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Numerical solution of a nonlinear integral equation for the determination of the unknown time-dependent diffusivity of radioactive materials

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Abstract

The thermal diffusion coefficient in radioactive materials is not a constant value, and this makes the heat transfer problem different in such materials and materials that are not homogeneous and have been destroyed or are disintegrating in some way. In the problem under discussion, the heat diffusion coefficient is time-dependent and satisfies a nonlinear integral equation. The existence and uniqueness of the solution to the integral equation in question are discussed in detail in Chapter 13 of the Book "Encyclopedia of the One-Dimensional Heat Equation" by Cannon, J. R. The integral equation in question is not a standard Volterra integral equation and therefore has not been studied much from a numerical perspective. For example, if we apply the fixed point method, which is a powerful tool in the analysis of existence and uniqueness, discussed in Chapter 13 of the aforementioned Book, to a numerical solution, we cannot go even one step forward with this method. Since the unknown function is located at the kernel of a nested integral, applying canonical methods becomes difficult. Therefore, in this paper, we have discussed a hybrid method of numerical integration and iterative methods that solves the problem with sufficient accuracy. In Section 5, we have extracted several sample problems using the properties of the heat equation in the case where the thermal diffusivity is Time-dependent. The numerical solution of these sample problems in the Section 6 demonstrates the efficiency and accuracy of the proposed method.

Keywords: Nonlinear integral equation, Heat equation, Numerical solution, Time-dependent thermal diffusivity.

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1. Introduction

In this paper, we consider the problem of determining the thermal diffusivity of a radioactive conductor, which is changing with time. More presisely, we consider the determination of a positive continuous function $a(t)$ defined on the interval $0 \leqslant t < T$ and a function $u = u(x,t)$ defined on $0 \leqslant x < \infty,\ 0 \leqslant t < T$, such that the pair (a, u) satisfies

$$\begin{aligned} & u_t = a(t)u_{xx}, \qquad 0 < x < \infty, \quad 0 < t < T, \\ & u(x,0) = 0, \qquad 0 < x < \infty, \\ & u(0,t) = \psi(t), \qquad 0 \leqslant t < T,\ \psi(0) = 0, \\ & -a(t)u_x(0,t) = g(t), \quad 0 < t < T, \end{aligned} \tag{1.1}$$

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where ψ and g are given functions that are defined respectively on $0 \leqslant t < T$ and $0 < t < T$. A Physical example giving rise to a system of this form is a heat conduction situation in which the medium is undergoing radioactive decay, so that its thermal conductivity varies with the degree of decay. In turn, the degree of decay may be related to the time, so that the thermal conductivity may be considered a function of time alone [12]. To state the problem precisely, we present a definition and a lemma regarding this issue.

Definition 1.1. By a solution (a, u) of (1.1), we mean that

1. $a(t)$ is positive and continuous on $0 \leqslant t < T$;
2. $u(x, t)$ is continuous for $0 \leqslant t < T$, $0 \leqslant x < \infty$;
3. u_x, u_t and u_{xx} exists and are continuous for $0 < x < \infty$, $0 < t < T$;
4. $\lim_{x \downarrow 0} u_x(x, t)$ exists for $0 < t < T$;
5. a and u satisfy (1.1), and
6. For any t , $0 < t < T$, u and u_x cannot grow faster than $C \exp(x^{\alpha})$, $\alpha < 2$, as x tends to infinity.

According to Definition 1.1, we have the following lemma.

Lemma 1.2. *Under the assumption that ψ is continuously differentiable and $\psi(0) = 0$, problem (1.1) possesses a unique solution if and only if the nonlinear integral equation*

$$a(t) = \frac{\pi^{1/2} g(t)}{\int_0^t \left[\psi'(\tau) / \left(\int_\tau^t a(y) dy \right)^{1/2} \right] d\tau}, \quad 0 < t < T, \tag{1.2}$$

possesses a unique positive solution $a(t)$ that is continuous for $0 \leqslant t < T$.

Proof. See Lemma 13.4.1 of [8]. □

Before turning to obtain the numerical solution of the integral equation (1.2), it is convenient to list the following restrictions upon the data.

Assumption A

We assume that

1. ψ is continuously differentiable on every compact subset of $0 \leqslant t < T$;
2. $\psi(0) = 0$;
3. $\psi^{'}(0) > 0$, $0 < t < T$;
4. g is continuous for $0 \leqslant t < T$, and positive for $0 < t < T$; and
5. The function

$$h(t) = \frac{\pi^{1/2} g(t)}{\int_0^t \left[\psi'(\tau) / (t - \tau)^{1/2} \right] d\tau}, \quad 0 < t < T,$$

satisfies

$$\lim_{t \downarrow 0} h(t) = h_0 > 0$$

Theorem 1.3. *Under **Assumption A**, there exists a solution to the nonlinear integral equation (1.2).*

Proof. See Theorem 13.4.1 of [8]. □

Theorem 1.4. *Under **Assumption A**, the solution to the nonlinear integral equation (1.2) is unique.*

Proof. See Theorem 13.4.2 of [8]. □

According to Theorem 1.4, for the known functions g and ψ, the unique solution to Eq. (1.2) gives us the function $a(t)$, which is in fact the thermal diffusivity of the system under study. Many heat transfer problems involve integral equations or, equivalently, a system of integral equations [1, 2, 3, 4]. Some of these nonlinear integral equations, which are in the form of nonlinear Volterra integral equations or Similar ones, can be solved by linearizing the system of governing equations [1, 2]. However, for singular Kernels, the fractional variation method is also suitable [11, 15].

Given that Eq. (1.2) is not in standard Volterra form or similar, the methods mentioned above, Encounter algorithmic complexity, so that if the same fixed point method relied on in [8] in the analysis of stability, existence, and uniqueness of the solution is used as a numerical method; not even a single Step is taken forward. Most authors who have worked on such equations have analyzed the problem from a theoretical perspective, examining existence, uniqueness, and stability [8, 6, 7, 12]. In one case, Douglas, J. Jr. and Jones, B. F., presented ideas about numerical solutions that did not lead to any sample problems to be compared [9]. With some sample problems that we will design in Section 5, we will see that the presented method is an efficient tool with a high degree of accuracy in numerically solving the problem in question. Therefore, we organize the paper as follows: In Section 2, we present a high-precision tool. For numerically computing the definite integral when the integrand is known over the entire integration interval, but the analytical solution of the integral is not available. In Section 3, we present a numerical integration method for computing improper integrals. Section 4, using Sections 2 and 3, presents a hybrid method for solving the problem, which ultimately leads to a numerical algorithm for solving the problem. In Section 5, we design samples of problems that have analytical solutions for a range of functions, ψ, and g. Finally, in Section 6, we show the applicability and accuracy of the method in some sample problems.

