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Solving non-homogeneous non-linear difference and differential equations by using additive and multiplicative derivative and integral with applications

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Abstract

Difference equations are discrete analogy of the differential equations. These equations appear in mathematical modeling of physics and engineering problems, economic and population subjects that deal with discrete data and variables. In this article, we consider and solve these types of equations that are non-homogeneous and non-linear. The solving method is performed using by multiplicative discrete and continuous differential equations. Then by using the concept of discrete derivative, the analytical method and analytical solutions are given to linear and non-linear non-homogeneous difference and differential equations. Finally some examples about applications of multiplicative models with numerical calculations and geometrical graphs are presented.

Keywords: Invariant Functions, Non-Linear Difference Equations, Non-homogenuous difference equation, Discrete additive derivative, Discrete multiplicative derivative.

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1. Introduction

A common way to create a new mathematical system is to change the axioms of a known system. Non-Newtonian calculus provides alternative approaches to conventional Newtonian calculus [10, 25].

multiplicative calculus also known as non-Newtonian calculus, geometric calculus, or multiplicative analysis is a family of alternative calculi developed to model phenomena where relative change, ratios, growth factors, and exponential behavior are more natural than classical additive change. Its history spans more than a century, evolving from early ideas in geometric means to modern applications in physics, finance, and differential equations.

This non-Newtonian calculus, first introduced by Grossman and Katz (1933-2010) in the period between 1967 and 1970[11, 12, 13].

Recently, it has been shown that the non-Newtonian calculus (multiplicative) is more suitable for some problems than the ordinary Newtonian calculus (additive). For example, in statistics, finance, economics,

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biology, demography, pattern recognition in images, signal processing, thermostats, and quantum information theory.

More ever the multiplicative derivative measures relative change, making it ideal for exponential growth, financial returns, biological growth, and systems with geometric structure. The multiplicative integral accumulates proportional changes and is useful in economics, ecology, and control theory.

Considering the importance of calculus of changes in applications, for example, in physics, economics and biology, and the importance that non-newtonian calculus usually has in these fields, surely the discussed calculus of changes will attract the attention of the research community to research in these fields. ,[6, 1, 19, 20, 2]

Later, mathematicians such as D. Stanley in [7], Shahi. A, et al in [24] for nonlinear equations, Akmak. A et al in [3] for matrix transformation, Bin bashioglu et al in [4] for fixed point theory, and A. Bashirov in [1, 2], extended and applied this theory for continuous case.

On the other hand, N. Aliyev et al. studied additive arithmetic in discrete mode. They introduced the additive discrete calculus and presented some basic formulas for the discrete additive derivative and integral[17]. Then N. Aliyev and M. Jahanshahi presented and extended multiplication calculus in discrete and continuous cases [14, 21].

In the last papers [21, 16], the authors introduced some invariant functions for discrete and continuous multiplicative derivatives. Using these invariant multiplicative functions, one can solve linear and non-linear difference and differential equations. Also, they expanded analytical and numerical methods to solve linear and non-linear and differential equations through multiplicative calculations.

We will also show that various problems from different sciences can be modeled more efficiently using multiplicative calculus, rather than Newtonian calculus. Since multiplicative calculus is a relatively new discovery, attempts have been made to explain its basic principles, such as exponential calculus, multiplicative calculus, and multiplicative differential equations. [5, 9, 23]

Also multiplicative calculus can linearize functions and differential equations that are nonlinear in standard calculus. As we will see in next sections of this paper.

In this paper, the authors expand and generalize these methods for solving non- homogeneous discrete and continuous multiplicative difference and differential equations. The extended methods provide useful ways to solve non-homogeneous and non-linear difference, differential and integral equations.

In the final section of paper, some important applications of multiplicative calculus with numerical calculations and geometrical graphs will be presented.

This paper is organized in 4 parts: In the first part, some elementary concepts, definitions and basic formulas about non-newtonian calculus are presented. In the second part, solving non-homogeneous linear difference equations by additive differential equations are discussed. Third part devotes to solve non-linear and non-homogeneous difference equations by using discrete multiplicative differential equations. Fourth part discusses non-homogeneous multiplicative differential equations in order to solve non-linear non-homogeneous ordinary differential equations.

2. Multiplicative calculus as a non-Newtonian calculus

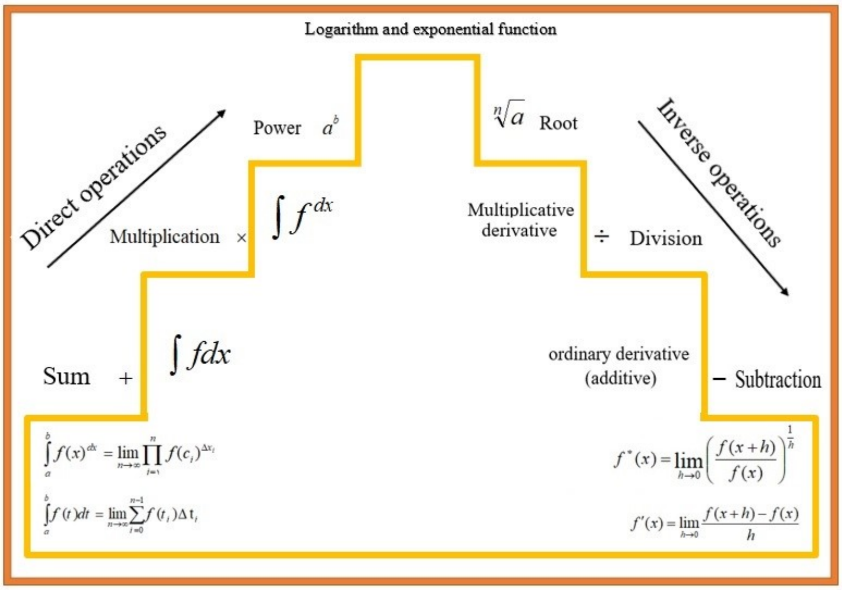
This definition of integral operation and derivative operation is basis of normal continuous additive calculus which is also known as Newtonian calculus. Actually, the operation of limit in definition of integrals and derivative such that they are related to the continually.

