Math. & Comput. Sci., 7(2) (2026), 21–30
Online: ISSN 2717-2708
Mathematics and Computational Sciences
Journal Homepage: http://mcs.qut.ac.ir/mcs

Numerical solutions of fractional optimal control problems based on RBF methods

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Abstract

This study presents a numerical method based on Radial Basis Functions (RBFs) for solving a class of fractional optimal control problems. First, the necessary optimality conditions are derived in the form of a system of two fractional differential equations. Then, by solving an associated system of algebraic equations, an approximate solution to the problem is obtained. The fractional derivative considered in this study is the Caputo fractional derivative. Several examples are provided to demonstrate the effectiveness of the proposed method and to compare the accuracy of the resulting numerical solutions.

Keywords: Optimal control, Fractional calculus, Radial basis functions, Meshless method.

2020 MSC: 49M05, 26A33

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1. Introduction

Optimal control, a fundamental branch of control theory and mathematical optimization, aims to determine control and state variables that satisfy the system dynamics and constraints while optimizing a performance functional. By employing advanced analytical tools, particularly differential equations and optimization principles, this field provides a powerful framework for the analysis and design of complex systems. The significance of optimal control is evident not only in the enrichment and expansion of theoretical mathematical foundations but also in its diverse applications across engineering, economics, biomedical sciences and emerging technologies. Within this framework, the central problem lies in determining a control function that, while ensuring the satisfaction of the systems dynamical constraints, achieves the optimization of a prescribed performance functional.
In this context, the state equations typically describe the dynamic behavior of the system, while the control variables serve as the decision functions that influence this evolution. The objective is to design a control law that balances feasibility with optimality, thereby providing an effective framework for analyzing and managing complex dynamical processes.

Among the computational tools in various fields of science and mathematics that have recently attracted significant attention from researchers is fractional calculus. Leibniz and Riemann-Liouville were

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doi: 10.30511/mcs.2026.2070811.1462

Received: 31 January 2026 Accepted: 06 February 2026

the first to study the concept of fractional differentiation in the 19th century. In the real world, many physical systems can be observed that exhibit fractional-order dynamics. In other words, their behavior is governed by fractional differential equations. These physical phenomena, due to memory and hereditary effects, can be more effectively described and modeled using fractional differential equations than with integer-order models. In recent decades, fractional calculus has found numerous applications in various areas of physics and engineering, such as signal processing, control engineering [3, 7], electromagnetics [9], bioscience [14], fluid mechanics [15], electrochemistry [20], diffusion processes [10], dynamic of viscoelastic materials [12], continuum and statistical mechanics [15] and propagation of spherical ames [16].

Given the advantages of fractional models in accurately capturing system behaviors with memory and hereditary characteristics, the field of fractional optimal control has emerged as a powerful framework for analyzing and designing control strategies in complex systems. Unlike classical optimal control problems, fractional-order systems introduce additional mathematical challenges due to the nonlocal nature of fractional operators and the lack of standard solution techniques. This has led to the development of various analytical and numerical methods tailored specifically for fractional systems. The present study aims to explore a class of fractional optimal control problems, formulate the corresponding necessary conditions and implement efficient computational techniques to obtain the optimal solutions. Our goal is to contribute to the theoretical advancement and practical applicability of optimal control in fractional dynamic systems [1, 2, 4, 5, 6, 8, 11, 18, 19, 21, 22].

While several numerical approaches, including spectral methods [17], Ritz methods [18], and operational matrix techniques [13], have been successfully applied to Fractional Optimal Control Problems (FOCPs), meshless methods based on Radial Basis Functions (RBFs) offer distinct advantages in handling complex geometries and high-dimensional problems without the need for mesh generation. Existing RBF-based approaches for FOCPs often rely on direct collocation or Galerkin formulations, which may lead to dense and potentially ill-conditioned systems when enforcing fractional derivative constraints directly. The primary novelty of this work lies in the development of a least-squares framework coupled with RBF approximations for FOCPs. This approach transforms the problem into an unconstrained optimization problem where the dynamical constraints are incorporated via penalty terms, thereby avoiding the direct solution of a coupled system of fractional differential equations and often resulting in better numerical stability. Furthermore, we provide a detailed analysis of shape parameter selection and its impact on both accuracy and the condition number of the resulting algebraic system, a crucial aspect often glossed over in similar works.