2. Gaussian integration quadrature procedure

Suppose f is a given function on the interval $[a, b]$, then the Gaussian integration quadrature reads as follows:

$$\int_a^b f(t)dt \simeq \sum_{i=1}^{n} c_i f(t_i), \tag{2.1}$$

where the nodes $t_1, \dots, t_n$ in the interval $[a, b]$ and coefficients $c_1, c_2, \dots, c_n$ are chosen to minimize the expected error obtained in the approximation (2.1). Since for every interval $[a, b]$ and $n \in \mathbb{N}$ the pairs

$$\{(t_i, c_i) : i = 1, 2, \dots, n\}, \tag{2.2}$$

can be obtained by any mathematical programming software. We denote the nodes in (2.2) by

GaussianQuadratureWeights(n, a, b) and the approximation value in (2.1) by $I_G(f(t), \{t, a, b\})$. These notations are similar to the symbols and procedure definitions in the Mathematica programming language. For example, GaussianQuadratureWeights$[n, a, b]$ is exactly a Mathematica command. In this language, [ ] should be used instead of () for functions. Note that this is a programming command and words should not be written separately. This rule is constructed in such away that it is exact for any polynomial of degree $2n - 1$. This high convergence speed, combined with the high accuracy provided by the iteration applied to the composite Simpson and trapezoidal methods mentioned in Section 4, leads to excellent numerical results in the numerical output. The convergence analysis of such methods for Eq. (1.2) is given in detail in [9]. For further information about Gaussian integration quadrature, see [5, 10, 13, 14].

3. Numerical solution of an improper integral

Suppose $a, b \in \mathbb{R}$, $0 < c < 1$, then

$$\int_a^b \frac{dt}{(b - t)^c} = \frac{(b - a)^{1-c}}{1 - c}.$$

If f is a function that can be written in the form

$$f(t) = \frac{g(t)}{(b-t)^c},$$

where g is continuous on $[a, b]$, then the improper integral $\int_a^b f(x)dx$ also exists. We will approximate this integral using the Gaussian rule. We can construct the first Taylor polynomial, $P(x)$, for g about b,

$$P(t) = g(b) + g'(b)(t-b) = g(b) - g'(b)(b-t),$$

and write

$$\int_a^b f(t)dt = \int_a^b \frac{g(t)-P(t)}{(b-t)^c}dt + \int_a^b \frac{P(t)}{(b-t)^c}dt.$$

Since $P(t)$ is a polynomial, we can exactly determine the value of

$$\int_a^b \frac{P(t)}{(b-t)^c}dt = g(b)\frac{(b-a)^{1-c}}{1-c} - g'(b)\frac{(b-a)^{2-c}}{2-c}.$$

Suppose $g \in C^2[a, b]$ and define

$$G(t) = \begin{cases} \frac{g(t)-P(t)}{(b-t)^c} & a \leqslant t < b, \\ 0 & t = b. \end{cases}$$

Since $g \in C^2[a, b]$, then for all $t \in [a, b)$ there exists some $\xi_t \in [a, b)$ such that $g(t) - P(t) = \frac{g''(\xi_t)}{2!}(b-t)^2$. Hence $G(t) = \frac{g''(\xi_t)}{2!}(b-t)^{2-c}$, and so $\lim_{t\to b^-} G(t) = 0 = G(b)$. This means that G is continuous at $t = b$, and hence $G \in C[a, b]$. So we can approximate $\int_a^b G(t)dt \simeq I_G(G(t), \{t, a, b\})$ and obtain

$$\int_a^b \frac{g(t)}{(b-t)^c}dt \simeq g(b)\frac{(b-a)^{1-c}}{1-c} - g'(b)\frac{(b-a)^{2-c}}{2-c} + I_G(G(t), \{t, a, b\}). \tag{3.1}$$

We denote the approximation formula (3.1) by $\mathtt{Improper}(g(t), \{t, a, b, c\})$.

4. Composite Simpson-trapezoidal iteration procedure with Gaussian integration iteration

We divide the interval $[0, T]$ in to N_P subintervals, with common length $h = \frac{T}{N_P}$. Suppose $N_P = 2n_1 + 1$ is an odd positive integer, where $n_1 \in \mathbb{N}$. Let $t_i = ih$, $i = 0, 1, \ldots, N_P$, and substitute $t = t_i$ in Eq (1.2) to obtain

$$a(t_i) = \frac{\sqrt{\pi}g(t_i)}{\int_0^{t_i} \frac{\psi'(\tau)}{\sqrt{\int_\tau^{t_i} a(y)dy}}d\tau} \quad i = 1, \ldots, N_P. \tag{4.1}$$

For $i = 1$ we approximate $a(y)$ by $p_{0,1}(y)$, where $p_{j,j+1}(y) = a_j + \frac{y-jh}{h}(a_{j+1} - a_j)$, $t_j \leqslant y \leqslant t_{j+1}$ is the linear interpolation of $a(y)$ passes through the two points $(t_j, a_j), (t_{j+1}, a_{j+1})$ and a_j is the approximation of $a(t_j)$ $j = 0, 1, \ldots, N_p - 1$. Let

$$\phi(\tau, t) = \frac{\psi'(\tau)}{\sqrt{\int_\tau^t p(y)dy}}, \tag{4.2}$$

where $p(y)$ is a piecewise polynomial of linear and quadratic interpolations. So

$$a_1 = \frac{\sqrt{\pi}g(t_1)}{\int_0^{t_1} \phi(\tau, t_1)d\tau} = \frac{\sqrt{\pi}g(t_1)}{\int_0^{t_1} \frac{\psi'(\tau)}{\sqrt{\int_\tau^{t_1} p_{0,1}(y)dy}}d\tau}, \tag{4.3}$$

which is an equation with the only unknown a_1. This is a nonlinear equation in terms of a_1. To solve this equation, we assume that n_I is a natural number, which is the number of iterations of an iterative process and assume that $a_1^{(0)} = a_0$ is the initial estimate of a_1. After n_I iterations, $a_1^{(n_I)} = a_1$ is the final estimated value for a_1. To avoid these complex notations, we have algorithmized this issue in Step 4.1 of Subsection 4.1 with an "If" statement. The same is repeated in Steps 6.1, 7.1, 8.3, and 9.1 of the algorithm of Subsection 4.1 to solve the corresponding nonlinear equations using an iterative technique. These equations are described later in this Section. For $i \geqslant 2$ suppose $p_{i-2,i-1,i}(y)$, $(i-2)h \leqslant y \leqslant ih$ be the quadratic interpolation of $a(y)$ passes through the three points $((i-2)h, a_{i-2}), ((i-1)h, a_{i-1})$, and (ih, a_i). For $i = 2k,\ k = 1, \dots, n_1$ we obtain

