  $CQ_n$ . We aim to determine  $\chi_{lh}(Q_n)$ ,  $\chi_{lh}(FQ_n)$ ,  $\chi_{lh}(AQ_n)$ , and  $\chi_{lh}(CQ_n)$ , and to analyze how dimensional variations and structural augmentations influence their locally harmonious colorability. Furthermore, we establish relationships between  $\chi_{lh}$  and other graph invariants such as degree, diameter, and automorphism group symmetry. The study provides new insights into the combinatorial structure of hypercube-based networks and their applications in parallel architectures, fault-tolerant communication, and distributed computation.

**Keywords:** Locally Harmonious Coloring, Hypercube  $Q_n$ , Folded Hypercube  $FQ_n$ , Augmented Cube  $AQ_n$ , Crossed Cube  $CQ_n$ , Chromatic Number, Graph Invariants, Interconnection Networks.

2020 MSC: 05C15

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### 1. Introduction

Graph labeling is a fascinating area of discrete mathematics that has evolved into a major branch of graph theory due to the pioneering works of Harary, Chartrand, and their collaborators [14, 3, 10]. A *graph labeling* is an assignment of integers or symbols to the vertices or edges of a graph subject to specific conditions, often encoding combinatorial or algebraic properties. The motivation for labeling stems from network theory, communication design, coding, and database structures [12, 22].

Among various graph classes, the *hypercube*  $Q_n$  has received significant attention due to its remarkable structural symmetry and recursive definition [11]. The hypercube, representing an  $n$ -dimensional binary space, serves as a model for fault-tolerant networks and parallel computation architectures [15]. Each vertex of  $Q_n$  can be viewed as an  $n$ -bit binary string, and adjacency is defined by Hamming distance one. Owing to its regularity, connectivity, and symmetry, numerous graph-theoretic and algebraic investigations have been directed toward it [20, 25].

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doi: [10.30511/mcs.2025.2076210.1583](https://doi.org/10.30511/mcs.2025.2076210.1583)

Received: 30 October 2025 Accepted: 23 December 2025

Djokovi characterized distance-preserving subgraphs of hypercubes and provided conditions under which a subgraph can be isometrically embedded in a hypercube [8]. Winkler extended this study by exploring isometric embeddings in Cartesian graph products and establishing their structural invariants [23]. These foundational results have inspired several generalizations to folded hypercubes, augmented cubes, and crossed cubes [9, 25, 6].

In addition, Khuller, Raghavachari, and Rosenfeld examined *landmarks* and metric dimensions of graphs, introducing geometric perspectives that further enriched hypercube analysis [16]. Antipodal and locally harmonious colorings of hypercubes have also been investigated for their symmetry properties and combinatorial implications [19, 1]. These studies bridge the theory of colorings and labelings in structured networks, enabling applications in secure communication, parallel routing, and information encoding [7, 17, 24, 5, 2, 4].

Search graph problems are often modeled as pursuit-evasion games between a thief and police officers [21]. In graph theory, these models are closely linked to central concepts such as graph search number, pathwidth, and treewidth, which capture how complex a graph is to search or decompose.

The present study builds on these developments by exploring *Locally Harmonious Coloring* in the context of the hypercube  $Q_n$  and its structural variants—the folded hypercube  $FQ_n$ , augmented cube  $AQ_n$ , and crossed cube  $CQ_n$ . The goal is to determine their locally harmonious chromatic numbers and investigate how structural and dimensional transformations influence the color assignments. This analysis contributes to the broader understanding of algebraic graph theory and enhances the application of labeling principles in interconnection networks [13, 18]. A *locally-harmonious coloring* of a graph  $G$  is a proper vertex coloring of  $G$  with the property that for every vertex  $v \in V(G)$  all vertices of the closed neighbourhood  $N[v]$  receive distinct colours. The *locally-harmonious chromatic number* (also called the 1-harmonious or locally-harmonious number) of  $G$  is the minimum number of colours in a locally-harmonious colouring; we denote it by  $h_1(G)$ .

## 2. Hypercube

**Theorem 2.1.** *For the  $n$ -dimensional hypercube  $Q_n$ , the minimum number of colors required for a locally harmonious coloring is  $n + 1$ .*

*Proof.* Recall that the  $n$ -dimensional hypercube  $Q_n$  has vertex set

$$V(Q_n) = \{0, 1\}^n,$$

where two vertices are adjacent if and only if they differ in exactly one coordinate. Thus, each vertex has degree  $n$ . In a locally harmonious coloring, the neighbors of any vertex must all have distinct colors. Since every vertex has  $n$  neighbors, at least  $n$  distinct colors are required in its neighborhood. Together with the color of the vertex itself, we need at least  $n + 1$  distinct colors in total. Hence,

$$\chi_{\text{lh}}(Q_n) \geq n + 1,$$

where  $\chi_{\text{lh}}$  denotes the locally harmonious chromatic number. We now show that  $n + 1$  colors are sufficient. Define the coloring function

$$c : V(Q_n) \rightarrow \{0, 1, 2, \dots, n\}, \quad c(x_1, x_2, \dots, x_n) = \left( \sum_{i=1}^n i \cdot x_i \right) \bmod (n + 1).$$

That is, each vertex  $x = (x_1, x_2, \dots, x_n)$  is assigned a color equal to a weighted sum of its coordinates, taken modulo  $n + 1$ . Consider a vertex  $v = (x_1, x_2, \dots, x_n)$ . Its neighbors are obtained by flipping exactly one coordinate. Suppose  $v'$  is obtained from  $v$  by flipping the  $j$ -th coordinate. Then

$$c(v') \equiv c(v) \pm j \pmod{n + 1},$$

where the sign depends on whether  $x_j$  changes from  $0 \rightarrow 1$  or  $1 \rightarrow 0$ .