In the present paper, we consider the following fractional optimal control problem involving the Caputo fractional derivative [17].

$$\begin{aligned} \min J &= \int_{t_0}^{t_f} f(x(t),u(t),t)\ dt, \qquad (1.1)\\ s.t{}_{t_0}D_{t_f}^{\alpha}x(t) &= g(x(t),u(t),t), \qquad t\in[t_0,t_f]\\ x(t_0) &= x_0 \end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \times [t_0, t_f] \to \mathbb{R}$ and $g : \mathbb{R}^n \times U \times [t_0, t_f] \to \mathbb{R}$ are continuously differentiable vector functions and t_0, t_f and x_0 are assumed to be real and constant values and $n-1 < \alpha < n\,(n \geqslant 1)$ is the order of the fractional derivative.

In the proposed method, radial basis functions are employed to approximate the state and control functions. The main advantage of this approach is the reduction of the problem to a system of algebraic equations. The proposed RBF-based approach is a meshless numerical technique, which eliminates the need for grid generation and offers high flexibility in handling complex domains. The structure of this paper is organized as follows: Section 2 provides an introduction to fractional calculus and radial basis functions. In Section 3, a numerical method for solving the problem (1.1) is presented. Section 4 includes illustrative examples to demonstrate the application and efficiency of the method. Finally, a concise conclusion is given in Section 5.

2. Preliminaries and Notations

In this section, some fundamental definitions and properties of fractional calculus are presented.

Definition 2.1. The Caputo fractional derivative of a function $x(t)$ is defined as follows [17]:

$$D_t^{\alpha} x(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{dx^n} x(\tau)\, d\tau, & n-1 < \alpha < n \\ x^{(n)}(t) & \alpha = n \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt$$

The Caputo derivative is adopted in this study because it allows the use of classical initial conditions involving integer-order derivatives, making it more practical for modeling real-world systems. In contrast, the Riemann-Liouville form requires fractional initial conditions, which are often difficult to interpret physically.

Definition 2.2. A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is called a radial basis function if there exists a univariate function $\phi : [0, +\infty) \to \mathbb{R}^+$ such that for every vector $x \in \mathbb{R}^d$, the function satisfies $\Phi(x) = \phi(\|x\|)$, where $\|.\|$ enotes an arbitrary norm on $\mathbb{R}^d$.

A family $\{\Phi_i(x)\}_{i=1}^N$ of radial basis functions associated with the function $\phi : [0, +\infty) \to \mathbb{R}^+$ is defined as follows:

$$\Phi_i(x) = \phi(\|x - x_i\|), \qquad i = 1, 2, ..., N$$

Using radial basis functions, a method for function approximation can be considered. The dimension-independence of these functions makes them efficient and accurate for high-dimensional problems. Some of the most commonly used radial basis functions are listed in the following table. Suppose that the

Table 1: Type of basis function $\phi(r)(r \geqslant 0)$

Infinitely smooth RBFs	-
Gaussian(GA)	$e^{-(\epsilon r)^2}$
Inverse quadratic(IQ)	$\frac{1}{1+(\epsilon r)^2}$
Inverse multiquadric(IMQ)	$\frac{1}{\sqrt{1+(\epsilon r)^2}}$
Multiquadric(MQ)	$\sqrt{1+(\epsilon r)^2}$
Picewise smooth RBFs	-
Linear	r
Cubic	r^3
Thin Plate Spline(TPS)	$r^2 \log r$

approximation of $f(x)$ at an arbitrary point x can be expressed as a linear combination of N radial basis functions listed in the table.

