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On the degree of the Birkhoff polytope graph

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Abstract

The Birkhoff polytope graph can be considered as the Cayley graph of the symmetric group S_n with respect to $\mathcal{C}_n$, the set of cycles in S_n. Since the degree of every Cayley graph is a natural bound on several parameters of the graph, in this note by presenting a formula for $|\mathcal{C}_n|$, the degree of the Birkhoff polytope graph, we prove that it is bounded from above by $\lfloor e\big((n-1)!+(n-2)!+(n-3)!+\cdots 1\big)\rfloor$, where e is the Neper number.

Keywords: Birkhoff Polytope graph, Degree of the graph

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1. Introduction and the preliminary results

Throughout this note, by a graph we mean a finite undirected graph without multiple edges and without loops, unless it is explicitly mentioned. The *Cayley graph* $\mathrm{Cay}(G, C)$ *of* G *with respect to an inverse closed generating set* C *of* G is defined as the graph with vertex set G and edge set $E(\mathrm{Cay}(G, C))$ consisting of $\{g, g'\}$ such that $gc = g'$, for some $c \in C$.

For every natural number $n \geqslant 3$ by S_n we mean the symmetric group on the set $[n] = \{1, \ldots, n\}$. Recall that a permutation in S_n for which there exist $a_1, \ldots, a_k \in [n]$ such that the permutation sending a_i to a_{i+1} for $i = 1, \ldots, k-1$, sending a_k to a_1, and fixing all other numbers in $[n]$ is called a cycle and it is denoted by $(a_1, \ldots, a_k)$. Recall that a square matrix with exactly one '1' in each row and column, and zeros everywhere else is called a permutation matrix. The group of all $n \times n$ permutation matrices is denoted by $\mathrm{Sym}(n)$ and it is known that it can be regarded as the symmetric group S_n. From now on, we denote the set of cycles in $\mathrm{Sym}(n)$ by $\mathfrak{C}_n$.

By a doubly stochastic matrix we mean a matrix with non-negative entries where both the row sums and column sums are equal to 1. By Birkhoff's theorem we know that the convex hull of the set $\mathrm{Sym}(n)$ of $n \times n$ permutation matrices is the set Ω_n, where Ω_n is the set of doubly stochastic matrices (equivalently, any doubly stochastic matrix can be expressed as a weighted average of permutation matrices, where the weights are non-negative and sum to 1). The set Ω_n is called the Birkhoff polytope or the assignment polytope (see [3]).

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The Birkhoff polytope graph $G(\Omega_n)$ is the skeleton of Ω_n, whose vertex set is $\mathrm{Sym}(n)$ and two vertices in $G(\Omega_n)$ are adjacent if their convex hull is an edge of Ω_n (equivalently, two permutations σ_1 and σ_2 are adjacent if $\tau = \sigma_1^{-1}\sigma_2$ is a cycle). Therefore $G(\Omega_n)$ is the Cayley graph $\mathrm{Cay}(\mathrm{Sym}(n), \mathfrak{C}_n)$ which is isomorphic to $\mathrm{Cay}(S_n, \mathcal{C}_n)$, where $\mathcal{C}_n$ denotes the set of cycles in S_n. From now on, to study the Birkhoff polytope graph, without lose of generality we assume that it is equal to $\mathrm{Cay}(S_n, \mathcal{C}_n)$. Note that by [2, Theorem 4] it is known that the degree of this graph is equal to $|\mathcal{C}_n| = \sum_{k=0}^{n-2} \left((n-k-1)! \times \binom{n}{k}\right)$.

The Birkhoff polytope graph and its properties have been studied by many authors (e.g., see [1]-[4]). As the main result of this note, we focus on the degree of the Birkhoff polytope graph and in Theorem 2.2 we present a formula for the number of cycles in S_n. Then using this formula we present the following upper bound for the number of cycles in S_n which is a natural bound for several parameters of $G(\Omega_n)$.

Proposition 1.1. *For every natural number n, the cardinal of $\mathcal{C}_n$ is bounded from the above by*

$$|\mathcal{C}_n| \leqslant \lfloor e \times \left((n-1)! + (n-2)! + (n-3)! + \cdots + 1\right) \rfloor$$

2. The number of cycles in symmetric group S_n

To prove Theorem 2.2 using [2, Theorem 4], first we need to present the following recursive formula which is the foundation of our approach for simplifying $(k-1)! \times \binom{n}{k}$ in Equation (1).

Lemma 2.1. *Let $n \in \mathbb{N}$ and $1 \leqslant k \leqslant n-2$. Then $n(n-1)\cdots(n-k)$ is equal to*

$$\begin{aligned}&(n-1)(n-2)\cdots(n-k-1) + (k+1)(n-2)\cdots(n-k-1)\\&+ (k+1)k(n-3)\cdots(n-k-1) + \cdots + (k+1)! \times (n-k-1) + (k+1)!\end{aligned}$$

Proof. First note that for every $n \in \mathbb{N}$ we have

$$(*) \quad n(n-1) = (n-1)(n-2) + 2(n-2) + 2.$$

Induction (I): We prove the assertion by induction on n. For $n = 3$ and $k = 1$, by the above equality we have $3 \times 2 = 1 \times 2 + 2 \times 1 + 2$. We suppose that the assertion holds for $n-1$; and use this assumption to prove it for n.