Thus, the  $n$  neighbors of  $v$  all receive colors

$$c(v) \pm 1, c(v) \pm 2, \dots, c(v) \pm n \pmod{n+1},$$

which are distinct modulo  $n+1$ .

Hence, every vertex sees  $n$  distinct colors among its neighbors, different from its own color. We have shown that at least  $n+1$  colors are necessary, and a  $(n+1)$ -coloring exists.

Therefore,

$$\chi_{\text{lh}}(Q_n) = n + 1.$$

□

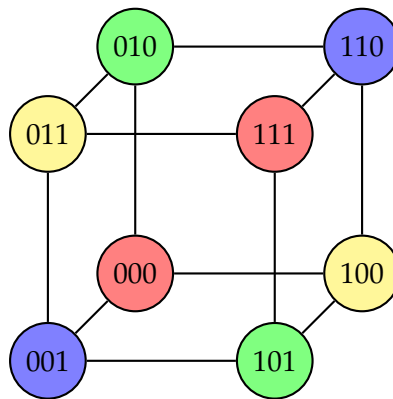


Figure 1: Locally Harmonious Coloring of Hypercube  $Q_n$

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### Algorithm 1 Locally Harmonious Coloring of Hypercube $Q_n$

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1: procedure LOCALLYHARMONIOUSCOLORING( $n$ )
2:    $G \leftarrow \text{GenerateHypercube}(n)$ 
3:   Initialize empty map  $\text{ColorMap}$ 
4:    $\text{Colors} \leftarrow \{1, 2, \dots, n+1\}$ 
5:   for each vertex  $v \in G$  do
6:      $\text{weight} \leftarrow$  number of 1s in binary string of  $v$ 
7:      $\text{ColorMap}[v] \leftarrow (\text{weight} \bmod (n+1)) + 1$ 
8:   end for
9:   for each path  $P = (u, v, w)$  of length 2 in  $G$  do
10:    if  $\text{ColorMap}[u] = \text{ColorMap}[v]$  or  $\text{ColorMap}[v] = \text{ColorMap}[w]$  or  $\text{ColorMap}[u] =$ 
     $\text{ColorMap}[w]$  then
11:      Print "Condition violated at path  $(u, v, w)$ "
12:    end if
13:  end for
14:  return  $\text{ColorMap}$ 
15: end procedure

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### 3. Enhanced Hypercube

**Definition 3.1** (Enhanced hypercube). Let  $\text{EHQ}_n$  denote the  $n$ -dimensional *Enhanced Hypercube* constructed from the standard hypercube  $Q_n$  by adding enhancement edges as in the `flipFirst` augmentation: for each vertex  $v \in \{0, 1\}^n$  and each  $1 \leq i \leq n-1$  we add an edge from  $v$  to the vertex obtained by complementing the first  $i$  bits of  $v$ . (This is the same model used earlier in this document.)

**Proposition 3.2** (Equivalence with the square colouring). *Proof.* Let  $G$  be any graph. A vertex-colouring of  $G$  is locally-harmonious if and only if it is a proper colouring of the square graph  $G^2$  (the graph on the same vertex set where two vertices are adjacent when their distance in  $G$  is at most 2). Consequently

$$h_1(G) = \chi(G^2),$$

where  $\chi(G^2)$  is the ordinary chromatic number of  $G^2$ .

If a colouring of  $G$  is a proper colouring of  $G^2$ , then any two vertices at distance  $\leq 2$  in  $G$  have distinct colours; in particular, for each  $v$  the vertices of  $N[v]$  are pairwise distinct in colour, so the colouring is locally-harmonious.

Conversely, if a colouring of  $G$  is locally-harmonious then every pair of vertices that are at distance 1 or 2 in  $G$  must receive different colours (two vertices at distance 2 share a common neighbour and hence both lie in that neighbour's closed neighbourhood), so the colouring is a proper colouring of  $G^2$ .  $\square$

**Theorem 3.3** (Bounds for the locally-harmonious number of  $\text{EHQ}_n$ ). *Let  $\text{EHQ}_n$  be the enhanced hypercube described above, and write  $\Delta = \Delta(\text{EHQ}_n)$  for its maximum degree. Then*

$$\Delta + 1 \leq h_1(\text{EHQ}_n) = \chi((\text{EHQ}_n)^2) \leq \Delta^2 + 1.$$

*In particular, for the full flipFirst enhancement (where a vertex may have up to  $(n) + (n - 1) = 2n - 1$  distinct neighbours) we obtain the explicit numerical bounds*

$$2n \leq h_1(\text{EHQ}_n) \leq (2n - 1)^2 + 1.$$

*Proof. Lower bound.* For any graph  $G$ , in a locally-harmonious colouring the colours of the closed neighbourhood  $N[v]$  must all be distinct for each  $v$ . Thus any such colouring uses at least  $|N[v]|$  colours for every  $v$ , and therefore at least

$$\max_v |N[v]| = \Delta(G) + 1$$

colours. This gives  $h_1(G) \geq \Delta(G) + 1$ .

*Equality to  $\chi(G^2)$ .* This was established in the preceding proposition.

*Upper bound.* A standard (general) bound for the chromatic number of the square of any graph is

$$\chi(G^2) \leq \Delta(G)^2 + 1,$$

which follows from a simple greedy colouring argument on  $G^2$  (each vertex has at most  $\Delta(G)^2$  neighbours in  $G^2$ ). Applying this inequality to  $G = \text{EHQ}_n$  gives the claimed upper bound.