$$f(x) \approx s(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|),$$

Here, N is the number of data points $\mathbf{x} = (x_1, x_2, ..., x_d)$, d is the dimension of the problem and λ represents the coefficient. In order for the above combination to interpolate the values of f_j at the positions x_j, $j = 1, ..., N$, the expansion coefficients λ_j must satisfy the following relation:

$$\mathbf{A}\lambda = f,$$

where the entries of the matrix $\mathbf{A}$ are given by

$$A_{ij} = \phi(\|x_j - x_i\|), \qquad 1 \leqslant i, j \leqslant N$$

and

$$f = [f_1, ..., f_N]^T, \qquad \lambda = [\lambda_1, ..., \lambda_N]^T$$

The interpolation of $f(x)$ is unique if and only if the matrix $\mathbf{A}$ is nonsingular.

3. Numerical Method

In this section, a numerical method based on radial basis functions and the least squares approach is presented for solving fractional optimal control problems involving Caputo fractional derivatives. It is assumed that the optimal control problem is defined over a finite time interval and consists of a standard performance index along with a state equation involving a fractional derivative of order $\alpha \in (0, 2)$.

Definition of RBF Approximations for the Unknown Functions

To approximate the state functions $x(t)$ and control functions $u(t)$, we employ a linear combination of radial basis functions as follows:

$$x(t) \approx \sum_{i=1}^{N} a_i \phi_i(t),$$
$$u(t) \approx \sum_{i=1}^{N} b_i \phi_i(t),$$

where $\{t_i\}_{i=1}^{N}$ are a set of scattered points selected within the interval $[0, T]$, $\phi_i(t) = \phi(\|t - t_i\|)$ is a selected radial basis function and a_i and b_i are the unknown coefficients to be determined.

3.1. Analysis of Shape Parameter and Numerical Stability

The shape parameter c or ϵ critically influences both the accuracy of the RBF interpolation and the conditioning of the matrix $\mathbf{A}$. A small value leads to a flat basis function and an ill-conditioned, nearly singular matrix. A large value sharpens the function, improving interpolation accuracy but can also increase matrix condition number and amplify numerical errors in derivative computations. We conducted a systematic sensitivity analysis for the IMQ basis ($\phi(r) = 1/\sqrt{1 + (cr)^2}$) across a practical range $c \in [0.5, 15]$. The results revealed a clear trade-off: smaller c reduces accuracy, while larger c increases the condition number dramatically. The value $c = 7$ was selected as it provides an optimal balance, delivering satisfactory solution accuracy while maintaining a manageable condition number (approximately 10^{10}), which ensures numerical stability in double-precision arithmetic. This choice is supported by the consistent performance of the method across all numerical examples.

Fractional derivative of radial basis functions

Given that $x(t)$ is approximated using radial basis functions $\phi_i(t)$, the Caputo fractional derivative of $x(t)$ can be expressed as follows:

$$^{C}D_t^{\alpha}x(t) \approx \sum_{i=1}^{N} \alpha_i {}^{C}D_t^{\alpha}\phi_i(t)$$

The fractional derivative of the radial basis function $^{C}D_t^{\alpha}\phi_i(t)$ is computed either analytically (in special cases) or numerically. In this work, numerical integration-based methods are employed to evaluate the Caputo fractional derivative. Specifically, the Simpson numerical quadrature rule was used to approximate the integral definition of the Caputo derivative, providing second-order accuracy for smooth RBFs. In practice, for each point t_k, we have:

$$^{C}D_t^{\alpha}x(t) \approx \sum_{i=1}^{N} a_i D_{ki}^{(\alpha)}$$

where $D_{ki}^{(\alpha)}$ represents the numerical approximation of the fractional derivative of the basis function ϕ_i at the point t_k.

Within the framework of the least squares method, the constraint of the optimal control problem, which is a fractional differential equation of Caputo type, is assumed as follows:

$$^{C}D_t^{\alpha}x(t) = f(x(t), u(t), t), \qquad t \in (0, T],$$

which must be satisfied together with the following initial condition:

$$x(0) = x_0.$$

These constraints are incorporated into the objective function in the form of numerical residuals.