$$\int_0^{ih} \phi(\tau, ih)d\tau = \int_0^{2kh} \phi(\tau, 2kh)d\tau = \sum_{j=1}^{k} \int_{2(j-1)h}^{2jh} \phi(\tau, 2kh)d\tau$$
$$= \begin{cases} \int_0^{2h} \phi(\tau, 2h)d\tau & k = 1, \\ \sum_{j=1}^{k-1} \int_{2(j-1)h}^{2jh} \phi(\tau, 2kh)d\tau + \int_{2(k-1)h}^{2kh} \phi(\tau, 2kh)d\tau & 2 \leqslant k \leqslant n_1. \end{cases} \tag{4.4}$$

And for $i = 2k+1,\ k = 1, \dots, n_1$ we have

$$\int_0^{ih} \phi(\tau, ih)d\tau = \int_0^{(2k+1)h} \phi(\tau, (2k+1)h)d\tau \tag{4.5}$$
$$= \begin{cases} \int_0^{2h} \phi(\tau, 3h)d\tau + \int_{2h}^{3h} \phi(\tau, 3h)d\tau & k = 1, \\ \sum_{j=1}^{k-1} \int_{2(j-1)h}^{2jh} \phi(\tau, (2k+1)h)d\tau + \int_{2(k-1)h}^{2kh} \phi(\tau, (2k+1)h)d\tau + \int_{2kh}^{(2k+1)h} \phi(\tau, (2k+1)h)d\tau & 2 \leqslant k \leqslant n_1. \end{cases}$$

For $i = 2k+1,\ k = 0, 1, 2, \dots, n_1$ in the last integral of Eq. (4.5) and Eq. (4.3), we obtain

$$\int_{2kh}^{(2k+1)h} \phi(\tau, (2k+1)h)d\tau = \int_{2kh}^{(2k+1)h} \frac{\psi'(\tau)}{\sqrt{\int_\tau^{(2k+1)h} p_{2k,2k+1}(y)dy}}d\tau$$
$$= \int_{2kh}^{(2k+1)h} \frac{K_T(\tau, i)}{\sqrt{(2k+1)h - \tau}}d\tau = \text{Improper}\left(K_T(\tau, i), \left\{\tau, 2kh, (2k+1)h, \frac{1}{2}\right\}\right), \tag{4.6}$$

where

$$K_T(\tau, i) = \frac{\sqrt{2}\psi'(\tau)}{\sqrt{ia_{i-1} + (2-i)a_i + \frac{\tau}{h}(a_i - a_{i-1})}}.$$

For $k = 1, \dots, n_1$, we see that

$$p_{2k-2,2k-1,2k}(y) = \alpha_k y^2 + \beta_k y + \gamma_k,$$

where

$$\alpha_k = \frac{1}{2h^2}(a_{2k-2} + a_{2k}) - \frac{1}{h^2}a_{2k-1},$$
$$\beta_k = \frac{1}{h}\left(\frac{1-4k}{2}a_{2k-2} - (2-2k)a_{2k-1} + \frac{3-4k}{2}a_{2k}\right),$$
$$\gamma_k = k(2k-1)a_{2k-2} - 2k(2k-2)a_{2k-1} + \frac{(2k-1)(2k-2)}{2}a_{2k}.$$

Hence for $\tau \in [2(k-1)h, 2kh]$, we have

$$\begin{aligned}&\int_{\tau}^{2(k+1)h} p_{2k,2k+1}(y)dy = (2kh-\tau)\left\{\frac{\alpha_k}{3}(4k^2h^2+2kh\tau+\tau^2)+\frac{\beta_k}{2}(2kh+\tau)+\gamma_k\right\}\\ &=(2kh-\tau)\left\{\frac{1}{3h^2}\left(\frac{a_{2k-2}+a_{2k}}{2}-a_{2k-1}\right)\tau^2\right.\\ &+\frac{1}{h}\left(\left(\frac{1}{4}-\frac{2k}{3}\right)a_{2k-2}+\left(\frac{4k}{3}-1\right)a_{2k-1}+\left(\frac{3}{4}-\frac{2k}{3}\right)a_{2k}\right)\tau\\ &\left.+\left(\frac{2k^2}{3}-\frac{k}{2}\right)a_{2k-2}+\left(2k-\frac{4k^2}{3}\right)a_{2k-1}+\left(\frac{2k^2}{3}-\frac{3k}{2}+1\right)a_{2k}\right\}=:\mathrm{simp}(\tau,k).\end{aligned} \tag{4.7}$$

Now by defining

$$K_S(\tau,k):=\frac{\psi'(\tau)}{\sqrt{\frac{\alpha_k}{3}(4k^2h^2+2kh\tau+\tau^2)+\frac{\beta_k}{2}(2kh+\tau)+\gamma_k}},$$

we obtain

$$\int_{2(k-1)h}^{2kh}\phi(\tau,2kh)d\tau=\int_{2(k-1)h}^{2kh}\frac{K_S(\tau,k)}{\sqrt{2kh-\tau}}d\tau=\mathrm{Improper}\left(K_S(\tau,k),\left\{\tau,2(k-1)h,2kh,\frac{1}{2}\right\}\right),$$

which are the improper integrals of (4.4).

To compute the remaining integrals in (4.4), suppose $k=2,\dots,n_1$ and let $1\leqslant j\leqslant k-1$, then we have

$$\int_{2(j-1)h}^{2jh}\phi(\tau,2kh)d\tau=\int_{2(j-1)h}^{2jh}\frac{\psi'(\tau)}{\sqrt{\int_{\tau}^{2kh}p(y)dy}}d\tau,$$

and for $\tau\in[2(j-1)h,2jh]$ we have

$$\begin{aligned}\int_{\tau}^{2kh}p(y)dy&=\int_{\tau}^{2jh}p_{2j-2,2j-1,2j}(y)dy+\sum_{s=j+1}^{k}\int_{(2s-2)h}^{2sh}p_{2s-2,2s-1,2s}(y)dy\\ &=\mathrm{simp}(\tau,j)+\frac{h}{3}\sum_{s=j+1}^{k}\left(a_{2s-2}+4a_{2s-1}+a_{2s}\right),\end{aligned}$$