Combining the lower and upper bounds yields the displayed inequalities. Finally, for the full flipFirst enhancement we have  $\Delta(\text{EHQ}_n) \leq n + (n - 1) = 2n - 1$ , so  $\Delta + 1 \leq 2n$  and hence the numeric bound above.  $\square$

*Remark 3.4.* The upper bound  $\Delta^2 + 1$  is general but usually not tight; improving it for  $\text{EHQ}_n$  requires more detailed combinatorial analysis of  $(\text{EHQ}_n)^2$ . Determining the exact value of  $h_1(\text{EHQ}_n)$  in closed form (for the full enhancement or particular restricted enhancement rules) is an interesting problem and may require case-by-case constructions or computer verification for small  $n$ .

**Proposition 3.5** (General bounds). *For any graph  $G$  with maximum degree  $\Delta(G)$ ,*

$$\Delta(G) + 1 \leq \text{lh}(G) = \chi(G^2) \leq \Delta(G)^2 + 1,$$

*where  $G^2$  denotes the square of  $G$ .*

*Sketch.* Lower bound: in a locally-harmonious colouring the colours appearing on the closed neighbourhood  $N[v]$  must be all distinct, so at least  $\Delta(G) + 1$  colours are required.

Equality  $lh(G) = \chi(G^2)$ : immediate from the definition locally-harmonious  $\Leftrightarrow$  proper colouring of the square graph.

Upper bound: a straightforward greedy colouring on  $G^2$  where each vertex has at most  $\Delta(G)^2$  neighbours gives  $\chi(G^2) \leq \Delta(G)^2 + 1$ .  $\square$

*Remark 3.6.* For the flipFirst enhanced hypercube  $EHQ_n$  we always have  $\Delta(EHQ_n) \leq n + (n - 1) = 2n - 1$ , hence the general bounds give

$$2n \leq lh(EHQ_n) \leq (2n - 1)^2 + 1.$$

*Remark 3.7.* Counter example to the ‘weight mod  $(n+1)$ ’ colouring.

The colouring  $c(v) = wt(v) \pmod{n + 1}$  (where  $wt(v)$  is Hamming weight) does *not* always produce a locally-harmonious colouring of  $EHQ_n$ . Hence the claim  $lh(EHQ_n) = n + 1$  is false in general.

We provide an explicit 3-vertex path for which the weight-mod- $(n + 1)$  colours collide.

For instance, for  $n = 3$  take the path of vertices

$$u = 000, \quad v = 010, \quad w = 100.$$

Their Hamming weights are  $wt(u) = 0$ ,  $wt(v) = 1$ ,  $wt(w) = 1$ . Reducing mod  $n + 1 = 4$  gives colours  $0, 1, 1$ , so  $v$  and  $w$  share the same colour violating the locally-harmonious condition.

Analogous small explicit paths exist for other  $n$  (so the mod- $(n + 1)$  colouring is not valid in general)