Definition of the residual function for the fractional differential equation

By expressing the functions $x(t)$ and $u(t)$ in terms of radial basis functions $\phi_i(t)$, the Caputo fractional derivative is also computed numerically. Then, the residual of the fractional differential equation is defined as follows:

$$R_1(t) :=^{C} D_t^{\alpha}x(t) - f(x(t), u(t), t),$$

This residual function is evaluated at the nodal points $\{t_k\}_{k=1}^{M}$ and the weighted squares of these residuals are incorporated into the objective function.

Definition of the residual function for the initial condition

The initial condition residual is defined as the difference between the approximate value of $x(t)$ at time zero and the given known value of x_0.

$$R_2(t) := x(0) - x_0$$

This residual is computed at a single point (the beginning of the domain) and is incorporated into the objective function.

Ultimately, these two residuals are incorporated into the numerical objective function in the form of squared terms.

$$J_{\text{total}} = \int_0^T L(x(t), u(t), t)\, dt + \beta \sum_{k=1}^{M} w_k |R_1(t_k)|^2 + \lambda |R_2|^2,$$

where w denotes the numerical integration weight for the constraint and β and λ are adjustable penalty coefficients.

To determine the unknown coefficients $\{a_i\}$ and $\{b_i\}$, the derivative of the objective function J with respect to these coefficients is computed and set equal to zero.

$$\frac{\partial J}{\partial a_i} = 0, \qquad i = 1, 2, ..., N$$
$$\frac{\partial J}{\partial b_j} = 0, \qquad j = 1, 2, ..., N$$

The formulation of the optimization problem gives rise to a system of equations derived from the necessary optimality conditions. These conditions are obtained by applying the calculus of variations (or an equivalent optimality framework) to the defined objective functional, while explicitly accounting for the governing fractional differential equation and the associated initial condition through their corresponding residual terms.
By incorporating these residuals into the optimization framework, the problem is transformed into a constrained minimization problem, where the constraints are enforced in a weak or residual sense. The resulting optimality conditions lead to a coupled system of algebraic equations involving the unknown coefficients of the approximate solution.
Upon discretization, this system can be systematically assembled and represented in a compact matrix form, which facilitates both theoretical analysis and efficient numerical implementation.

$$\mathbf{A} \cdot \mathbf{c} = \mathbf{b}$$

In this system, the vector **c** contains the unknown coefficients $[a_1, ..., a_N, b_1, ..., b_N]^T$,
the matrix **A** is derived from the partial derivatives of the objective function J and the vector **b** consists of constant values and data related to the objective function and constraints. Solving this system of equations yields the optimal coefficients, through which the state and control functions are approximately computed.

4. Numerical Examples

In this section, to evaluate the efficiency and accuracy of the proposed numerical method, several illustrative examples are presented. In each example, a fractional optimal control problem is first formulated, then the least squares method is applied to the problem and finally, the numerical results are reported.

Example 4.1. Consider the following optimization problem [17].

$$\min J = \int_0^1 \left[(x(t) - t^2)^2 + \left(u(t) + t^4 - \frac{20t^{0.9}}{9\Gamma(0.9)} \right)^2 \right] dt$$
$$\text{s.t.} D_t^{1.1} x(t) = t^2 x(t) + u(t),$$
$$x(0) = x'(0) = 0.$$

The functions $x(t) = t^2$ and $u(t) = -t^4 + \dfrac{20t^{0.9}}{9\Gamma(0.9)}$ are chosen such that they minimize the objective functional J, yielding the optimal value $J = 0$.
We apply the method proposed in this paper and present the obtained results. The numerical results obtained from the proposed method are summarized in the table below. In addition, the graphs of the approximate and exact solutions are depicted to illustrate the accuracy of the approach.