In classic calculus (Newtonian or continuous additive calculus) we saw that the operation of integration was defined via multiplication and addition and then limitation. And also, we saw that normal derivative operation has been done by the two inverse operations division and subtraction and then limitation.

In multiplicative calculus as a non-Newtonian calculus, we will show that multiplicative integral is defined by the two direct operations multiplication and power. Also continuous derivative multiplicative is basically defined by the two inverse operations $n'th$ root and division, then the operation of limit is being used.

2.1. Schematic diagram about extension of Newtonian and non-Newtonian calculus

In the following illustration, we show that the extension and generalization of Newtonian derivative and integral and non-Newtonian (multiplicative) derivative and integral. Exponential function and Logarithmic function play important roles in this process.



Part1

3. Preliminary concepts and Basic Definitions

3.1. Continuous additive and multiplicative derivative

In this section, we introduce the continuous additive and multiplicative derivative.

Definition 3.1. Suppose the function $f\colon A \subset R^+ \longrightarrow R$ is a positive value function, similar to the expression of the definition of the continuous additive (ordinary derivative) of the function

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{3.1}$$

The continuous multiplicative derivative can be defined with the following limit.

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}. \tag{3.2}$$

If this limit exists, then the function is multiplicative differentiable and we denote it by $f^*(x)$.

We see that this new operation for the derivative (which is an inverse operation) is made by the two inverse operations of root and division.

Now we try to get a practical formula to calculate the multiplicative derivative of the function.

For this, according to the above definition, we have:

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)} - \frac{f(x)}{f(x)} + 1\right)^{\frac{1}{h}}$$
$$= \lim_{h\to 0}\left(1 + \frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}\frac{f(x+h)-f(x)}{h}\frac{1}{f(x)}}$$
$$= e^{\frac{f'(x)}{f(x)}} = e^{(Lnf)'}. \tag{3.3}$$

Regarding the last relation of above calculations. We consider the following considerations

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0}\left(1+h\right)^{\frac{1}{h}} = e.$$

Note that the all of calculations be done in continous case. For more consideration see [1, 2].

For the second order multiplicative derivative, we have:

$$f^{**}(x) = e^{(Lnf^*)'(x)} = e^{(Lnf)''(x)}. \tag{3.4}$$

By mathematical induction, we can calculate the arbitrary n order.

$$f^{(n)}(x) = e^{(Lnf)^{(n)}(x)} \quad n = 0, 1, 2, \cdots. \tag{3.5}$$

Theorem 3.2. *[17, 21, 15] Basic formula for continuous multiplicative derivative :*

1. $(cf)^*(x) = f^*(x)$
2. $(f \cdot g)^*(x) = f^*(x) \cdot g^*(x)$
3. $(\frac{f}{g})^*(x) = \frac{f^*(x)}{g^*(x)}$
4. $((f)^g)^*(x) = f^*(x)^{g(x)} \cdot f(x)^{g'(x)}$
5. $(fog)^*(x) = f^*(g(x))^{g'(x)}$.

Remark 3.3. The invariant function of the continuous multiplicative derivative is: [17, 21, 15]

$$y = e^{e^{\lambda x}} \tag{3.6}$$

3.2. Definitions of discrete derivative and integral

Suppose $y = f(x)$ is a function which is defined from integer numbers $\mathbb{Z}$ to real numbers $\mathbb{R}$; that is $f : A \subset \mathbb{Z} \longrightarrow \mathbb{R}$ Where is A subset of $\mathbb{Z}$.

The discrete additive derivative of f is defined by:

$$f'(x) = f(x+1) - f(x).$$

which is shown by $f'(x)$. Note that the additive discrete derivative is shown by (′) .

This definition is obtained from continuous additive derivative without limit operation:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{3.7}$$

Where the length step h is assumed equal one, $h = 1$.

Remark 3.4. The invariant function for this kind of derivetive is:

$$y(x) = 2^x.$$

If we want to use to solve discerete differential equationswe will use its parametric

$$y(n) = (1+\lambda)^n$$

3.3. Discerete additive integration

Let the two functions f and g satisfy in the following relation,

$$f'(t) = g(t), \quad t \in \mathbb{Z}.$$

Which this derivative for the function f is additive discrete derivative. Suppose x_0, x are the arbitrary points in $\mathbb{Z}$, then by the definition of additive discrete derivative of f , we can write:

$$f(x_0+1) - f(x_0) = f'(x_0) = g(x_0)$$

$$f(x_0+2) - f(x_0+1) = f'(x_0+1) = g(x_0+1)$$

$$\vdots$$

$$f(x-1) - f(x-2) = f'(x-2) = g(x-2)$$

$$f(x) - f(x-1) = f'(x-1) = g(x-1).$$

Now by adding these relations side-by-side, we will have:

$$f(x) - f(x_0) = \sum_{t=x_0}^{x-1} g(t).$$

or

$$f(x) = f(x_0) + \sum_{t=x_0}^{x-1} g(t).$$

Now by considering these relations, we define the discrete additive integration of function f as follows.