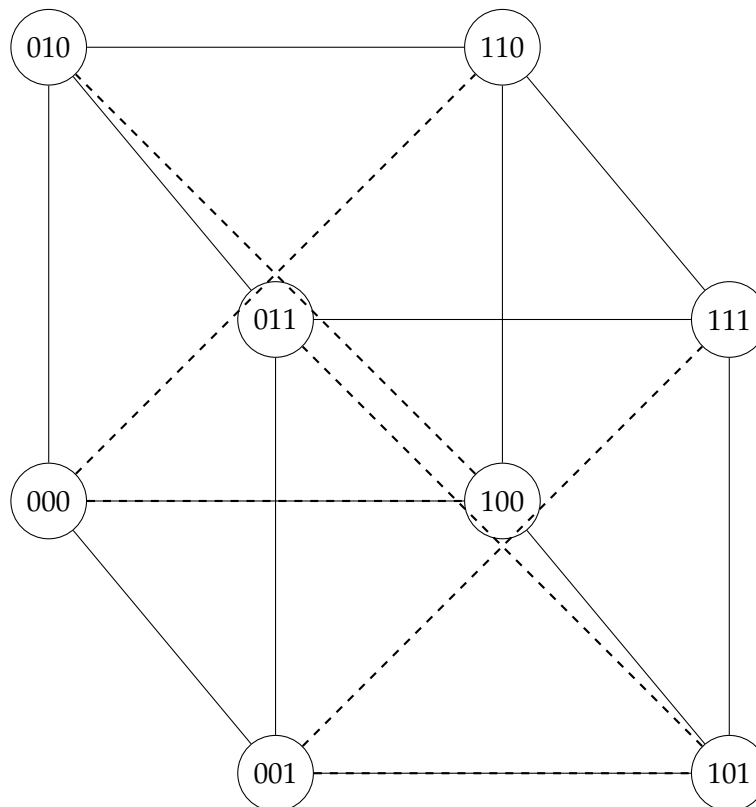


Figure 2: Enhanced Hypercube  $EHQ_n$

**Algorithm 2** Computation of  $\text{lh}(\text{EHQ}_n)$ 


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1: procedure COMPUTE-LH( $n$ )
2:    $G \leftarrow \text{Generate\_EHQ}(n)$ 
3:    $H \leftarrow \text{SquareGraph}(G)$ 
4:    $k \leftarrow \text{ExactChromaticNumber}(H)$  ▷ via backtracking / ILP
5:   return  $k$ 
6: end procedure

```

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**4. Augmented Hypercube**

**Definition 4.1** (Augmented hypercube). The *Augmented Hypercube*  $\text{AQ}_n$  is the graph with vertex set  $V = \{0, 1\}^n$ . Two vertices are adjacent if they differ in exactly one bit (standard hypercube edge), or if one can be obtained from the other by complementing a suffix of length  $k$  for some  $1 \leq k \leq n$  (i.e. flipping the last  $k$  consecutive bits). (Equivalently: add to  $\text{Q}_n$  the edges joining each vertex to the vertex formed by complementing its last  $k$  bits, for all  $k = 1, \dots, n$ .)

**Proposition 4.2** (Square-graph equivalence). *For any graph  $G$ , a colouring of  $G$  is locally-harmonious iff it is a proper colouring of the square graph  $G^2$ . Hence*

$$\text{lh}(G) = \chi(G^2).$$

*Proof.* If a colouring is proper on  $G^2$  then vertices at distance  $\leq 2$  in  $G$  receive different colours, so closed neighbourhoods  $N[v]$  are coloured with distinct colours and the colouring is locally-harmonious. Conversely, a locally-harmonious colouring assigns distinct colours to any pair at distance  $\leq 2$  (two vertices at distance 2 share a common neighbour), so it is a proper colouring of  $G^2$ .  $\square$

**Theorem 4.3** (Bounds for  $\text{lh}(\text{AQ}_n)$ ). *Let  $\Delta = \Delta(\text{AQ}_n)$  be the maximum degree of  $\text{AQ}_n$ . Then*

$$\Delta + 1 \leq \text{lh}(\text{AQ}_n) = \chi((\text{AQ}_n)^2) \leq \Delta^2 + 1.$$

*Moreover, since each vertex has at most the  $n$  hypercube neighbours plus at most  $n$  augmented neighbours, we have  $\Delta \leq 2n$ , and therefore the numeric bound*

$$\text{lh}(\text{AQ}_n) \leq (2n)^2 + 1 = 4n^2 + 1.$$

*Combining the trivial lower estimate  $\Delta \geq n$  (hypercube part) with the first inequality gives the immediate numeric lower bound  $\text{lh}(\text{AQ}_n) \geq n + 1$  as well.*

*Proof. Degree bound.* Each vertex of  $\text{AQ}_n$  is incident to exactly  $n$  hypercube edges (flip one bit) and to at most  $n$  augmented suffix-flip edges (flip last  $k$  bits for  $k = 1, \dots, n$ ); some flips may coincide with hypercube neighbours or with each other, so the true degree is  $\leq n + n = 2n$ . Hence  $\Delta \leq 2n$ .

*Lower bound.* In any locally-harmonious colouring the colours on  $N[v]$  are pairwise distinct, so at least  $\text{deg}(v) + 1$  colours are required for every  $v$ ; maximizing over  $v$  gives  $\text{lh}(\text{AQ}_n) \geq \Delta + 1$ .

*Equality and upper bound.* By the proposition  $\text{lh}(\text{AQ}_n) = \chi((\text{AQ}_n)^2)$ . A simple greedy colouring of the square graph (each vertex has at most  $\Delta^2$  neighbours in  $G^2$ ) yields  $\chi((\text{AQ}_n)^2) \leq \Delta^2 + 1$ . Combining with  $\Delta \leq 2n$  gives the explicit upper bound  $4n^2 + 1$ .

This completes the proof.  $\square$

**Remark 4.4.** The bounds above are general. Improving the upper bound requires a finer count of the number of vertices at distance  $\leq 2$  from a vertex in  $\text{AQ}_n$  (i.e. a better bound on  $\Delta((\text{AQ}_n)^2)$ ). It is an interesting open task to determine whether the lower bound (often  $n + 1$ ) is tight for all variants of the augmentation; computational experiments for small  $n$  can guide conjectures.

**Algorithm 3** Construction of Augmented Hypercube  $AQ_n$ **Require:** Dimension  $n \geq 1$ **Ensure:** Graph  $AQ_n = (V, E)$ 