Table 2: Numerical results

Present method			Method of [17]	
n	m	J	N	J
15	100	3.38242×10^{-6}	3	5.82938×10^{-6}
17	120	3.56396×10^{-6}	4	1.46226×10^{-6}
17	122	3.46682×10^{-6}	5	4.3337×10^{-6}
10	40	4.00886×10^{-6}	7	3.5498×10^{-6}

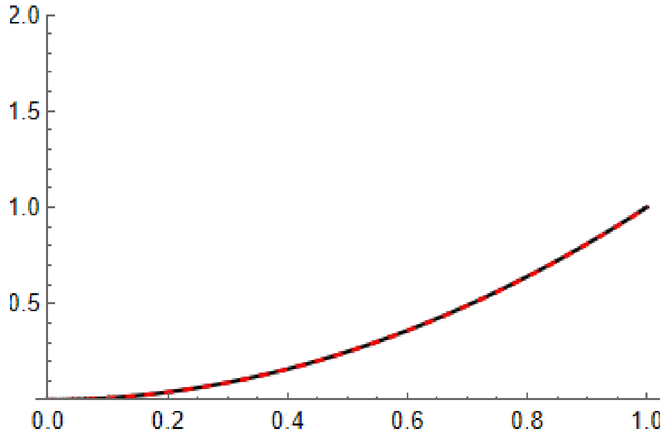

Figure 1: Exact and approximate functions of $x(t)$

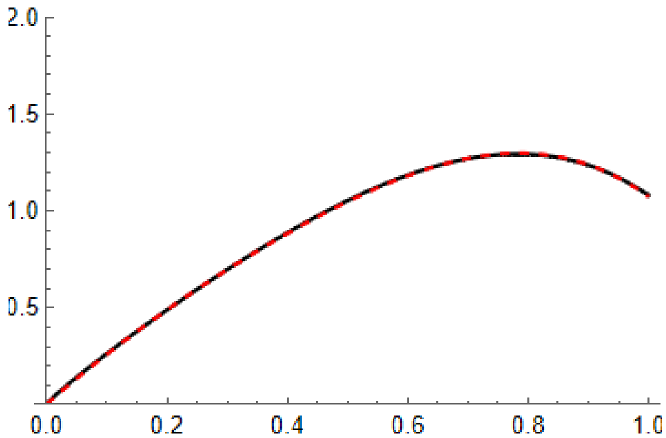

Figure 2: Exact and approximate functions of $u(t)$

Example 4.2. Consider the following optimization problem [17].

$$\min J = \frac{1}{2}\int_0^1 [x(t)^2 + u(t)^2]\,dt$$
$$\text{s.t.} D_t^{\alpha} x(t) = -x(t) + u(t), \qquad 0 < \alpha \leqslant 1,$$
$$x(0) = 0.$$

The exact solution of this problem for $\alpha = 1$ is given as follows [17].

$$x(t) = \beta \sinh(\sqrt{2}t) + \cosh(\sqrt{2}t),$$
$$u(t) = (\beta + \sqrt{2}) \sinh(\sqrt{2}t) + (\sqrt{2}\beta + 1) \cosh(\sqrt{2}t),$$

where

$$\beta = -\frac{\sqrt{2}\sinh(\sqrt{2}) + \cosh(\sqrt{2})}{\sinh(\sqrt{2}) + \sqrt{2}\cosh(\sqrt{2})}.$$

To illustrate the applicability of the proposed method, it is applied to a representative example. The value of the objective function obtained from the present numerical solution is 0.136984. The corresponding optimal cost for this exact solution is $J_{exact} \approx 0.192909$ and the relative error in the objective functional is approximately 0.18%, demonstrating excellent agreement with the exact optimum. The corresponding graphs show a close agreement between the approximate and exact solutions. These results highlight the accuracy and reliability of the proposed approach.