Definition 3.5. Suppose $f(t)$ is a function which is defined on $A \subset \mathbb{Z}$. Then the additive discrete integration of f in the interval $[x_0, x)$ is defined by the sum $\sum_{t=x_0}^{x-1} g(t)$ and is denoted by:

$$\int_{x_0}^{x} g(t)\Delta t = \sum_{t=x_0}^{x-1} g(t)$$

Note that Δt is equal to unit. According to this definition, we can write:

$$\int_{x_0}^{x} g(t)\Delta t = f(x) - f(x_0)$$

where $f'(t) = g(t)$

Example 3.6. For the identity function $f(x) = x$, we have:

$$\int_{x_0}^{x} t\Delta t = \sum_{t=x_0}^{x-1} t \quad = x_0 + (x_0+1) + (x_0+2) + \cdots + +(x_0+x-1)$$

$$= \frac{(x+x_0-1)}{2}(x-x_0) = \frac{x^2-{x_0}^2-x+x_0}{2} = \frac{\overset{2}{x}}{2} - \frac{\overset{2}{x_0}}{2}.$$

Where the ordered power function is defined by:

$$\overset{n}{x} = x(x-1)\cdots[x-(n-1)]$$

3.4. Discrete multiplicative derivative

Suppose the function is defined as follows:

$$f : A \subset \mathbb{Z} \longrightarrow \mathbb{R} \quad , \quad x \in A \subset \mathbb{Z}$$

The discrete multiplicative derivative of the function is defined as follows:

$$f^{[1]}(x) = \frac{f(x+1)}{f(x)}. \tag{3.8}$$

Where the notation[1] shows the discrete multiplicative derivative.

Similar to discrete additive derivative, discrete multiplicative derivative is resulted from continuous case with out limit operation and choosing $h = 1$.

Remark 3.7. The discrete multiplicative derivative of a constant function is equal to unity. (The normal derivative of the constant function is equal to zero.)

Theorem 3.8. *[21, 15] It is easy to see that the following relationships hold.*

1. $\Big(f(x) \cdot g(x)\Big)^{[1]} = f^{[1]}(x) \cdot \ g^{[1]}(x)$

2. $\left(\frac{f(x)}{g(x)}\right)^{[1]} = \frac{f^{[1]}(x)}{g^{[1]}(x)}$

3. $\left(f^n(x)\right)^{[1]} = \left(f^{[1]}(x)\right)^n$

Remark 3.9. [17, 21, 15] The Invariant function of the discrete multiplicative derivative is:

$$f(x) = c^{2^x} \tag{3.9}$$

Remark 3.10. It should be noted that when we want to use the invariant function of this type of derivative to solve multiplicative discrete differential equations, we use its parametric form as follow: [17, 21, 15]

$$y_n = C^{(1+\lambda)^n}.$$

Where C is a constant.

3.5. Discrete multiplicative integral

In this section , we give the concept of multiplicative discrete integration of a function . For this, suppose for the two functions f and g, we would have the following relation:

$$f^{[1]}(x) = g(x).$$

where the notation [1] is used commenly for first order multiplicative discrete and continuous derivative. Note that, when we want to use this notation for solving difference equations, we will write and show this function by discrete variable form as $f(n)$.

According to the definition of multiplicative derivative, we can write:

$$\frac{f(x+1)}{f(x)} = g(x)$$

$$\Rightarrow f(x+1) = f(x) \cdot g(x) \quad , \quad x \geqslant x_0$$

If we begin from the point x_0 and with $h = 1$, we will have:

$f(x_0+1) = g(x_0)\cdot f(x_0)$
$f(x_0+2) = g(x_0+1)\cdot f(x_0+1) = g(x_0+1)\cdot g(x_0)\cdot f(x_0)$
$f(x_0+3) = g(x_0+2)\cdot f(x_0+2) = g(x_0+2)\cdot g(x_0+1)\cdot g(x_0)\cdot f(x_0)$
$\vdots$
$f(x_0+n) = g(x_0+n-1)\cdot f(x_0+n-1) = f(x_0)\cdot \prod_{t=x_0}^{x_0+n-1} g(t)$

Now by multiplying these relations together side-by-side, we obtain:

$$f(x_0+1)\cdot f(x_0+2)\cdots f(x_0+n) = g(x_0)\cdot g(x_0+1)\cdots g(x_0+n-1)\cdot f(x_0)\cdot f(x_0+1)\cdots f(x_0+n-1).$$

If we remove the expression

$$f(x_0)\cdot f(x_0+1)\cdots f(x_0+n-1).$$

from each sides of above relation , we have:

$$f(x_0+n) = f(x_0)\cdot \prod_{j=0}^{n-1} g(x_0+j).$$

If we use the notation $\mathop{\prod\!\!\!\!\!\!\int}$ for this kind of integration , we can write:

$$f(x) = c \mathop{\prod\!\!\!\!\!\!\int}_{x_0}^{x} g(\xi) = c\prod_{\xi=x_0}^{x-1} g(\xi).$$

where the constant c is the arbitrary constant of this integration.
It is easy to see that the following relations hold.

$$\mathop{\prod\!\!\!\!\!\!\int}_{x_0}^{x} f(t)\cdot g(t) = \mathop{\prod\!\!\!\!\!\!\int}_{x_0}^{x} f(t)\cdot \mathop{\prod\!\!\!\!\!\!\int}_{x_0}^{x} g(t)\ .$$

$$\mathop{\prod\!\!\!\!\!\!\int}_{x_0}^{x} \frac{f(t)}{g(t)} = \mathop{\prod\!\!\!\!\!\!\int}_{x_0}^{x} f(t) \Big/ \mathop{\prod\!\!\!\!\!\!\int}_{x_0}^{x} g(t)\ . \tag{3.10}$$

Part2

4. Solving Non-homogenuous Linear Difference Equations by Using Non-Homogeneous Additive Differential Equations

4.1. *Discrete additive differential equations*

We consider the following second order non-homogeneous differential equation:

$$y''(x) + 2ay'(x) + by(x) = f(x) \quad x > x_0. \tag{4.1}$$

where the derivative is a discrete additive derivative as follows:

$$y'(x) = y(x+1) - y(x).$$

The related homogeneous equation is :

$$y''(x) + 2ay'(x) + by(x) = 0.$$

We consider the similar homogeneous proposed solution as invariant function which has been introduced in [17, 21]:
$y(x) = (\lambda+1)^x$.