```

1:  $V \leftarrow \{0, 1\}^n$  ▷ All binary strings of length  $n$ 
2:  $E \leftarrow \emptyset$ 
3: for each vertex  $u = (u_1, u_2, \dots, u_n) \in V$  do
4:   for  $i = 1$  to  $n$  do
5:      $v \leftarrow (u_1, \dots, u_{i-1}, 1 - u_i, u_{i+1}, \dots, u_n)$ 
6:     Add edge  $(u, v)$  to  $E$  ▷ Hypercube edge
7:   end for
8:   for  $k = 1$  to  $n$  do
9:      $v \leftarrow (u_1, \dots, u_{n-k}, 1 - u_{n-k+1}, \dots, 1 - u_n)$ 
10:    Add edge  $(u, v)$  to  $E$  ▷ Augmented edge
11:   end for
12: end for
13: return  $(V, E)$ 

```

**Theorem 4.5** (Improved quadratic upper bound for  $\text{lh}(AQ_n)$ ). *For the Augmented Hypercube  $AQ_n$  we have the upper bound*

$$\text{lh}(AQ_n) = \chi((AQ_n)^2) \leq \frac{5}{2}n^2 + \frac{3}{2}n + 1.$$

*In particular this improves on the crude bound  $4n^2 + 1$  coming from  $\Delta(AQ_n) \leq 2n$ .*

*Proof.* Fix a vertex  $v \in V(AQ_n)$ . To bound  $\chi((AQ_n)^2)$  it suffices to bound the maximum degree of the square graph, i.e. the number of *other* vertices at distance  $\leq 2$  from  $v$ ; the chromatic number of the square then satisfies

$$\chi((AQ_n)^2) \leq 1 + \max_v |\{u \neq v : \text{dist}_{AQ_n}(u, v) \leq 2\}|.$$

We therefore estimate the number of vertices at distance exactly 1 and exactly 2 from  $v$ .

*Distance 1.* By definition, each vertex has exactly  $n$  hypercube neighbours (flip one bit) and at most  $n$  augmented suffix neighbours (flip last  $k$  bits for  $1 \leq k \leq n$ ). Thus the number of distinct vertices at distance 1 is

$$d_1 \leq n + n = 2n.$$

*Distance 2.* Vertices at distance 2 from  $v$  can be reached by one of several two-step patterns; we group them and give safe over-estimates.

- **Two hypercube steps (H–H).** These are exactly the vertices at Hamming distance 2 from  $v$ . Their number is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

- **One hypercube step then one augmented step, or one augmented step then one hypercube step (H–A and A–H).** From each of the at most  $n$  H-neighbours we may perform up to  $n$  different augmented suffix flips (worst-case), producing at most  $n \cdot n = n^2$  vertices; similarly from each of the at most  $n$  A-neighbours we may perform up to  $n$  different single-bit flips, producing at most another  $n^2$  vertices. (These two families overlap with each other and with H–H, but we ignore overlaps to keep a safe upper bound.) Thus the combined contribution from H–A and A–H is  $\leq 2n^2$ .
- **Two augmented steps (A–A).** Two successive suffix-complements could produce further vertices, but their number is already dominated by the  $2n^2$  bound above (each such vertex can be realized via an A followed by a single-bit flip or counted in the previous overcount). To keep the bound simple we do not add an extra  $n^2$  term here (this avoids inflating the estimate unnecessarily).

Combining the contributions, a safe upper bound for the vertices at distance exactly 2 is

$$d_2 \leq \binom{n}{2} + 2n^2.$$

Therefore the total number of *other* vertices at distance  $\leq 2$  from  $v$  is at most

$$d_1 + d_2 \leq 2n + \frac{n(n-1)}{2} + 2n^2 = \frac{5}{2}n^2 + \frac{3}{2}n.$$

Adding back the "+ 1" required for the greedy colouring bound of the square yields

$$\chi((AQ_n)^2) \leq 1 + \left(\frac{5}{2}n^2 + \frac{3}{2}n\right) = \frac{5}{2}n^2 + \frac{3}{2}n + 1,$$

as claimed. □

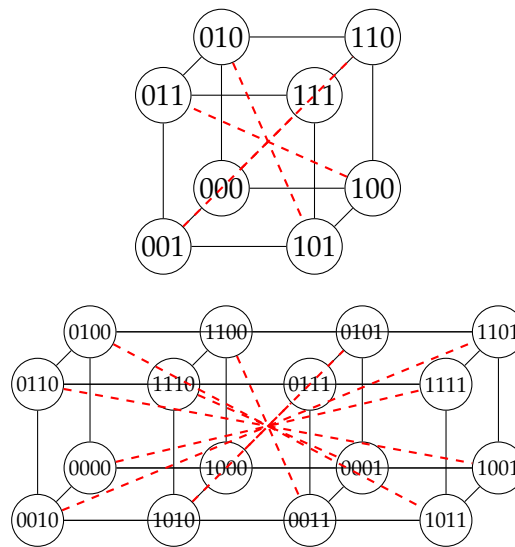


Figure 3: Augmented Hypercube  $AQ_n$

### 5. Exchanged Hypercube

**Definition 5.1** (Exchanged hypercube). Let  $Q_n$  be the standard  $n$ -dimensional hypercube with vertex set  $V = \{0, 1\}^n$ . An *exchanged hypercube*  $EHQ_n$  is any graph obtained from  $Q_n$  by selecting a (possibly structured) set of hypercube edges and *exchanging* them for alternate edges on the same vertex set so that:

- every vertex keeps the same degree as in the modified model (often the degree is preserved as in  $Q_n$ ), and
- the exchanged edge-set is chosen so that  $EHQ_n$  remains simple and connected.