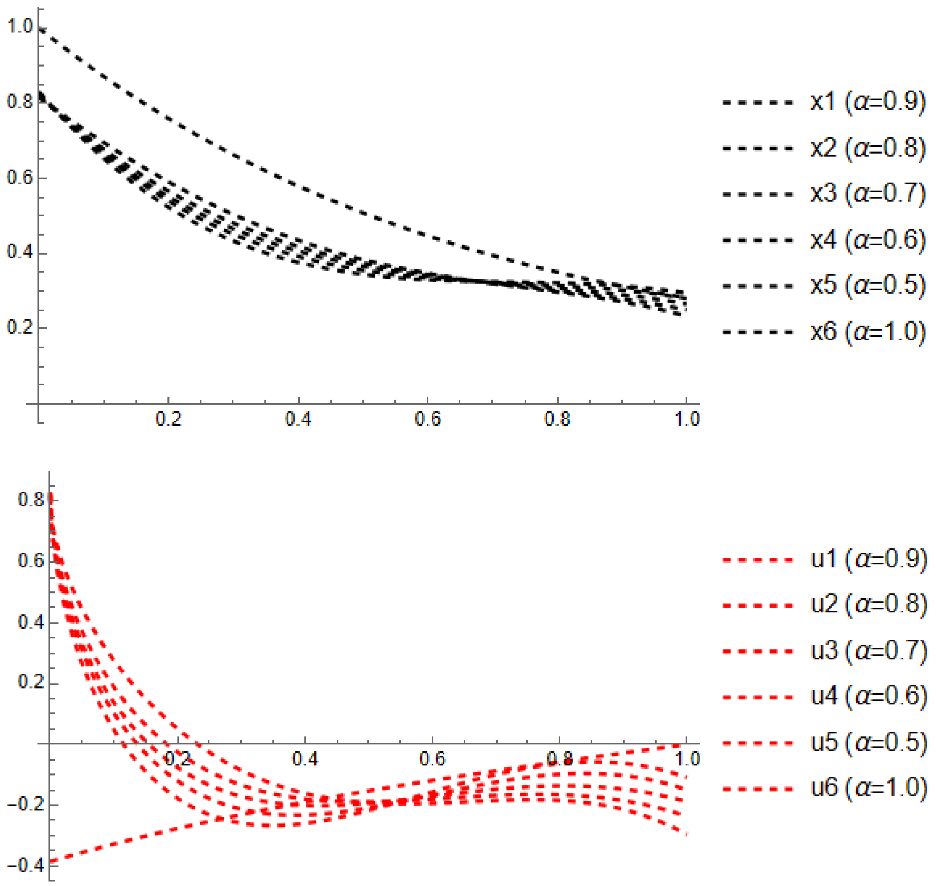

Figure 3: Exact and approximate functions of $x(t)$

Example 4.3. Consider the following minimiation problem [17]

$$\min J = \int_0^1 \left[e^t (x(t) - t^4 + t - 1)^2 + (t^2+1)^2 \left(u(t) + 1 - t + t^4 - \frac{8000 t^{\frac{21}{10}}}{77\Gamma(\frac{1}{10})} \right) \right] dt$$

$$\begin{aligned} &\text{s.t.} D_t^{1.9} x(t) = x(t) + u(t), \qquad 0 < \alpha \leqslant 1, \\ &\quad x(0) = 1, \qquad x'(0) = -1. \end{aligned}$$

In this problem the performance index J takes its minimum value when $x(t) = 1 - t + t^4$ and the minimum value is $J = 0$. In this example, we implement the method introduced in this paper and report the corresponding results. Furthermore, to better demonstrate the precision of the method, we provide plots comparing the approximate and exact solutions. Using the proposed method, the objective function attains a value of 0.00277796. The result obtained by the present method can be compared with those reported in other references for further evaluation [17]. This example features a high fractional order ($\alpha = 1.9$), posing a greater challenge for numerical approximation. The achieved cost is $J \approx 0.00278$. While this is not as close to zero as in previous examplesreflecting the increased difficultythe approximate state and control functions closely follow the exact trajectories, as shown in Figures 4 and 5. This demonstrates the method's robustness even for challenging high-order fractional problems.