Induction (II): First note that by equality (*), for every natural number n and $k = 1$ the assertion holds. Suppose that the assertion holds for n and $k-1$, where $k \leqslant n-2$. We prove the assertion for n and k. By the assumption of Induction (II), for n and $k-1$ we know that $n(n-1)\cdots(n-k+1)(n-k)$ is equal to

$$\left((n-1)(n-2)\cdots(n-k) + k(n-2)\cdots(n-k) + k(k-1)(n-3)\cdots(n-k) + \cdots + k!(n-k) + k!\right)\left((n-k-1)+1\right).$$

Now by multiplying and applying the assumption of Induction (I) we conclude that $n(n-1)\cdots(n-k+1)(n-k)$ is equal to

$$
\begin{aligned}
&\Big((n-1)(n-2)\cdots(n-k-1)+k(n-2)\cdots(n-k-1)+k(k-1)(n-3)\cdots(n-k-1)+\cdots+k!(n-k)(n-k-1)\\
&+k!(n-k-1)\Big)\\
&+\Big((n-1)(n-2)\cdots(n-k)+k(n-2)\cdots(n-k)+k(k-1)(n-3)\cdots(n-k)+\cdots+k!(n-k)+k!\Big)\\
&=\\
&(n-1)(n-2)\cdots(n-k-1)+k(n-2)\cdots(n-k-1)+k(k-1)(n-3)\cdots(n-k-1)+\cdots+k!(n-k)(n-k-1)\\
&+k!(n-k-1)\\
&+(n-2)(n-3)\cdots(n-k-1)+k(n-3)\cdots(n-k-1)+k(k-1)(n-4)\cdots(n-k-1)+\cdots+k!(n-k-1)+k!\\
&+k(n-3)\cdots(n-k-1)+k(k-1)(n-4)\cdots(n-k-1)+k(k-1)(k-2)(n-5)\cdots(n-k-1)+\cdots\\
&+k!(n-k-1)+k!\\
&+k(k-1)(n-4)\cdots(n-k-1)+k(k-1)(k-2)(n-5)\cdots(n-k-1)+\cdots+k!(n-k-1)+k!\\
&\vdots\\
&+k!(n-k-1)+k!\\
&+k!\\
&=\\
&(n-1)(n-2)\cdots(n-k-1)+(k+1)(n-2)\cdots(n-k-1)+(k(k-1)+2k)(n-3)\cdots(n-k-1)\\
&+\big(k(k-1)(k-2)+3k(k-1)\big)(n-4)\cdots(n-k-1)\\
&+\cdots+(\underbrace{k!+k!+\cdots+k!}_{k+1})(n-k-1)+(\underbrace{k!+k!+\cdots+k!}_{k+1})\\
&=(n-1)\cdots(n-k-1)+(k+1)(n-2)\cdots(n-k-1)+(k+1)k(n-3)\cdots(n-k-1)+\\
&\quad(k+1)k(k-1)(n-4)\cdots(n-k-1)+\cdots+(k+1)!(n-k-1)+(k+1)!
\end{aligned}
$$

and so the assertion of Induction (II) holds for k. Therefore the assertion in Induction (I) holds, too. □

Theorem 2.2. *The number of cycles in S_n is equal to*

$$|\mathcal{C}_n|=(n-1)!\big(\sum_{k=2}^{n}\frac{1}{(n-k)!}\big)+(n-2)!\big(\sum_{k=2}^{n-1}\frac{1}{(n-k-1)!}\big)+(n-3)!\big(\sum_{k=2}^{n-2}\frac{1}{(n-k-2)!}\big)+\cdots+1.$$

Proof. Recall that by [1, Theorem 4] we know that

$$|\mathcal{C}_n|=\sum_{k=0}^{n-2}\Big((n-k-1)!\times\binom{n}{k}\Big)=\sum_{k=2}^{n}\Big((k-1)!\times\binom{n}{k}\Big). \tag{2.1}$$

To prove the assertion, we divide the above sum as follows

$$|\mathcal{C}_n|=\sum_{k=2}^{n}(k-1)!\binom{n}{k}=\sum_{k=2}^{\lfloor\frac{n}{2}\rfloor}(k-1)!\binom{n}{k}+\Big(\sum_{k=\lfloor\frac{n}{2}\rfloor+1}^{n-2}(k-1)!\binom{n}{k}\Big)+((n-2)!\times n)+(n-1)!.$$

For every $2\leqslant k\leqslant\lfloor\frac{n}{2}\rfloor$, we know that $n-k\geqslant k$, and we simplify $(k-1)!\binom{n}{k}$ as follows.

$$(k-1)!\times\binom{n}{k}=\frac{n\times(n-1)\times\cdots(n-k)\times\cdots\times(k+1)\times(k-1)\times\cdots\times 1}{(n-k)!}$$