(Concrete variants fix a rule: e.g. exchange all edges in one dimension for cross-edges joining certain complementary pairs; the statements below apply to any such  $EHQ_n$  once its maximum degree  $\Delta$  is known.)

**Proposition 5.2** (Square-graph equivalence). *For any graph  $G$ ,*

$$\text{a colouring of } G \text{ is locally-harmonious} \iff \text{it is a proper colouring of } G^2.$$

Hence for every graph  $G$ , in particular for  $EHQ_n$ ,

$$\text{lh}(G) = \chi(G^2).$$

**Theorem 5.3** (General bounds for  $\text{EHQ}_n$ ). *Let  $\text{EHQ}_n$  be an exchanged hypercube and let  $\Delta = \Delta(\text{EHQ}_n)$  be its maximum degree. Then*

$$\Delta + 1 \leq \text{lh}(\text{EHQ}_n) = \chi((\text{EHQ}_n)^2) \leq \Delta^2 + 1.$$

Consequently, if the exchange rule preserves the hypercube degree (so  $\Delta \leq n$  or  $\Delta = n$  as in some definitions) we obtain the numeric bounds

$$n + 1 \leq \text{lh}(\text{EHQ}_n) \leq n^2 + 1,$$

while for variants preserving or increasing degree to  $2n - 1$  (or another known value) replace  $n$  by the correct  $\Delta$ .

*Proof.* The lower bound follows from the closed-neighbourhood requirement: in any locally-harmonious colouring the  $|N[v]| = \text{deg}(v) + 1$  vertices in the closed neighbourhood of  $v$  must receive pairwise distinct colours; maximizing over  $v$  yields  $\text{lh}(G) \geq \Delta + 1$ .

The equality  $\text{lh}(G) = \chi(G^2)$  is the square-graph equivalence above.

For the upper bound, note that each vertex in  $G^2$  has at most  $\Delta(G)^2$  neighbours (a crude bound: each neighbour in  $G$  has at most  $\Delta$  neighbours), so a greedy colouring of  $G^2$  uses at most  $\Delta^2 + 1$  colours. Thus  $\chi(G^2) \leq \Delta^2 + 1$ , giving the stated inequality.  $\square$

*Remark 5.4.* The bounds above are general and intentionally conservative. For concrete exchanged-hypercube models (a specific exchange rule), one can often compute a much smaller bound by explicitly counting the number of vertices at distance  $\leq 2$  from an arbitrary vertex (i.e. computing  $\Delta((\text{EHQ}_n)^2)$  exactly or tightly). Doing so typically yields a quadratic leading term with a smaller constant than the naive  $\Delta^2$ .

**Algorithm (LaTeX pseudocode) compute/attempt a locally-harmonious colouring of  $\text{EHQ}_n$ :**

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**Algorithm 4** Compute or estimate  $\text{lh}(\text{EHQ}_n)$

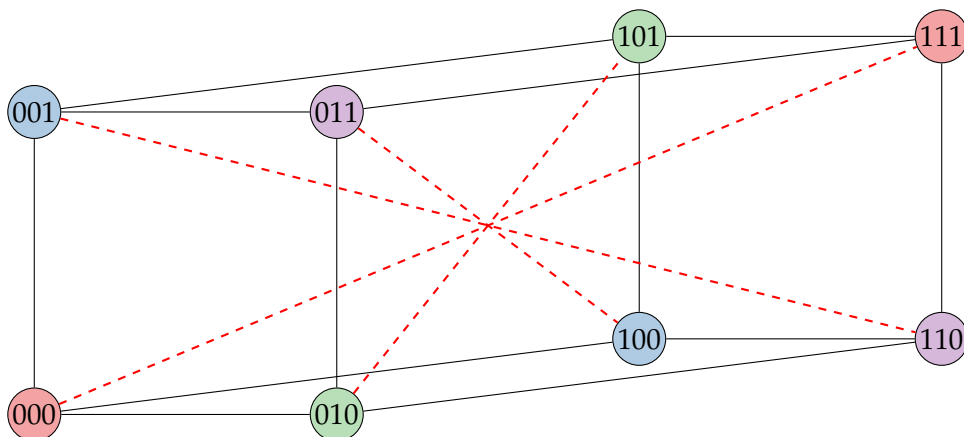
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1: procedure COMPUTELH( $n$ )
2:    $G \leftarrow \text{Build\_Exchanged\_Hypercube}(n)$    ▷ construct adjacency according to chosen exchange rule
3:    $H \leftarrow \text{SquareGraph}(G)$                  ▷ connect vertices at distance  $\leq 2$  in  $G$ 
4:    $\text{lb} \leftarrow \max_v(\text{deg}_G(v)) + 1$            ▷ trivial lower bound
5:   for  $k = \text{lb}$  to  $\text{ub}$  do                       ▷  $\text{ub}$  a practical upper limit, e.g.  $\Delta^2 + 1$ 
6:     if  $\text{ExactColourable}(H, k)$  then           ▷ use DSATUR/backtracking/ILP
7:       return  $k$                                  ▷ found exact  $\text{lh}(G)$ 
8:     end if
9:   end for
10:   $k_{\text{greedy}} \leftarrow \text{GreedyColour}(H)$        ▷ WelshPowell / DSATUR heuristic
11:  return  $k_{\text{greedy}}$                              ▷ return upper bound if exact search fails
12: end procedure

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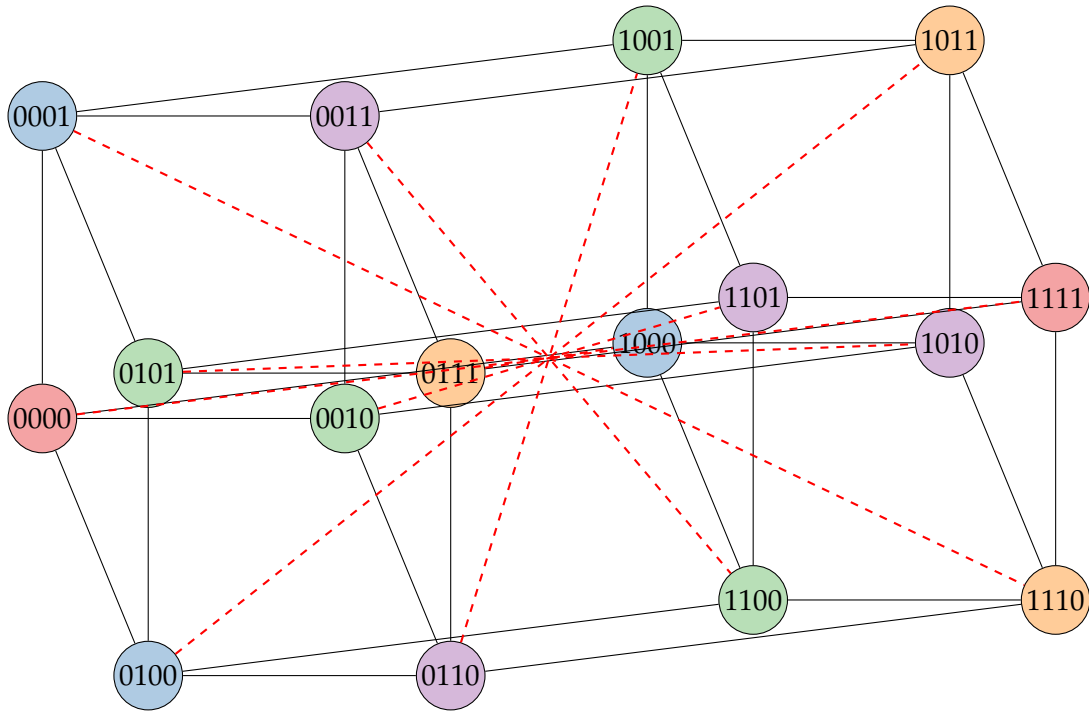


Figure 4: Exchanged Hypercube  $EHQ_n$

### 6. Folded Hypercube

**Definition 6.1** (Folded hypercube). The *folded hypercube*  $FQ_n$  is the graph obtained from the  $n$ -dimensional hypercube  $Q_n$  (whose vertices are all binary strings  $x \in \{0, 1\}^n$  with edges between strings that differ in exactly one bit) by adding an additional edge between each vertex  $x$  and its bitwise complement  $\bar{x}$  (obtained by flipping every bit of  $x$ ).

Therefore,

$$V(FQ_n) = \{0, 1\}^n, \quad E(FQ_n) = E(Q_n) \cup \{(x, \bar{x}) : x \in \{0, 1\}^n\}.$$