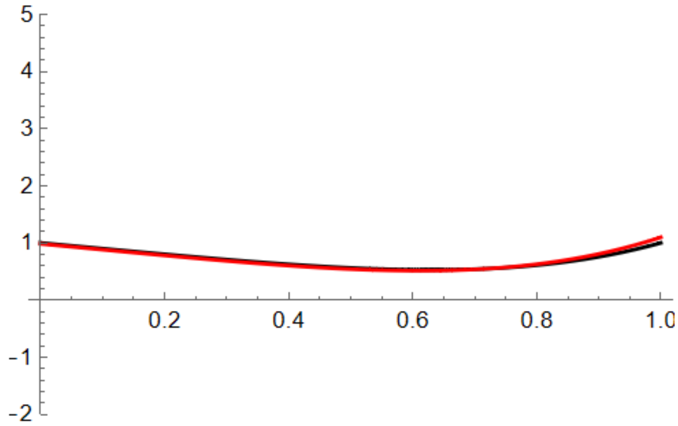

Figure 4: Exact and approximate functions of $x(t)$

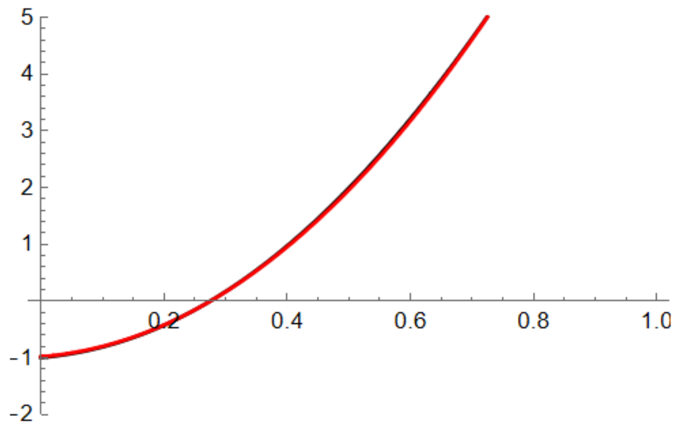

Figure 5: Exact and approximate functions of $u(t)$

5. Conclusion

In this paper, a numerical method based on the least squares approach was presented for solving optimal control problems governed by fractional differential equations of Caputo type. The state and control functions were approximated using linear combinations of appropriate basis functions. By defining a residual function and incorporating it into the objective functional, the dynamic constraints of the system were implicitly enforced within the optimization framework. The residual was evaluated at a set of discrete time points over the domain, resulting in a discretized form of the objective functional and a corresponding algebraic system for determining the unknown coefficients. The Gaussian method was also implemented for this problem; however, it did not yield satisfactory results. Therefore, by employing the multiquadric and inverse multiquadric approaches with the shape parameter set to $c = 7$, favorable outcomes were achieved. The shape parameter c was chosen as 7 based on a sensitivity analysis, where values in the range $5 \leqslant c \leqslant 9$ were tested. This range provided a good balance between stability and accuracy.

The numerical results demonstrated that the proposed method exhibits high accuracy and favorable convergence properties, while also maintaining a relatively simple implementation structure. Furthermore, the flexibility of the method in selecting basis functions and residual evaluation points makes it a robust and efficient tool for solving a wide range of optimal control problems involving nonlocal fractional dynamics.

Future Work: Future research may focus on extending the proposed meshless RBF approach to multi-dimensional fractional optimal control problems, incorporating inequality constraints and exploring adaptive strategies for selecting basis functions and shape parameters.

References

[1] Abdulaziz, O., Hashim, I., Ismail, E. S. (2009). Approximate analytical solution to fractional modified KdV equations. Mathematical and Computer Modelling, 49(1-2), 136-145. 1

[2] Adel, M., Khader, M. M. (2017). Approximate solutions for a certain class of fractional optimal control problems using laguerre collocation method. In International Mathematical Forum (Vol. 12, No. 8, pp. 379-389). 1

[3] Agrawal, O. P. (2004). A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dynamics, 38(1), 323-337. 1

[4] Baleanu, D., Defterli, O., Agrawal, O. P. (2009). A central difference numerical scheme for fractional optimal control problems. Journal of Vibration and Control, 15(4), 583-597. 1