where $\mathrm{simp}(\tau,j)$ is defined in Eq. (4.7). By defining

$$\phi_S(\tau,k,j):=\frac{\psi'(\tau)}{\sqrt{\mathrm{simp}(\tau,j)+\frac{h}{3}\sum_{s=j+1}^{k}\left(a_{2s-2}+4a_{2s-1}+a_{2s}\right)}},$$

we obtain

$$\int_{2(j-1)h}^{2jh}\phi(\tau,2kh)d\tau\simeq I_G\left(\phi_S(\tau,k,j),\{\tau,2(j-1)h,2jh\}\right),$$

and the associated integral equation is

$$a_{2k}=\frac{\sqrt{\pi}g(2kh)}{\mathrm{Improper}\left(K_S(\tau,k),\left\{\tau,2(k-1)h,2kh,\frac{1}{2}\right\}\right)+\sum_{j=1}^{k-1}I_G\left(\phi_S(\tau,k,j),\{\tau,2(j-1)h,2jh\}\right)},$$

where $k=1,\dots,n_1$. Similarly in the Eq. (4.5), for $k=2,\dots,n_1$ and $1\leqslant j\leqslant k-1$, the integrals inside Σ are

$$\int_{2(j-1)h}^{2jh}\phi(\tau,(2k+1)h)d\tau=\int_{2(j-1)h}^{2jh}\frac{\psi'(\tau)}{\sqrt{\int_{\tau}^{(2k+1)h}p(y)dy}}d\tau.$$

Hence for $\tau \in [2(j-1)h, 2jh]$ we have

$$\int_{\tau}^{(2k+1)h} p(y)dy = \int_{\tau}^{2jh} p_{2j-2,2j-1,2j}(y)dy + \sum_{s=j+1}^{k} \int_{(2s-2)h}^{2sh} p_{2s-2,2s-1,2s}(y)dy + \int_{2kh}^{(2k+1)h} p_{2k,2k+1}(y)dy$$
$$= simp(\tau, j) + \frac{h}{3} \sum_{s=j+1}^{k} (a_{2s-2} + 4a_{2s-1} + a_{2s}) + \frac{h}{2}(a_{2k+1} + a_{2k}),$$

where $\int_{(2s-2)h}^{2sh} p_{2s-2,2s-1,2s}(y)dy = \frac{h}{3}(a_{2s-2} + 4a_{2s-1} + a_{2s})$ and $\int_{2kh}^{(2k+1)h} p_{2k,2k+1}(y)dy = \frac{h}{2}(a_{2k+1} + a_{2k})$ are the Simpson's and Trapezoidal rules respectively. And by defining

$$\phi_0(\tau, k, j) := \frac{\psi'(\tau)}{\sqrt{simp(\tau, j) + \frac{h}{3}\sum_{s=j+1}^{k}(a_{2s-2} + 4a_{2s-1} + a_{2s}) + \frac{h}{2}(a_{2k+1} + a_{2k})}},$$

we obtain

$$\int_{2(j-1)h}^{2jh} \phi(\tau, (2k+1)h)d\tau \simeq I_G\left(\phi_0(\tau, k, j), \{\tau, 2(j-1)h, 2jh\}\right).$$

For complete the computes, for $k = 1, \dots, n_1$ we have

$$\int_{2(k-1)h}^{2kh} \phi(\tau, (2k+1)h)d\tau = \int_{2(k-1)h}^{2kh} \frac{\psi'(\tau)}{\sqrt{\int_{\tau}^{(2k+1)h} p(y)dy}}d\tau,$$

where for $\tau \in [2(k-1)h, 2kh]$ we have

$$\int_{\tau}^{(2k+1)h} p(y)dy = \int_{\tau}^{2kh} p_{2k-2,2k-1,2k}(y)dy + \int_{2kh}^{(2k+1)h} p_{2k,2k+1}(y)dy$$
$$= simp(\tau, k) + \frac{h}{2}(a_{2k+1} + a_{2k}),$$

and hence

$$\int_{2(k-1)h}^{2kh} \phi(\tau, (2k+1)h)d\tau = \int_{2(k-1)h}^{2kh} \frac{\psi'(\tau)}{\sqrt{simp(\tau, k) + \frac{h}{2}(a_{2k+1} + a_{2k})}}d\tau$$
$$=\simeq I_G\left(\frac{\psi'(\tau)}{\sqrt{simp(\tau, k) + \frac{h}{2}(a_{2k+1} + a_{2k})}}, \{\tau, 2(k-1)h, 2kh\}\right).$$

The associated integral equation for Eq. (4.5) is as follows

$$a_{2k+1} = \frac{\sqrt{\pi}g((2k+1)h)}{D_k},$$

where

$$D_k = Improper\left(K_T(\tau, 2k+1), \left\{\tau, 2kh, (2k+1)h, \frac{1}{2}\right\}\right) + \sum_{j=1}^{k-1} I_G\left(\phi_0(\tau, k, j), \{\tau, 2(j-1)h, 2jh\}\right)$$
$$+ I_G\left(\frac{\psi'(\tau)}{\sqrt{simp(\tau, k) + \frac{h}{2}(a_{2k+1} + a_{2k})}}, \{\tau, 2(k-1)h, 2kh\}\right), \quad k = 2, \dots, n_1,$$

and for $k = 1$, we have

$$a_3 = \frac{\sqrt{\pi}g(3h)}{I_G\left(\frac{\psi'(\tau)}{\sqrt{simp(\tau,1) + \frac{h}{2}(a_3 + a_2)}}, \{\tau, 0, 2h\}\right) + Improper\left(K_T(\tau, 3), \{\tau, 2h, 3h, \frac{1}{2}\}\right)}.$$

4.1. The iteration procedure

In this Subsection, we establish a procedure $TGS(n,c)$ with inputs: integer n and $a(0)=c$. This procedure returns the vector $[a_0, a_1, \ldots, a_{2n+1}]$ as the approximation of $[a(0), a(h), \ldots, a((2n+1)h)]$, where $(2n+1)h = T$ and using n_I times of iterations as mentioned after (4.3). We do the following Steps.