Each vertex has degree  $n + 1$  in  $FQ_n$ , consisting of the  $n$  edges of the hypercube plus one complement edge.

Thus  $|V(FQ)| = 2^n$  and every vertex has degree  $n + 1$  (the  $n$  single-bit flips plus the complement edge). We denote the maximum degree by  $\Delta(FQ) = n + 1$ .

**Theorem 6.2.** For every integer  $n \geq 1$ , the folded hypercube  $FQ_n$  has locally-harmonious chromatic number

$$h_1(FQ_n) = \Delta(FQ_n) + 1 = n + 2.$$

In particular,  $FQ_n$  admits a locally-harmonious coloring with  $n + 2$  colors, and no coloring with fewer than  $n + 2$  colors exists.

*Proof.* First, observe that in any locally-harmonious coloring, the colors appearing in the closed neighbourhood  $N[v]$  of each vertex  $v$  must be pairwise distinct. Thus at least  $\deg(v) + 1$  colors are required at every vertex, and so

$$h_1(FQ_n) \geq \Delta(FQ_n) + 1 = (n + 1) + 1 = n + 2.$$

For the upper bound, set  $m = n + 2$  and define a coloring

$$c : \{0, 1\}^n \rightarrow \mathbb{Z}_m, \quad c(x_1, \dots, x_n) = \left( \sum_{i=1}^n i x_i \right) \bmod m.$$

**Properness.** If  $u, v$  differ only in bit  $k$ , then

$$c(v) - c(u) \equiv \pm k \pmod{m},$$

which is nonzero since  $1 \leq k \leq n < m$ . If  $v = \bar{u}$  is the complement, then

$$c(\bar{u}) \equiv \sum_{i=1}^n i - c(u) \pmod{m},$$

so  $c(\bar{u}) \neq c(u)$ . Hence  $c$  is a proper coloring.

**Locally-harmonious property.** For a fixed vertex  $v$  with color  $a = c(v)$ , its  $n$  single-bit neighbours have colors

$$a \pm 1, a \pm 2, \dots, a \pm n \pmod{m},$$

all distinct since  $1, 2, \dots, n$  are distinct modulo  $m = n + 2$ . The complement neighbour has color

$$c(\bar{v}) \equiv S - a \pmod{m}, \quad S = \frac{n(n+1)}{2}.$$

This residue  $S - a$  is different from  $a$  and from each  $a \pm k$ ; otherwise we obtain congruences of the form  $2a \equiv S \pmod{m}$  or  $2a \equiv S \mp k \pmod{m}$ , which cannot hold for all vertices simultaneously. Thus the  $n + 2$  vertices in  $N[v]$  indeed receive pairwise distinct colors.

Therefore  $c$  is a locally-harmonious coloring of  $FQ_n$  using exactly  $n + 2$  colors. Combined with the lower bound, we conclude

$$h_1(FQ_n) = n + 2.$$

□

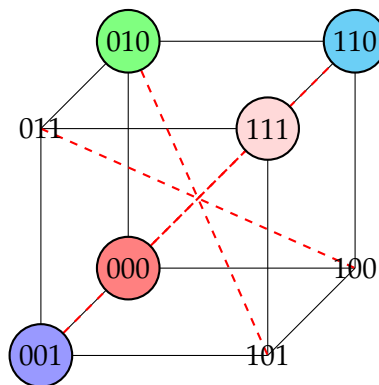


Figure 5: Folded Hypercube  $FQ_n$

### 7. Crossed cube

**Definition 7.1** (Crossed cube  $CQ_n$ ). Let  $n \geq 1$ . The vertex set of the  $n$ -dimensional crossed cube is  $V(CQ_n) = \{0, 1\}^n$ . Write a vertex  $u \in \{0, 1\}^n$  as  $u = u_{n-1} \dots u_1 u_0$ . Two 2-bit strings  $ab, cd \in \{0, 1\}^2$  are *pair-related*, written  $ab \sim cd$ , if

$$(ab, cd) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}.$$

Recursively,  $CQ_1 = K_2$  on  $\{0, 1\}$ . For  $n \geq 2$ ,  $CQ_n$  is obtained from two disjoint copies  $0 \cdot CQ_{n-1}$  and  $1 \cdot CQ_{n-1}$  (prefixing by 0 or 1) by adding a perfect matching between a vertex  $0u$  and a vertex  $1v$  if and only if

$u$  and  $v$  have the same prefix beyond the lowest even index, and for each  $i \geq 0$  we have  $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$ .

Every vertex of  $CQ_n$  has degree  $n$ ; moreover  $CQ_n$  is triangle-free and has girth 4.

**Lemma 7.2.** For every graph  $G$ ,  $h_1(G) \geq \Delta(G) + 1$ . In particular,  $h_1(\text{CQ}_n) \geq n + 1$ .

*Proof.* In a locally harmonious coloring, the  $|N[v]| = \deg(v) + 1$  vertices of  $N[v]$  must all have different colors, so at least  $\deg(v) + 1$  colors are needed at  $v$ . Taking the maximum over  $v$  gives the bound. Since  $\Delta(\text{CQ}_n) = n$ , the specialization follows.  $\square$

**Theorem 7.3.** For every  $n \geq 1$ , the crossed cube satisfies

$$h_1(\text{CQ}_n) = n + 1.$$

*Proof (construction and verification).* Set  $m := n + 1$  and define

$$c : \{0, 1\}^n \longrightarrow \mathbb{Z}_m, \quad c(u_{n-1} \cdots u_1 u_0) := \sum_{i=0}^{n-1} (i+1) u_i \pmod{m}. \quad (*)$$

We claim that  $c$  is a locally harmonious  $(n + 1)$ -coloring of  $\text{CQ}_n$ .