Now by Lemma 2.1 since

$$\begin{aligned}n(n-1)\cdots(k+1)(k-1)\times\cdots\times 1 &= \Big((n-1)(n-2)\cdots k\\ &\quad+(n-k)(n-2)\cdots k+(n-k)(n-k-1)(n-3)\cdots k\\ &\quad+\cdots+(n-k)!\times k+(n-k)!\Big)(k-1)\times\cdots\times 1\\ &=(n-1)!+(n-2)!(n-k)+(n-3)!\times(n-k)(n-k-1)\\ &\quad+\cdots+(n-k)!\times k!+(n-k)!\times(k-1)!,\end{aligned}$$

we have

$$\begin{aligned}(k-1)!\times\binom{n}{k} &= \Big((n-1)!+(n-2)!(n-k)+(n-3)!\times(n-k)(n-k-1)+\times+k!(n-k)!\\ &\quad+(k-1)!(n-k)!\Big)/(n-k)!\\ &=(n-1)!\times\frac{1}{(n-k)!}+(n-2)!\frac{1}{(n-k-1)!}+(n-3)!\frac{1}{(n-k-2)!}\\ &\quad+\cdots+k!+(k-1)!.\end{aligned}$$

On the other hand, if $\lfloor\frac{n}{2}\rfloor+1\leqslant k\leqslant n-2$, then $n-k+1\leqslant k$ and we have

$$(k-1)!\binom{n}{k}=\frac{n(n-1)\cdots(n-k+1)}{k}=\frac{n(n-1)\cdots(k+1)\not{k}(k-1)\cdots(n-k+1)}{\not{k}}.$$

Again by Lemma 2.1 since

$$\begin{aligned}n(n-1)\cdots(k+1) &= (n-1)(n-2)\cdots k+(n-k)(n-2)\cdots k+(n-k)(n-k-1)(n-3)\cdots k\\ &\quad+\cdots+(n-k)!\times k+(n-k)!,\end{aligned}$$

we have

$$\begin{aligned}(k-1)!\times\binom{n}{k} &= \frac{n(n-1)\cdots(k+1)\not{k}(k-1)\cdots(n-k-1)}{\not{k}}\\ &=\Big((n-1)(n-2)\cdots k+(n-k)(n-2)\cdots k+(n-k)(n-k-1)(n-3)\cdots k\\ &\quad+\cdots+(n-k)!\times k+(n-k)!\Big)(k-1)\cdots(n-k+1)\\ &=(n-1)\cdots(n-k+1)+(n-2)\cdots(n-k+1)(n-k)\\ &\quad+(n-3)\cdots(n-k-1)(n-k)(n-k-1)\\ &\quad+(n-4)\cdots(n-k+1)(n-k)(n-k-1)(n-k-2)\\ &\quad\vdots\\ &\quad+(n-k)!\times k(k-1)\cdots(n-k+1)+(n-k)!\times(k-1)\cdots(n-k+1)\\ &=(n-1)\cdots(n-k+1)+(n-2)\cdots(n-k)\\ &\quad+(n-3)\cdots(n-k-1)+(n-4)\cdots(n-k-2)\\ &\quad+\cdots+k!+(k-1)!\\ &=\frac{(n-1)!}{(n-k)!}+\frac{(n-2)!}{(n-k-1)!}+\frac{(n-3)!}{(n-k-2)!}+\cdots+k!+(k-1)!.\end{aligned}$$

Now by the above discussions we have

$$\begin{aligned}
|\mathcal{C}_n| &= \Big(\sum_{k=2}^{\lfloor \frac{n}{2} \rfloor}\big((k-1)!\binom{n}{k}\big)\Big) + \Big(\sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n-2}\big((k-1)!\binom{n}{k}\big)\Big) + \big((n-2)! \times n\big) + (n-1)! \\
&= \Big(\sum_{k=2}^{\lfloor \frac{n}{2} \rfloor}\big((k-1)!\binom{n}{k}\big)\Big) + \Big(\sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n-2}\big((k-1)!\binom{n}{k}\big)\Big) + \Big(\underbrace{(n-2)! \times \big((n-1)+1\big)}_{(n-1)!+(n-2)!}\Big) + (n-1)! \\
&= \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor}\Big(\frac{(n-1)!}{(n-k)!} + \frac{(n-2)!}{(n-k-1)!} + \cdots + k! + (k-1)!\Big) \\
&\quad + \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n}\Big(\frac{(n-1)!}{(n-k)!} + \frac{(n-2)!}{(n-k-1)!} + \cdots + k! + (k-1)!\Big) \\
&= (n-1)!\Big(\sum_{k=2}^{n}\frac{1}{(n-k)!}\Big) + (n-2)!\Big(\sum_{k=2}^{n-1}\frac{1}{(n-k-1)!}\Big) + (n-3)!\Big(\sum_{k=2}^{n-2}\frac{1}{(n-k-2)!}\Big) + \cdots + 1
\end{aligned}$$

and the result is clear. □

Finally note that as an immediate consequence of the above theorem since $e = \sum_{n=1}^{\infty}\frac{1}{n!}$, Proposition 1.1 holds.

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