Step 1. Define a local environment with local variables $v, h, K_T, K_S, simp, \phi_0, \phi_S$;

Step 2. Set $h = \frac{T}{2n+1}$, $V^T = \underbrace{[c, c, \cdots, c]}_{(2n+2)\text{times}}$;

Step 3. Define the functions

$$K_T(\tau, i) := \frac{\sqrt{2}\psi'(\tau)}{\sqrt{iv_i + (2-i)v_{i+1} + \frac{\tau}{h}(v_{i+1} - v_i)}},$$

where $i = 1, \ldots, 2n+1$, and v_i is the ith component of vector v. (Note that v_i approximates $a((i-1)h)$);

$$\begin{aligned} K_S(\tau, k) := \psi'(\tau) \Big/ \Bigg\{ & \left(\frac{2k^2}{3} - \frac{k}{2}\right) v_{2k-1} + \left(2k - \frac{4k^2}{3}\right) v_{2k} + \left(\frac{2k^2}{3} - \frac{3k}{2} + 1\right) v_{2k+1} \\ & + \frac{\tau}{h}\left[\left(\frac{1}{4} - \frac{2k}{3}\right) v_{2k-1} + \left(\frac{4k}{3} - 1\right) v_{2k} + \left(\frac{3}{4} - \frac{2k}{3}\right) v_{2k+1}\right] \\ & + \frac{\tau^2}{3h^2}\left(\frac{v_{2k-1} + v_{2k+1}}{2} - v_{2k}\right)\Bigg\}^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} simp(\tau, k) := (2kh - \tau) \Bigg\{ & \left(\frac{2k^2}{3} - \frac{k}{2}\right) v_{2k-1} + \left(2k - \frac{4k^2}{3}\right) v_{2k} + \left(\frac{2k^2}{3} - \frac{3k}{2} + 1\right) v_{2k+1} \\ & + \frac{\tau}{h}\left[\left(\frac{1}{4} - \frac{2k}{3}\right) v_{2k-1} + \left(\frac{4k}{3} - 1\right) v_{2k} + \left(\frac{3}{4} - \frac{2k}{3}\right) v_{2k+1}\right] \\ & + \frac{\tau^2}{3h^2}\left(\frac{v_{2k-1} + v_{2k+1}}{2} - v_{2k}\right)\Bigg\}, \end{aligned}$$

$$\phi_0(\tau, k, j) := \frac{\psi'(\tau)}{\sqrt{simp(\tau, j) + \frac{h}{2}(v_{2k+2} + v_{2k+1}) + \frac{h}{3}\sum_{s=j+1}^{k}(v_{2s-1} + 4v_{2s} + v_{2s+1})}};$$

where $k = 1, \cdots, n$, $1 \leqslant j \leqslant k-1$. Set $l = 1$;

Step 4.

Step 4.1 If $l \leqslant n_I$ set $v2 = \frac{\sqrt{\pi}g(h)}{Improper(K_T(\tau,1),\{\tau,0,h,\frac{1}{2}\})}$, else go to Step 5;

Step 4.2 Set $l = l+1$ and go to Step 4.1;

Step 5. Define

$$\phi_S(\tau, k, j) := \frac{\psi'(\tau)}{\sqrt{simp(\tau, j) + \frac{h}{3}\sum_{s=j+1}^{k}(v_{2s-1} + 4v_{2s} + v_{2s+1})}};$$

Step 6. Set $v_3 = v_2$, $l = 1$;

Step 6.1 If $l \leqslant n_I$ set $v3 = \frac{\sqrt{\pi}g(2h)}{Improper(K_S(\tau,1),\{\tau,0,2h,\frac{1}{2}\})}$, else go to Step 7;

Step 6.2 Set $l = l+1$ and go to Step 6.1;

Step 7. Set $v_4 = v_3$, $l = 1$;

Step 7.1 If $l \leqslant n_I$ set $v4 = \frac{\sqrt{\pi}g(3h)}{I_G\left(\frac{\psi'(\tau)}{\sqrt{simp(\tau,1)+\frac{h}{2}(v_4+v_3)}},\{\tau,0,2h\}\right)+Improper(K_T(\tau,3),\{\tau,2h,3h,\frac{1}{2}\})}$, else go to Step 8;

Step 7.2 Set $l = l+1$ and go to Step 7.1;

Step 8. Set $k = 2$;

Step 8.1 If $k \leqslant n$ set $v_{2k+1} = v_{2k}$, else go to Step 10;

Step 8.2 Set $l = 1$;

Step 8.3 If $l \leqslant n_I$ set $v2k+1 = \frac{\sqrt{\pi}g(2kh)}{\sum_{j=1}^{k-1} I_G(\phi_S(\tau,k,j),\{\tau,2(j-1)h,2jh\})+\mathrm{Improper}\left(K_S(\tau,k),\{\tau,2(k-1)h,2kh,\frac{1}{2}\}\right)}$, else go to Step 9;

Step 8.4 Set $l = l+1$ and go to Step 8.3;

Step 9. Set $v_{2k+2} = v_{2k+1}$, $l = 1$;

Step 9.1 If $l \leqslant n_I$ set

$$v2k+2 = \sqrt{\pi}g((2k+1)h) \Big/ \left\{ I_G\left(\frac{\psi'(\tau)}{\sqrt{\mathrm{simp}(\tau,k)+\frac{h}{2}(v_{2k+2}+v_{2k+1})}}, \{\tau, 2(k-1)h, 2kh\}\right) + \sum_{j=1}^{k-1} I_G\left(\phi_0(\tau,k,j), \{\tau, 2(j-1)h, 2jh\}\right) + \mathrm{Improper}\left(K_T(\tau, 2k+1), \left\{\tau, 2h, (2k+1)h, \frac{1}{2}\right\}\right) \right\},$$

else go to Step 9.3;