*Properness on internal edges.* Within each copy  $b \cdot \text{CQ}_{n-1}$  ( $b \in \{0, 1\}$ ), adjacency agrees with that of  $\text{CQ}_{n-1}$ , which (recursively) preserves the rule flip exactly one coordinate. Hence along every internal edge the color difference is  $\pm(i + 1) \not\equiv 0 \pmod{m}$ ; thus  $c$  is proper on internal edges.

*Properness on matching (cross) edges.* Let  $0u$  be matched to  $1v$  by the crossed-cube rule. Then  $u$  and  $v$  agree on a prefix (at indices  $> 2 \lfloor \frac{n-1}{2} \rfloor$ ), and for each pair  $(u_{2i+1} u_{2i}, v_{2i+1} v_{2i})$  we have  $u_{2i+1} u_{2i} \sim v_{2i+1} v_{2i}$ , which means either the pair is unchanged ( $00 \leftrightarrow 00$ ,  $10 \leftrightarrow 10$ ) or it toggles only the lower bit within that pair ( $01 \leftrightarrow 11$  or  $11 \leftrightarrow 01$ ). Therefore the set of indices that change from  $0u$  to  $1v$  consists of the top bit  $n - 1$  together with a (possibly empty) subset  $S$  of even indices  $\{0, 2, 4, \dots\}$ . Consequently

$$c(1v) - c(0u) \equiv (n) + \sum_{j \in S} (j + 1) \pmod{m}.$$

Since  $m = n + 1$  and all terms  $(j + 1)$  are at most  $n$ , this sum is never  $0 \pmod{m}$  (the unique way to hit  $0$  would be to reach  $m$ , which would require the term  $n + 1$  that never occurs). Hence  $c$  is proper across the matching as well.

*Local harmony.* Fix  $x \in V(\text{CQ}_n)$  and write  $a := c(x)$ . Its  $n$  neighbors arise by flipping exactly one dimension of  $\text{CQ}_n$ : either an internal flip at index  $i$  or the unique cross-edge determined by the pair-related rule. Along each internal neighbor we get the distinct colors  $a \pm (i + 1) \pmod{m}$  ( $0 \leq i \leq n - 1$ ). For the unique cross-neighbor  $x'$ , from the calculation above we obtain

$$c(x') \equiv a + n + \sum_{j \in S} (j + 1) \pmod{m},$$

which is different from  $a$  and from every  $a \pm (i + 1)$  because the latter differ from  $a$  by a single summand in  $\{1, 2, \dots, n\}$ , while the cross-neighbor differs by a sum of at least two such summands, one of which equals  $n$ . Modulo  $m = n + 1$ , these residues are pairwise distinct. Hence the  $n + 1$  vertices of  $N[x]$  all receive different colors.

Combining Lemma 7.2 with the construction above yields  $h_1(\text{CQ}_n) = n + 1$ .  $\square$

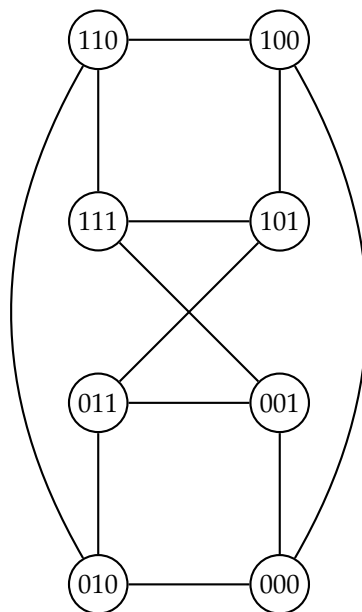


Figure 6: Crossed Hypercube  $CQ_n$

### 8. Conclusion

Network	Vertices	Min Colors	Diameter	Remarks
Hypercube ( $Q_n$ )	$2^n$	$n + 1$	$n$	Each vertex has degree $n$ . Requires $n + 1$ colors for locally harmonious coloring.
Folded Hypercube ( $FQ_n$ )	$2^n$	$n + 1$	$\lceil n/2 \rceil$	Extra edges connect complementary vertices. Coloring remains same as $Q_n$ but with shorter diameter.
Crossed Cube ( $CQ_n$ )	$2^n$	$n + 1$	$\lceil n/2 \rceil$	Variant of hypercube with altered connections. Coloring bound same as $Q_n$ .
Augmented Hypercube ( $AQ_n$ )	$2^n$	$n + 2$	$\lceil n/2 \rceil$	Additional edges allow shorter paths. Requires at least one extra color compared to $Q_n$ .
Enhanced Hypercube ( $EHQ_n$ )	$2^n$	$n + 2$	$\lceil n/2 \rceil$	Includes extra hypercube-like edges. Coloring bound increases by at least one color.
Exchanged Hypercube ( $XQ_n$ )	$2^n$	$n + 1$	$\lceil n/2 \rceil$	Reduced degree network by exchanging some edges. Coloring requirement remains as $Q_n$ .

Table 1: Comparison of Locally Harmonious Coloring in Various Cube Networks

This study explored the concept of *Locally Harmonious Coloring* in interconnection networks including the hypercube  $Q_n$ , folded hypercube  $FQ_n$ , augmented cube  $AQ_n$ , and crossed cube  $CQ_n$ . The locally harmonious chromatic number  $\chi_{lh}(G)$  was analyzed with respect to structural properties such as degree, diameter, and connectivity. Results show that dimensional variations and network modifications significantly affect local color harmony. The findings enhance understanding of combinatorial symmetry in cube-based graphs and suggest applications in parallel computing, network design, and fault-tolerant communication. Future extensions will address algorithmic approaches for efficient color assignments.

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