$$y'(x) = y(x+1) - y(x) = (\lambda+1)^{x+1} - (\lambda+1)^x = \lambda(\lambda+1)^x$$

$$\Rightarrow y''(x) = \lambda^2(\lambda+1)^x.$$

After placing in the homogeneous part, we will have:

$$\lambda^2(\lambda+1)^x + 2a\lambda(\lambda+1)^x + b(\lambda+1)^x \equiv 0.$$

The characteristic equation is:

$$\lambda^2 + 2a\lambda + b = 0$$

And the roots of the characteristic equation are as follows:

$$\lambda = -a \pm \sqrt{a^2 - b}.$$

The general solution of the homogeneous part is:

$$y(x) = c_1(\lambda_1+1)^x + c_2(\lambda_2+1)^x.$$

For example the solution of homogeneous part of the following discrete additive differential equation is:

$$y''(x) - 8y'(x) + 7y(x) = f(x) \quad x > 0. \tag{4.2}$$

$$y''(x) - 8y'(x) + 7y(x) = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 7 = 0$$

$$\Rightarrow \lambda = 4 \pm \sqrt{16-7} \Rightarrow \lambda_1 = 1, \lambda_2 = 7.$$

General solution of the homogeneous part is:

$$y(x) = c_1 2^x + c_2 2^{3x}.$$

By using the method of changing variable parameter, we obtain the following algebraic system:

$$\begin{cases} c_1'(x)2^{x+1} + c_2'(x)2^{3x+3} = f(x) \\ c_1'(x)2^{x+1} + c_2'(x)2^{3x+3} = 0 \end{cases}$$

The system of two equations with two unknowns with respect to $c_1'(x), c_2'(x)$ from which we have:

$$c_1'(x) = -\frac{1}{6}2^{-x-1}f(x) = -\frac{1}{3}2^{-x-2}f(x) \qquad \& \qquad c_2'(x) = \frac{1}{6}2^{-3x-4}f(x)$$

According to the concept of derivative and discrete additive integral, we will have:

$$c_1(x) = c_1(x_1) - \tfrac{1}{3}\int_{x_1}^{x} 2^{-\xi-2}f(\xi)$$

$$c_2(x) = c_1(x_2) + \tfrac{1}{3}\int_{x_2}^{x} 2^{-3\xi-4}f(\xi)$$

$$y(x) = c_1(x_1).2^x + c_2(x_2).2^{3x} - \tfrac{1}{3}\int_{x_1}^{x} 2^{-\xi-2}f(\xi) \; + \tfrac{1}{3}\int_{x_2}^{x} 2^{-3\xi-4}f(\xi)$$

Finally, the general solution of the homogeneous equation with assumption $x_1 = x_2 = x_0$ will be as follows:

$$y(x) = c_1.2^x + c_2.2^{3x} - \frac{1}{3}\left[\int_{x_0}^{x} (2^{x-\xi-2} - 2^{-3(x-\xi)-4})f(\xi)\right] \tag{4.3}$$

by choosing $y(x) = y_n, f(x) = f_n$, the final solution for the non-homogeneous equation:

$$y''(x) - 8y'(x) + 7y(x) = f(x) \quad x > 0.$$

with the integer variable n , the answer will be the following sequence.

$$y_n = c_1.2^n + c_2.2^{3n} - \sum_{k=0}^{n-1} \frac{1}{3}\left[(2^{n-k-2} - 2^{-3(n-k)-4})f_k\right]. \tag{4.4}$$

Regarding the definition 3.4 of discrete additive integral, the last relation of 4.4 replaced by the pervious relation 4.3.

Part3

5. Solving non-linear, non-homogenuous difference equations by discrete non-homogeneous multiplicative equation

5.1. Non-homogenuous difference equations with constant term

The discrete non-homogeneous multiplicative equation in the form:

$$\left(y_n{}^{[11]}\right)^a \cdot \left(y_n{}^{[1]}\right)^b \cdot y_n{}^c = A. \tag{5.1}$$

It is given that $A \neq 1$ corresponds to the difference equation where the constant A is the inhomogeneous term of the equation.

$$ay'' + by' + cy = A$$

We consider the similar homogeneous proposed solution as invariant function which has been introduced in [21]:

$$y_n = c^{(1+\lambda)^n}$$

we multiply the proposed answer by an unknown constant α .

$$y_n = \alpha \cdot c^{(1+\lambda)^n}$$

In this case, to find the indeterminate constant α , we have:

$$y_n{}^{[1]} = c^{\lambda(1+\lambda)^n} = c^{\left((1+\lambda)^n\right)^{\lambda}}.$$

$$y_n{}^{[11]} = c^{\lambda^2(1+\lambda)^n}.$$

By inserting in the above equation, we will have:

$$\alpha = \sqrt[c]{A}$$

that the solution of the homogeneous part will also be obtained from solving the characteristic polynomial $a\lambda^2 + b\lambda + c = 0$.

It should be noted that the inhomogeneous term in continuous multiplicative equations can be obtained from the general form of the discrete multiplicative equation (or the corresponding difference equation).

$$A\left(y^{[11]}\right)^a \cdot B\left(y^{[1]}\right)^b \cdot C\left(y^c\right) = 1. \tag{5.2}$$

This form can be written as equation 5.1 with non-homogeneous term as $\frac{1}{ABC}$.

5.2. Solving nonlinear non-homogeneous difference equations with arbitrary terms

In the previous section, we considered nonlinear inhomogeneous difference equations with a numerical constant term. In this section, we will examine and solve nonlinear difference equations with an optional inhomogeneous term f_n . For this purpose, we consider a second-order nonlinear difference equation with an inhomogeneous term as follows:

$$y_{n+2}^{\alpha} . y_{n+1}^{\beta} . y_n^{\gamma} = f_n \tag{5.3}$$

First, we transform this equation into the following non-homogenuous additive equation using a change of variable:

$$y_n = e^{Z_n}$$

$$e^{\alpha.Z_{n+2}+\beta.Z_{n+1}+\gamma.Z_n} = f_n. \tag{5.4}$$

Note that. Taking the logarithm of both sides of the last equation, we have:

$$\alpha.Z_{n+2} + \beta.Z_{n+1} + \gamma.Z_n = \ln f_n. \tag{5.5}$$

We note that the discrete additive derivative is as follows:

$$Z_n{}^{(1)} = Z_{n+1} - Z_n \Rightarrow Z_{n+1} = Z_n{}^{(1)} + Z_n$$

$$Z_n{}^{(11)} = Z_{n+2} - 2Z_{n+1} + Z_n = Z_{n+2} - 2(Z_n{}^{(1)} + Z_n) + Z_n$$

$$\Rightarrow Z_{n+2} = Z_n{}^{(11)} + 2Z_n{}^{(1)} + Z_n.$$

By substituting the values into the equation, we get:

$$\alpha.(Z_n{}^{(11)} + 2Z_n{}^{(1)} + Z_n) + \beta.(Z_n{}^{(1)} + Z_n) + \gamma.Z_n = \ln f_n \tag{5.6}$$

or

$$\alpha.Z_n{}^{(11)} + (2\alpha + \beta)Z_n{}^{(1)} + (\alpha + \beta + \gamma).Z_n = \ln f_n \tag{5.7}$$

Considering the discrete additive derivative invariant function:

$$Z_n = (\rho + 1)^n$$

the characteristic equation is solved as follows:

$$\alpha\rho^2 + (2\alpha + \beta)\rho + (\alpha + \beta + \gamma) = 0 \tag{5.8}$$

If we denote the roots of the characteristic equation by,ρ_1, ρ_2 , then the general solution is:

$$Z_n = C_{1n}(\rho_1 + 1)^n + C_{2n}(\rho_2 + 1)^n. \tag{5.9}$$

Now, by calculating discrete additive derivatives (similar to what is done in the continuous mode of the Lagrange method of parameter variation), we seek to solve non-homogeneous equation 5.7 and to form an algebraic system with respect to the coefficients C_{1n}, C_{2n}, and For this purpose:

$$Z_n^{(1)} = C_{1n}^{(1)}(\rho_1+1)^n\rho_1 + C_{1n}^{(1)}(\rho_1+1)^n + C_{1n}(\rho_1+1)^n\rho_1 + C_{2n}^{(1)}(\rho_2+1)^n\rho_2 + C_{2n}^{(1)}(\rho_2+1)^n + C_{2n}(\rho_2+1)^n\rho_2.$$

For the sake of simplicity of calculations, we put:

$$C_{1n}^{(1)}(\rho_1 + 1)^{n+1} + C_{2n}^{(1)}(\rho_2 + 1)^{n+1} = 0. \tag{5.10}$$

Again, by calculating the second-order discrete additive derivatives $Z_n{}^{(11)}$
and substituting them into the equation 5.7, we arrive at the following relation:

$$C_{1n}^{(1)}(\rho_1 + 1)^{n+1}\rho_1 + C_{2n}^{(1)}(\rho_2 + 1)^{n+1}\rho_2 = \ln f_n \tag{5.11}$$

Now we consider the relations 5.10, 5.11 and as an algebraic system with respect to and $C_{1n}^{(1)}, C_{2n}^{(1)}$.

$$\begin{cases} C_{1n}^{(1)}(\rho_1 + 1)^{n+1} + C_{2n}^{(1)}(\rho_2 + 1)^{n+1} = 0 \\ \\ C_{1n}^{(1)}(\rho_1 + 1)^{n+1}\rho_1 + C_{2n}^{(1)}(\rho_2 + 1)^{n+1}\rho_2 = \ln f_n \end{cases} \tag{5.12}$$

The determinant of the above device is:

$$\Delta = \begin{vmatrix} (\rho_1 + 1)^{n+1} & (\rho_2 + 1)^{n+1} \\ \\ (\rho_1 + 1)^{n+1}\rho_1 & (\rho_2 + 1)^{n+1}\rho_2 \end{vmatrix} = (\rho_1 + 1)^{n+1}(\rho_2 + 1)^{n+1}(\rho_2 - \rho_1) \tag{5.13}$$

Assuming the opposite of this determinant is zero, we have:

$$C_{1n}^{(1)} = \frac{1}{\Delta}\begin{vmatrix} 0 & (\rho_2 + 1)^{n+1} \\ \\ \ln f_n & (\rho_2 + 1)^{n+1}\rho_2 \end{vmatrix} = \frac{-(\rho_2 + 1)^{n+1}\ln f_n}{\Delta} = -\frac{\ln f_n}{(\rho_1 + 1)^{n+1}(\rho_2 - \rho_1)} \tag{5.14}$$

$$C_{2n}^{(1)} = \frac{1}{\Delta}\begin{vmatrix} (\rho_1+1)^{n+1} & 0 \\ (\rho_2+1)^{n+1}\rho_2 & \ln f_n \end{vmatrix} = \frac{(\rho_1+1)^{n+1}\ln f_n}{\Delta} = \frac{\ln f_n}{(\rho_2+1)^{n+1}(\rho_2-\rho_1)} \tag{5.15}$$

Now, by taking the discrete additive integral from both sides 5.14,5.15, we have the relations:

$$C_{1n} = C_{10} - \int_0^n \frac{\ln f_n}{(\rho_1+1)^{n+1}(\rho_2-\rho_1)} \quad , \quad C_{2n} = C_{20} + \int_0^n \frac{\ln f_n}{(\rho_2+1)^{n+1}(\rho_2-\rho_1)}$$

As a result, the final solution to the discrete additive equation 5.7 is given by the following equation:

$$Z_n = C_1(\rho_1+1)^n + C_2(\rho_2+1)^n + \int_0^n \left[(\rho_1+1)^{n-k-1}\frac{\ln f_k}{(\rho_2-\rho_1)} - (\rho_2+1)^{n-k-1}\frac{\ln f_k}{(\rho_2-\rho_1)}\right]$$

If we write the solution using the sigma symbol for the discrete integral, it will look like this:

$$Z_n = C_1(\rho_1+1)^n + C_2(\rho_2+1)^n + \sum_{k=0}^{n-1}\left[\left((\rho_1+1)^{n-k-1} - (\rho_2+1)^{n-k-1}\right)\frac{\ln f_k}{(\rho_2-\rho_1)}\right] \tag{5.16}$$

Now, using the change of variable:

$$y_n = e^{Z_n}$$

we arrive at the final solution to the non-homogeneous nonlinear difference equation5.3:

$$y_n = e^{C_1(\rho_1+1)^n}.e^{C_2(\rho_2+1)^n}.e^{\sum_{k=0}^{n-1}\left[\left((\rho_1+1)^{n-k-1}-(\rho_2+1)^{n-k-1}\right)\frac{\ln f_k}{(\rho_2-\rho_1)}\right]} \tag{5.17}$$

Remark 5.1. As we saw in the previous sections, the four basic operations +, -, ×, and ÷ in multiplication are converted to multiplication, division, power, root, and the symbol $\sum$ to $\prod$, respectively. Therefore, apart from the above method, the solution to equation 3 can be written using discrete multiplicative integral.