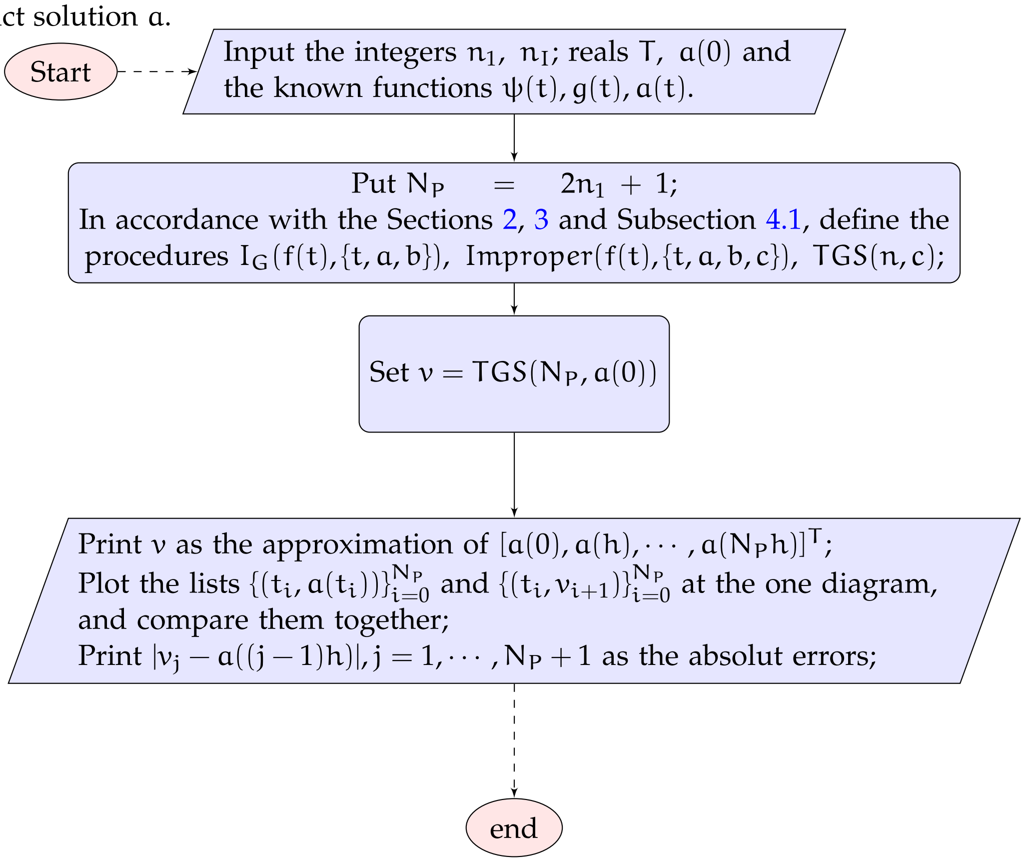
Step 9.2 Set $l = l+1$ and go to Step 9.1;

Step 9.3 Set $k = k+1$ and go to Step 8.1;

Step 10. Return v as the output of the procedure, and end the procedure.

4.2. *Total algorithm*

In accordance with Section 5, suppose we have a sample problem with exact solution $a(t)$ for Eq. (1.2). From the following flowchart, we obtain the approximated solution, v, and compare it with the exact solution a.



5. Some sample problems

We are trying to find a solution in the form of $u = \Phi(\lambda)$, $\lambda = \lambda(x, t)$ for problem (1.1) such that $a(t)$ is a continuous non-constant function. Using the chain rule, we observe that

$$\frac{\partial u}{\partial t} = \frac{d\Phi}{d\lambda}\frac{\partial \lambda}{\partial t} = \Phi^{'}\lambda_t, \tag{5.1}$$

$$\frac{\partial u}{\partial x} = \frac{d\Phi}{d\lambda}\frac{\partial \lambda}{\partial x} = \Phi^{'}\lambda_x,$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\Phi^{'}\lambda_x\right)\\ &= \left(\frac{\partial}{\partial x}\frac{d\Phi}{d\lambda}\right)\lambda_x + \Phi^{'}\frac{\partial}{\partial x}(\lambda_x)\\ &= \left(\frac{\partial}{\partial \lambda}\frac{d\Phi}{d\lambda}\right)\frac{\partial \lambda}{\partial x}\lambda_x + \Phi^{'}\lambda_{xx}\\ &= \frac{d^2\Phi}{d\lambda^2}(\lambda_x)^2 + \Phi^{'}\lambda_{xx}\end{aligned} \tag{5.2}$$

By substituting (5.1) and (5.2) into $u_t = a(t)u_{xx}$, we have

$$\Phi^{'}\lambda_t = a(t)\,(\lambda_x)^2\,\Phi^{''} + a(t)\lambda_{xx}\Phi^{'},$$

Or

$$\Phi^{'} = \frac{a(t)\,(\lambda_x)^2}{\lambda_t}\Phi^{''} + \frac{a(t)\lambda_{xx}}{\lambda_t}\Phi^{'}. \tag{5.3}$$

Therefore, function $\lambda = \lambda(x, t)$ should be chosen such that

$$\begin{cases} \frac{a(t)(\lambda_x)^2}{\lambda_t} = f(\lambda), \\ \frac{a(t)\lambda_{xx}}{\lambda_t} = h(\lambda), \end{cases} \tag{5.4}$$

where $f(\lambda)$ and $h(\lambda)$ are functions of λ and for $\lambda(x, t) = \frac{p(x)}{q(t)}$ we have

$$\begin{cases} \frac{a(t)\left(p^{'}(x)\right)^2}{-p(x)\dot{q}(t)} = f\left(\frac{p(x)}{q(t)}\right), \\ \frac{a(t)p^{''}(x)}{-p(x)q(t)} = h\left(\frac{p(x)}{q(t)}\right), \end{cases} \tag{5.5}$$

where $\dot{q}(t) = \frac{d}{dt}q(t)$. By choosing $f(\lambda) = -\frac{1}{2\lambda}$ in (5.5), we obtain

$$\frac{a(t)\left(p^{'}(x)\right)^2}{-p(x)\dot{q}(t)} = \frac{-q(t)}{2p(x)},$$

and hence

$$\left(p^{'}(x)\right)^2 = \frac{q(t)\dot{q}(t)}{2a(t)} = k, \tag{5.6}$$

where k is a constant. By choosing $k = 1$, $p^{'}(x) = \pm 1$ and hence $p(x) = \pm x + c$, where c is the integration constant. In this work, we choose $p(x) = x + 1$. So $p^{''} = 0$, and from Eq. (5.5) we obtain $h(\lambda) = 0$. Thus (5.3) reduces to $\Phi^{'} = -\frac{1}{2\lambda}\Phi^{''}$ and we obtain $-2\lambda = \frac{\Phi^{''}}{\Phi^{'}} = \left(\ln c_0|\Phi^{'}|\right)^{'}$, where $c_0 > 0$ is the integration constant. The latter equation yields $\Phi' = c\exp\left(-\lambda^2\right)$, where c is the integration constant. A choice for $\Phi(\lambda)$ is

$$\Phi(\lambda) = \mathrm{Erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} \exp\left(-\rho^2\right) d\rho. \tag{5.7}$$

Now we evaluate function $q(t)$ using Eq. (5.6) for $k = 1$. So $\frac{q(t)\dot{q}(t)}{2a(t)} = 1$, and one solution to this equation is $q^2 = 4\int_0^t a(\tau)d\tau$. We show that for $q(t) = 2\sqrt{\int_0^t a(\tau)d\tau}$, the following function

$$u(x,t) = \mathrm{Erfc}\left(\frac{x+1}{2\sqrt{\int_0^t a(\tau)d\tau}}\right), \tag{5.8}$$

is true for some functions $\psi(t)$ and $g(t)$ in equations (1.1). Initial condition $u(x,0) = 0$, is satisfies because

$$u(x,0) = \lim_{t\to 0^+} \mathrm{Erfc}\left(\frac{x+1}{2\sqrt{\int_0^t a(\tau)d\tau}}\right) = \lim_{\lambda\to\infty} \mathrm{Erfc}(\lambda) = 0.$$