[5] Canuto, C., Quarteroni, A., Hussaini, M. Y., Zang Jr, T. A. (2007). Spectral methods: evolution to complex geometries and applications to fluid dynamics. Berlin, Heidelberg: Springer Berlin Heidelberg. 1

[6] Cuahutenango-Barro, B., Taneco-Hernández, M. A., Lv, Y. P., Gómez-Aguilar, J. F., Osman, M. S., Jahanshahi, H., Aly, A. A. (2021). Analytical solutions of fractional wave equation with memory effect using the fractional derivative with exponential kernel. Results in Physics, 25, 104148. 1

[7] Duarte, F. B., Machado, J. T. (2002). Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators. Nonlinear Dynamics, 29(1), 315-342. 1

[8] Elnagar, G., Kazemi, M. A., Razzaghi, M. (1995). The pseudospectral Legendre method for discretizing optimal control problems. IEEE transactions on Automatic Control, 40(10), 1793-1796. 1

[9] Engheta, N. (2002). On fractional calculus and fractional multipoles in electromagnetism. IEEE Transactions on Antennas and Propagation, 44(4), 554-566. 1

[10] Gafiychuk, V., Datsko, B., Meleshko, V. (2008). Mathematical modeling of time fractional reactiondiffusion systems. Journal of Computational and Applied Mathematics, 220(1-2), 215-225. 1

[11] Ghomanjani, F. (2016). A numerical technique for solving fractional optimal control problems and fractional Riccati differential equations. Journal of the Egyptian Mathematical Society, 24(4), 638-643. 1

[12] Lederman, C., Roquejoffre, J. M., Wolanski, N. (2004). Mathematical justification of a nonlinear integro-differential equation for the propagation of spherical flames. Annali di Matematica Pura ed Applicata, 183(2), 173-239. 1

[13] Lotfi, A., Yousefi, S. A., Dehghan, M. (2013). Numerical solution of a class of fractional optimal control problems via the Legendre orthonormal basis combined with the operational matrix and the Gauss quadrature rule. Journal of Computational and Applied Mathematics, 250, 143-160. 1

[14] Magin, R. L. (2010). Fractional calculus models of complex dynamics in biological tissues. Computers and Mathematics with Applications, 59(5), 1586-1593. 1

[15] Mainardi, F. (1997). Fractional calculus: some basic problems in continuum and statistical mechanics. In Fractals and fractional calculus in continuum mechanics (pp. 291-348). Vienna: Springer Vienna. 1

[16] Meral, F. C., Royston, T. J., Magin, R. (2010). Fractional calculus in viscoelasticity: an experimental study. Communications in nonlinear science and numerical simulation, 15(4), 939-945. 1

[17] Nemati, S. (2016). A spectral method based on the second kind Chebyshev polynomials for solving a class of fractional optimal control problems. Sahand Communications in Mathematical Analysis, 4(1), 15-27. 1, 2.1, 4.1, 2, 4.2, 4.3

[18] Nemati, A., Yousefi, S. A. (2017). A numerical scheme for solving two-dimensional fractional optimal control problems by the Ritz method combined with fractional operational matrix. IMA Journal of Mathematical Control and Information, 34(4), 1079-1097. 1

[19] Oldham, K., Spanier, J. (1974). The fractional calculus theory and applications of differentiation and integration to arbitrary order (Vol. 111). Elsevier. 1

[20] Oldham, K. B. (2010). Fractional differential equations in electrochemistry. Advances in Engineering software, 41(1), 9-12. 1

[21] Rezazadeh, A., Avazzadeh, Z. (2023). Solving a category of twodimensional fractional optimal control problems using discrete Legendre polynomials. Asian Journal of Control, 25(1), 551-562. 1

[22] Rida, S. Z., El-Sherbiny, H. M., Arafa, A. A. M. (2008). On the solution of the fractional nonlinear Schrödinger equation. Physics Letters A, 372(5), 553-558. 1