Example 5.2. consider the following discrete multiplicative difference equation

$$y^{[11]}(x)\cdot\left(y^{[1]}(x)\right)^{-8}\cdot y(x)^7 = f(x). \tag{5.18}$$

similar to equation 4.2 and the invariant function for discrete multiplicative drivative, the general solution of homogeneous part of 5.18 will be as follows:

$$y(x) = C_1^{(1+\lambda_1)^x}.C_2^{(1+\lambda_2)^x} = C_1^{2^x}.C_2^{2^{3x}}. \tag{5.19}$$

and the final solution of non-homogeneous equation 5.18 will be the following form :

$$y(x) = C_1^{2^x}.C_2^{2^{3x}}\left[\prod \left(\frac{2^{x-\xi-2}}{2^{-3(x-\xi)-4}}\right)^{f(\xi)}\right]^{-\frac{1}{3}}$$

And using the product symbol $\prod$, the final answer will be as follows.

$$y(x) = C_1^{2^n}.C_2^{2^{3n}}\left(\prod_{\xi=0}^{n-1}\left(\frac{2^{n-\xi-2}}{2^{-3(n-\xi)-4}}\right)^{f(n)}\right)^{-\frac{1}{3}} = \frac{C_1^{2^n}.C_2^{2^{3n}}}{\sqrt[3]{\prod_{\xi=0}^{n-1}\left(\frac{2^{n-\xi-2}}{2^{-3(n-\xi)-4}}\right)^{f(n)}}}. \tag{5.20}$$

Note that the equation 5.18 is equivalent to the following non-homogeneous nonlinear difference equation:

$$y_{n+2}.y_{n+1}^{-10}.y_n^{16} = f_n$$

and its solution is given by relation 5.20

Also, note that the form of this solution was constructed by remark (5.1) and the similarity between this solution and relations 4.3, 4.4, and definition of discrete multiplicative integral which was presented in section (3.5).

In the following example , the solution is given by the process of section (5.2).

Example 5.3. consider the following non-homogeneous nonlinear difference equation:

$$y_{n+2}.y_{n+1}^{-1}.y_n^{-2} = 2^n$$

For this first we obtain the general solution of homogeneous part as follows:

$$y_{n+2}.y_{n+1}^{-1}.y_n^{-2} = 1$$

to abtain this solution, we consider the general form of equation $\left(y_n{}^{[11]}\right)^a \cdot \left(y_n{}^{[1]}\right)^b \cdot y_n{}^c = 1$
Using the concept of discrete multiplicative derivative, we have:

$$y^{[1]}(k) = \frac{y(k+1)}{y(k)} = \frac{y_{k+1}}{y_k} \text{ and } y^{[11]}(k) = \frac{\frac{y_{k+2}}{y_{k+1}}}{\frac{y_{k+1}}{y_k}} = \frac{y_{k+2}.y_k}{y_{k+1}^2}$$

$$\left(\frac{y_{k+2}.y_k}{y_{k+1}^2}\right)^a \cdot \left(\frac{y_{k+1}}{y_k}\right)^b \cdot y_k{}^c = 1 \Rightarrow (y_{k+2})^a.(y_{k+1})^{b-2a}.(y_k)^{a-b+c} = 1$$

Now we consider the general form of the following second-order difference equations:

$$(y_{k+2})^{\alpha}.(y_{k+1})^{\beta}.(y_k)^{\gamma} = 1$$

then we arrive at the following algebraic system:

$$\begin{cases} \alpha = a \\ \beta = b - 2a \\ \gamma = a - b + c \end{cases} \Rightarrow \begin{cases} 1 = a \\ -1 = b - 2a \\ -2 = a - b + c \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 1 \\ c = -2 \end{cases}$$

$$a\lambda^2 + b\lambda + c = 0 \Rightarrow \lambda^2 + \lambda - 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_1 = -2 \end{cases}$$

$$\Rightarrow \begin{cases} y_1(n) = c_1{}^{(1+(1))^n} \\ y_2(n) = c_2{}^{(1+(-2))^n} \end{cases} \Rightarrow \begin{cases} y_1(n) = c_1{}^{2^n} \\ y_2(n) = c_1{}^{(-1)^n} \end{cases}$$

we show that one of these solutions satisfies in the related homogeneous equation and the other does as well, in the same way.

$$y_1(n) = c_1{}^{2^n} \Rightarrow y_{n+2} = c_1^{2^{n+2}} = c_1^{2^n.4} = \left(c_1^{2^n}\right)^4 = y_n{}^4$$

$$y_{n+1} = c_1^{2^{n+1}} = c_1^{2^n.2} = \left(c_1^{2^n}\right)^2 = y_n{}^2$$

$$y_n{}^4.(y_n{}^2)^{-1}.y_n{}^{-2} = 1.$$

Now we use the change of variable formula $y_n = e^{Z_n}$ and write the answer formula from it:

$$y = c_1{}^{2^n}.c_1{}^{(-1)^n}$$

by using 5.17 we have:

$$y_n = e^{C_1(2)^n}.e^{C_2(-1)^n}.e^{\sum_{k=0}^{n-1}\left[\left((2)^{n-k-1}-(-1)^{n-k-1}\right)\frac{\ln 2^k}{-3}\right]} \tag{5.21}$$