To verify the validity of the second condition of (1.1), we consider that

$$u(0,t) = \mathrm{Erfc}\left(\frac{1}{2\sqrt{\int_0^t a(\tau)d\tau}}\right),$$

so by defining

$$\psi(t) := \begin{cases} \mathrm{Erfc}\left(\frac{1}{2\sqrt{\int_0^t a(\tau)d\tau}}\right), & 0 < t, \\ 0, & \text{Otherwise,} \end{cases} \tag{5.9}$$

the second condition also holds for the continuous and positive function $a(t)$, $t \in [0,T)$. To investigate the correctness of the final condition, we consider that

$$g(t) = -a(t)u_x(0,t) = \frac{a(t)}{\sqrt{\pi}} \frac{1}{\sqrt{\int_0^t a(\tau)d\tau}} \exp\left(-\frac{1}{4\int_0^t a(\tau)d\tau}\right),$$

so we define

$$g(t) := \begin{cases} \frac{a(t)}{\sqrt{\pi}} \frac{1}{\sqrt{\int_0^t a(\tau)d\tau}} \exp\left(-\frac{1}{4\int_0^t a(\tau)d\tau}\right), & 0 < t, \\ 0, & \text{Otherwise,} \end{cases} \tag{5.10}$$

The functions $\psi(t)$ and $g(t)$ defined by relations (5.9) and (5.10) also satisfy **Assumption A** and, therefore, for any given continuous and positive function $a(t)$, $t \in [0,T)$, ranges of sample problems are generated.

6. Numerical results

Example 6.1. For $a(t) = 1 + \cos t$, $t \in [0,1)$, in relations (5.9) and (5.10), from Section 5, we have

$$\psi(t) = \begin{cases} \mathrm{Erfc}\left(\frac{1}{2\sqrt{t+\sin t}}\right), & 0 < t, \\ 0, & \text{Otherwise,} \end{cases}$$

$$g(t) = \begin{cases} \frac{\exp\left\{-\frac{1}{4(t+\sin t)}\right\}(1+\cos t)}{\sqrt{\pi}\sqrt{t+\sin t}}, & 0 < t, \\ 0, & \text{Otherwise.} \end{cases}$$

Hence the integral equation (1.2) with the above data has the exact solution $a(t) = 1 + \cos t,\ t \in [0, 1)$. Table 1 shows absolute and relative errors of $\tilde{a}$ at $t_i = 10ih,\ i = 1, \ldots, 10, h = T/N_P, T = 1, N_P = 2n_1 + 1, n_1 = 50,\ n_I = 10$, a is the exact solution, and $\tilde{a}$ is evaluated by the proposed method. In this Table, the second column is absolute and the third column is relative errors. Figure 1 shows the variation of these solutions as functions of t for Example 6.1.

Table 1: Absolute and relative errors of $\tilde{a}$ for Example 6.1.

i	$\lvert a - \tilde{a}\rvert_{t=t_i}$	$\left\lvert \frac{a-\tilde{a}}{a} \right\rvert_{t=t_i}$
1	1.21×10^{-6}	6.08×10^{-7}
2	9.63×10^{-7}	4.86×10^{-7}
3	7.84×10^{-7}	4.01×10^{-7}
4	6.63×10^{-7}	3.45×10^{-7}
5	5.75×10^{-7}	3.06×10^{-7}
6	5.05×10^{-7}	2.76×10^{-7}
7	4.48×10^{-7}	2.53×10^{-7}
8	4.00×10^{-7}	2.35×10^{-7}
9	3.57×10^{-7}	2.19×10^{-7}
10	3.20×10^{-7}	2.07×10^{-7}

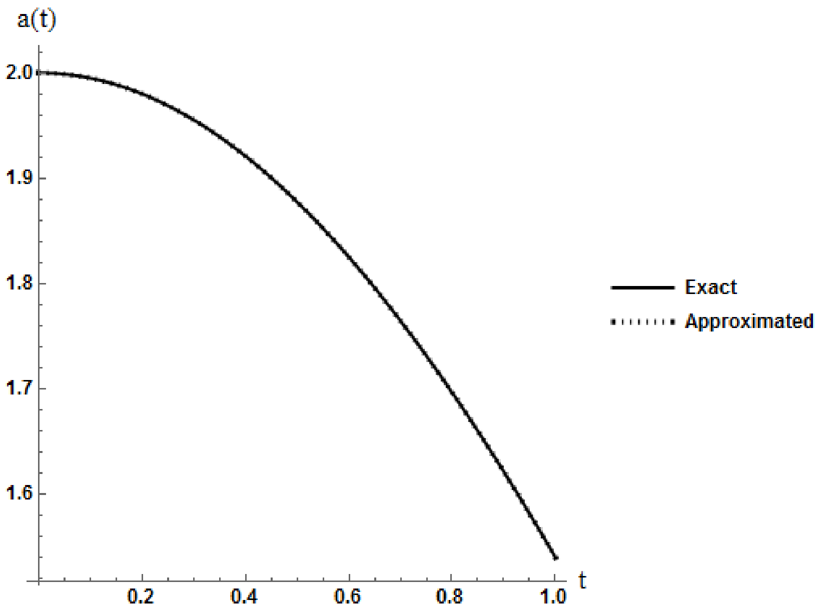


Figure 1: Variations of $a(t)$ and $\tilde{a}(t)$ as functions of t for Example 6.1.

Example 6.2. For $a(t) = 2 + \sin t,\ t \in [0, \pi)$, in relations (5.9) and (5.10), from Section 5, we have

$$\psi(t) = \begin{cases} \mathrm{Erfc}\left(\frac{1}{2\sqrt{2t+1-\cos t}}\right), & 0 < t, \\ 0, & \text{Otherwise}, \end{cases}$$

$$g(t) = \begin{cases} \frac{\exp\left\{\frac{1}{-4-8t+4\cos t}\right\}(2+\sin t)}{\sqrt{\pi}\sqrt{1+2t-\cos t}}, & 0 < t, \\ 0, & \text{Otherwise}. \end{cases}$$

Hence the integral equation (1.2) with the above data has the exact solution $a(t) = 2 + \sin t,\ t \in [0, \pi)$. Table 2 shows absolute and relative errors of $\tilde{a}$ at $t_i = 30ih,\ i = 1, \ldots, 10, h = T/N_P, T = \pi, N_P =$

$2n_1+1, n_1 = 150,\ n_I = 10$, a is the exact solution, and $\widetilde{a}$ is evaluated by the proposed method. In this Table, the second column is absolute and the third column is relative errors. Figure 2 shows the variation of these solutions as functions of t for Example 6.2.

Table 2: Absolute and relative errors of $\widetilde{a}$ for Example 6.2.

i	$\lvert a-\widetilde{a}\rvert_{t=t_i}$	$\left\lvert\frac{a-\widetilde{a}}{a}\right\rvert_{t=t_i}$
1	1.90×10^{-7}	8.25×10^{-8}
2	2.10×10^{-7}	8.14×10^{-8}
3	2.17×10^{-7}	7.72×10^{-8}
4	2.14×10^{-7}	7.25×10^{-8}
5	2.03×10^{-7}	6.78×10^{-8}
6	1.87×10^{-7}	6.33×10^{-8}
7	1.67×10^{-7}	5.92×10^{-8}
8	1.44×10^{-7}	5.54×10^{-8}
9	1.21×10^{-7}	5.20×10^{-8}
10	9.84×10^{-8}	4.89×10^{-8}

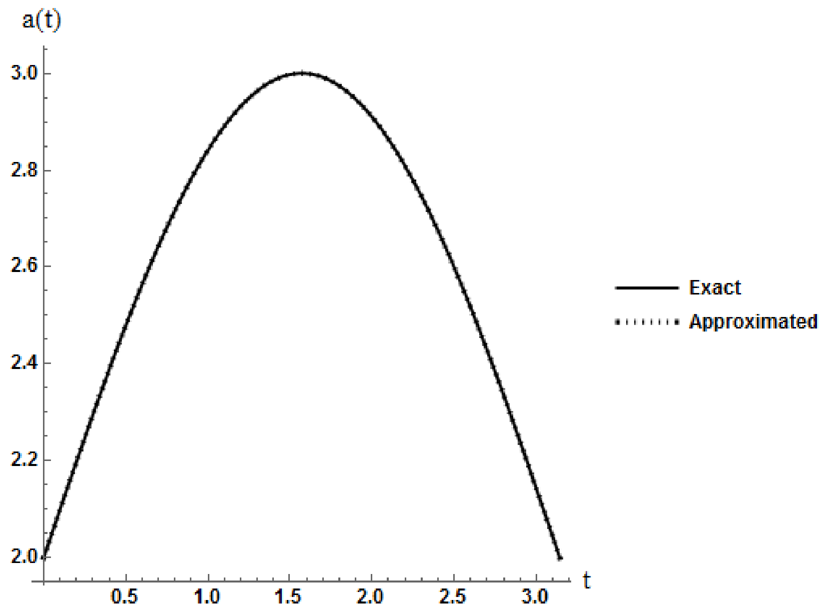


Figure 2: Variations of $a(t)$ and $\widetilde{a}(t)$ as functions of t for Example 6.2.

Example 6.3. For $a(t) = 2+2t,\ t\in[0,10)$, in relations (5.9) and (5.10), from Section 5, we have

$$\psi(t) = \begin{cases} \mathrm{Erfc}\left(\frac{1}{2\sqrt{t^2+2t}}\right), & 0<t, \\ 0, & \text{Otherwise,} \end{cases}$$

$$g(t) = \begin{cases} \frac{2(t+1)}{\sqrt{\pi}}\frac{1}{\sqrt{t^2+2t}}\exp\left(-\frac{1}{4t^2+8t}\right), & 0<t, \\ 0, & \text{Otherwise.} \end{cases}$$

Hence the integral equation (1.2) with the above data has the exact solution $a(t) = 2+2t,\ t\in[0,10)$. Table 3 shows absolute and relative errors of $\widetilde{a}$ at $t_i = 10ih,\ i = 1,\dots,10, h = T/N_P, T = 10, N_P = 2n_1+1, n_1 = 50,\ n_I = 10$, a is the exact solution and $\widetilde{a}$ is evaluated by the proposed method. In this Table, the second column is absolute and the third column is relative errors. Figure 3 shows the variation of these solutions as functions of t for Example 6.3.

Table 3: Absolute and relative errors of $\tilde{a}$ for Example 6.3.

i	$\|a-\tilde{a}\|$	$\frac{a-\tilde{a}}{a}$
1	5.17×10^{-6}	1.30×10^{-6}
2	4.05×10^{-6}	6.79×10^{-7}
3	3.71×10^{-6}	4.67×10^{-7}
4	3.55×10^{-6}	3.58×10^{-7}
5	3.46×10^{-6}	2.91×10^{-7}
6	3.40×10^{-6}	2.45×10^{-7}
7	3.37×10^{-6}	2.12×10^{-7}
8	3.34×10^{-6}	1.87×10^{-7}
9	3.32×10^{-6}	1.68×10^{-7}
10	3.31×10^{-6}	1.52×10^{-7}

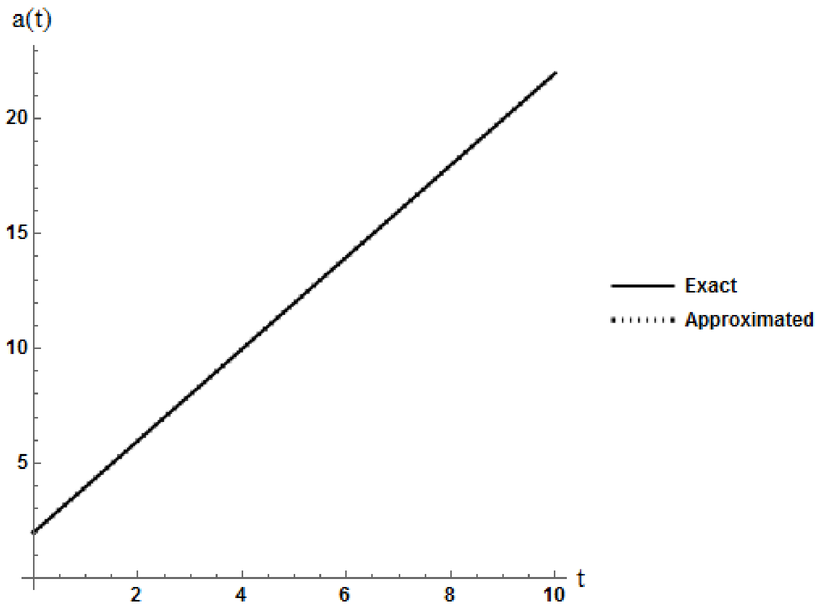


Figure 3: Variations of $a(t)$ and $\tilde{a}(t)$ as functions of t for Example 6.3.

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