Part4

6. Non-Homogeneous continuous multiplicative equations

We consider the boundary value problem including the following multiplicative second-order differential equation:

$$y^{[11]}(x)\cdot\left(y^{[1]}(x)\right)^a\cdot y(x)^b = f(x) \qquad x\in(x_0,x_1) \tag{6.1}$$

$$y(x_0)=\alpha \quad , \quad y(x_1)=\beta \tag{6.2}$$

where $f(x)$ is the inhomogeneous term of the known continuous function on the interval (x_0, x_1) and α, β, a, b are known positive real constants.

same as descrete case we use the following chenge variable :

$$y(x) = e^{z(x)} \tag{6.3}$$

By calculating $y^{[1]}(x)$, $y^{[11]}(x)$ and replacing theese values in 6.1 , we abtain the following boundary value problem:

$$z''(x) + az'(x) + bz(x) = \ln f(x), \qquad x\in(x_0,x_1). \tag{6.4}$$

$$z(x_0) = \ln\alpha \quad , \quad z(x_1) = \ln\beta. \tag{6.5}$$

We consider the homogeneous equation like equation 6.4 and write its characteristic equation:

$$z''(x) + az'(x) + bz(x) = 0 \tag{6.6}$$

Considering the suggested change variable $z(x) = e^{\lambda x}$
We have:

$$\lambda^2 + a\lambda + b = 0 \tag{6.7}$$

The roots of characteristic equation 6.7 are:

$$\lambda_1 = \frac{-a-\sqrt{a^2-4b}}{2}, \lambda_2 = \frac{-a+\sqrt{a^2-4b}}{2} \tag{6.8}$$

Supposing the following conditions,

$$a^2-4b>0 \quad , \quad \lambda_1 \neq \lambda_2 \tag{6.9}$$

Therefore the general solution of equation 6.6 is:

$$z(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{6.10}$$

Now by using Lagrange method for non-homogeneous equation and performing some simple operations, and Similar to relations 5.12,5.14, 5.15 we have the following solution:

following integrals

$$z(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \int_{x_1}^{x} \frac{e^{\lambda_1 (x-\xi)} \ln f(\xi)}{\lambda_2 - \lambda_1} d\xi - \int_{x_1}^{x} \frac{e^{\lambda_2 (x-\xi)} \ln f(\xi)}{\lambda_2 - \lambda_1} d\xi$$

or

$$z(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \int_{x_0}^{x_1} g(x-\xi) \ln f(\xi) d\xi \tag{6.11}$$

and as a result, the Green's function of the boundary value problem 6.4, 6.5 is as follows :

$$g(x-\xi) = \begin{cases} \frac{e^{\lambda_2 (x-\xi)} - e^{\lambda_1 (x-\xi)}}{2(\lambda_2 - \lambda_1)}, & x_0 < \xi < x \\ -\frac{e^{\lambda_2 (x-\xi)} - e^{\lambda_1 (x-\xi)}}{2(\lambda_2 - \lambda_1)}, & x < \xi < x_1 \end{cases} \tag{6.12}$$

The final solution of main problem equation 6.1, 6.2 will be as follows:

$$y(x) = e^{z(x)} = e^{C_1 e^{\lambda_1 x}} . e^{C_2 e^{\lambda_2 x}} . e^{\int_{x_0}^{x_1} g(x-\xi) \ln f(\xi) d\xi}. \tag{6.13}$$

Example 6.1. consider the following first and secend order multiplicative non-homogeneous differential equations:

$$y^{[1]}(x) \cdot y(x) = f(x). \tag{6.14}$$

$$y^{[11]}(x) \cdot y^{[1]}(x) \cdot y(x) = f(x). \tag{6.15}$$

They are equivalent to the following non-linear non-homogeneous ordinary differential equations respectivly:

$$y' + y \ln y = y \ln f(x). \tag{6.16}$$

$$y'' y - y'^2 + y y' + y^2 (\ln y - \ln f(x)) = 0. \tag{6.17}$$

Therefore their analytical solutions can be writen by using the related formula 6.13.

7. Some application and numerial calculations of multiplicative calculus

7.1. Multiplicative Model for The Problem of Hunting and Hunter (Hunter-Prey Problem)

As we know, the mathematical model of this problem in ordinary calculus (continuous additive calculus) is as follows:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta x y, & x(0) = x_0, \\ \frac{dy}{dt} = \sigma y + \gamma x y, & y(0) = y_0. \end{cases}$$

In which $x(t)$ is the population of hunting and $y(t)$ is the population of hunter. Numerical constants $\alpha, \beta, \sigma, \gamma$ depend on the isolated environment and the prey and predator populations of the problem.

The above system of ordinary differential equations can be written as follows, by using the multiplicative model due to the appearance of product sentences xy in the above system of differential equations.

(Choose for simplicity$\alpha = \beta = \sigma = \gamma = 1$)

$$\begin{cases} \frac{x'}{x} = 1 - y \\ \\ \frac{y'}{y} = 1 + x \end{cases} \Rightarrow \begin{cases} e^{\frac{x'}{x}} = e^{1-y} \\ \\ e^{\frac{y'}{y}} = e^{1+x} \end{cases} \Rightarrow \begin{cases} x^*(t) = e^{1-y(t)} \\ \\ y^*(t) = e^{1+x(t)}. \end{cases}$$

The solutions of this system of multiplicative differential equations can be written as a system of integral equations of the first type of Volterra by applying multiplicative integral and given initial conditions:

$$x(t) = x(0) \cdot e^{\int_0^t \ln e^{(1-y(\tau))} d\tau}.$$

$$y(t) = y(0) \cdot e^{\int_0^t \ln e^{(1+x(\tau))} d\tau}.$$

Now, to find approximate solutions and use the method of successive approximations, we make the following functional sequences as

$$x_n(t) = x(0) \cdot e^{\int_0^t (1-y_{n-1}(\tau)) d\tau}.$$

$$y_n(t) = y(0) \cdot e^{\int_0^t (1+x_{n-1}(\tau)) d\tau}.$$

It is clear that accurate approximate solutions can be obtained from the above sequences, which are much simpler than the analytical and approximate methods presented in various articles and books to solve the hunting and predator problem and we get better solutions with fewer repetitions.

The mathematics of hunting and predator problem can be extended to other phenomena such as the relationship between producer and consumer, police and criminals, diseases and humans. [22], [18]

7.2. Exponential Approxmation for Non-Linear Functions

In this section, we give the linear approximation as exponential approximation for nonlinear functions. Similarly, suppose $x(t)$ is a positive and differentiable function at a point $t = a$, then its linear approximation and exponential approximation are as follows:

$$L(t) = x(a) + x'(a)(t - a), \qquad E(t) = x(a).x^*(a)^{t-a}$$

In fact, this approximation is multiplicative analogy for linear approximation in additive calculus. As we see in Stanly [7] and the works of D. Filip and C. Piatecki [8], for the function $x(t) = \frac{1}{t}$ at the point $t = 2$, linear approximation in additive calculus and exponential approximation in multiplicative calculus are given by:

$$L(t) = x(2) + x'(2)(t - 2) = 1 - \frac{1}{4}t$$

$$E(t) = x(2).x^*(2)^{t-2} = \frac{1}{2}e^{-\frac{1}{2}(t-2)} = \frac{1}{2}e^{-\frac{t}{2}+1}$$

It is very interesting that the exponential approximation is a more near than the linear one. As it can be seen in the following Figure 1 .

7.3. Multiplicative Linearization of functions and differential equations

As we see in the following Figure 2, the multiplicative derivative transforms the exponential function into a straight line. This fact helps us to transform differential functions and nonlinear equations in multiplicative calculus into linear functions and linear differential equation with respect to production operation, which is not possible in ordinary Newtonian calculus.

In other words, we can transform the derivative and integral of the product of two functions and the division of two functions into the product of derivatives and division of derivatives and the product of integrals and division of integrals, as shown in Theorems 3.2 and 3.8 . [14, 16, 22]

This ability in multiplicative calculus allows us to study nonlinear differential equations involving parameters, as well as some nonlinear integral equations involving parameters, in the form of linear integral equations.

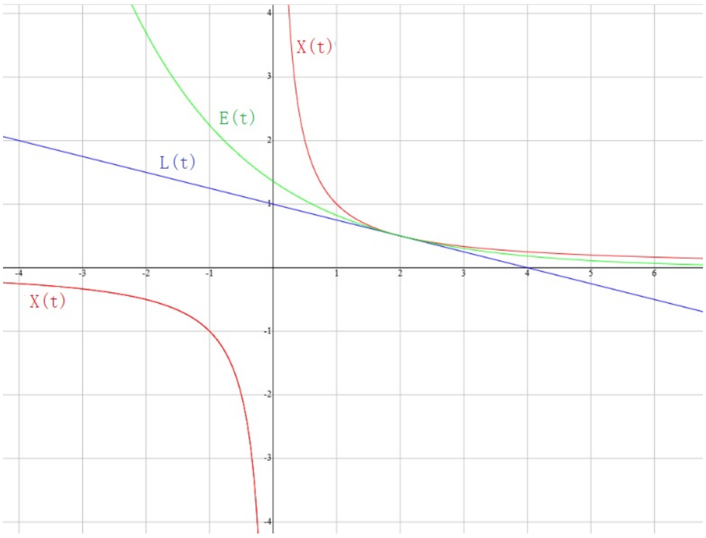


Figure 1: $x(t) = \frac{1}{t}$at the pointt $= 2$

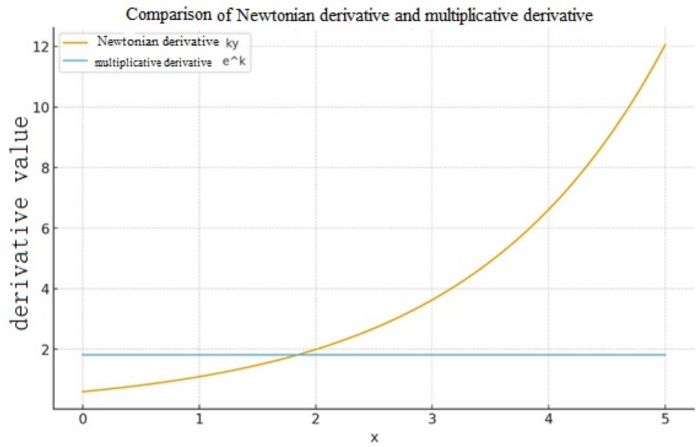


Figure 2: Comparison of Newtonian derivative and multiplicative derivative

7.4. *Advantages of Multiplicative Calculus for Some Population Modeling*

Models percentage growth directly, more accurate for exponential and near-exponential populations, better for small or large population scales (scale-invariant), avoids numerical issues of classical calculus (overflow/underflow), leads to simpler equations in many biological systems. Moreover, in the following cases, multiplicative calculus works better: Bacterial growth, viral infection models cell division, ecological colonization, compartmental disease models and ecosystem energy scaling. [9, 23]

Conclusion

In this paper, first, the solution of linear and nonlinear difference equations was presented using additive and multiplicative discrete differential equations, respectively. Then, these types of equations were solved in the non-homogeneous case, with a constant non-homogeneous term, then with an arbitrary non-homogeneous term. Finally, the solution of boundary value problems including non-homogeneous nonlinear differential equations (continuous case) was presented using non-homogeneous continuous multiplicative differential equations and the Green's function method.

The efficiency of multiplicative calculus in analytically solving nonlinear equations and its other applications in other fields of modeling physics and engineering problems, it is expected that this calculus and multiplicative difference and differential equations will play a greater role in phenomena in which changes occur rapidly.

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