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Fixed point theory approach in vector-valued metric spaces for the solvability of multivariate integral equation systems

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Abstract

This study investigates the existence of solutions for certain systems of integral equations in multi-component models with asymmetric and heterogeneous dynamics, using fixed point theory. Classical metric frameworks are not sufficiently flexible to adequately capture these systems, highlighting the need for more flexible approaches. To address these challenges, a Perov-type vector-valued metric space endowed with a triangle inequality controlled by two matrices is introduced, which extends classical metric frameworks by incorporating two independent control matrices. This double-controlled structure significantly enlarges the admissible class of mappings and allows component-wise control adapted to heterogeneous dynamics. Within this setting, the concept of $\mathbb{M}_\alpha$-admissible pairs of selfmaps is defined, and new common fixed point theorems under generalized matrix-type contraction conditions are established, extending several existing results in the literature. The proposed methodology is applied to a two-dimensional integral equation system, and a numerical example is presented to validate the theoretical results.

Keywords: Common fixed point, $\mathbb{M}_\alpha-$admissible pair of selfmaps, Perov-type vector-valued metric spaces with double-controlled triangle inequality, Integral equation systems

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1. Introduction

Fixed point theory (FPT) is a central tool in nonlinear analysis, providing fundamental criteria for the existence, uniqueness, and stability of invariant points of nonlinear mappings. Beyond its theoretical importance in pure mathematics, fixed point theory has found wide applicability in economics, computer science, engineering, and data-driven modeling, particularly in contexts involving stability, optimization, and learning dynamics, and continues to play a central role in nonlinear analysis and dynamical systems [7, 12, 16, 19, 11, 10, 9, 13, 15, 22].

One of the most prominent applications of fixed point theory lies in the analysis of differential and integral equations. Many problems in these areas can be reformulated as finding fixed points of specific operators. In this context, classical results such as the Banach contraction principle provide powerful tools for guaranteeing the existence and uniqueness of solution under certain conditions. This theorem is particularly effective in solving ordinary and partial differential equations, offering a systematic approach to proving the existence of solutions. Fixed point methods have also become indispensable in numerical

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analysis, where iterative schemes are employed to approximate solutions of linear and nonlinear differential and integral equations and to ensure convergence toward stable solutions [8, 14, 4, 1, 2, 23, 21].

Classical results of FPT have been extensively developed in standard metric spaces and their generalizations, such as b-metric spaces and Perov's vector-valued metric spaces. Perov introduced vector metric spaces (VMS) [17], Bakhtin proposed b-metric spaces by relaxing the triangle inequality using a positive coefficient [5], and Boriceanu integrated these ideas into vector-valued b-metric spaces [6]. Later, Ali and Kim advanced this framework further by developing Czerwik-VMS through the use of diagonal matrices [3]. Despite their success, these frameworks share a common structural restriction: the triangle inequality is governed by a single control parameter (scalar or matrix-valued), implicitly assuming uniform growth behavior across all components. Such uniform control is often inadequate for multi-component systems exhibiting heterogeneous growth rates, asymmetric coupling, or non-uniform stability, particularly in integral and nonlocal models where different components may simultaneously display contractive, neutral, or temporarily expansive behavior.

To address this limitation, this research introduces a Perov-type vector-valued metric space endowed with a triangle inequality controlled by two matrices. Here, the control is imposed on the triangle inequality of the vector-valued metric, rather than on the metric space itself. The presence of two independent control matrices substantially enlarges the admissible class of mappings and allows the geometry of the space to adapt flexibly to multi-directional or asymmetric dominance relations among components. In fact, by separating the control into two matrices, one can assign the necessary enlargement to the component where temporary growth occurs while keeping the remaining directions minimally impacted. As a result, the proposed framework in this study naturally captures coupled nonlinear models whose coordinates operate under distinct growth regimes.

The motivation for this extension is primarily application-driven. Many applied models require multi-scale and component-wise control mechanisms that cannot be captured by single-parameter metric frameworks. Typical examples arise in thermo-fluid and thermo-elastic systems, where temperature components may exhibit temporary amplification while strain or displacement variables remain contractive. Similar behavior occurs in reaction–diffusion and population dynamics systems, as well as in delayed and nonlocal control systems, where different state variables are governed by kernels with inherently distinct bounds or nonlinearities. Coupled integral systems arising in engineering and physics, where asymmetry in kernel behavior is typical–for instance, one operator exhibits mild contraction while another displays moderate nonlocal growth. In such settings, the double-controlled structure of Perov-type vector-valued metric space provides a natural and structurally essential framework, extending the scope of fixed point theory to classes of systems beyond the reach of existing approaches.

The concept of α-admissible mappings was introduced to relax classical contraction requirements by incorporating an auxiliary admissibility function, thereby extending fixed point results beyond standard contractive settings. The idea was first formalized by Samet et al. [20], who extended the Banach principle to situations with limited monotonicity. Subsequently, α-admissibility has been developed within various generalized metric and operator settings, with notable applications to differential and integral equations and to common fixed points for nonlinear and fractional-order systems. In vector-valued metric settings, α-admissibility can be naturally extended from scalar function to matrix-valued admissibility operators, allowing component-wise and heterogeneous control. Motivated by this perspective, this paper introduces $\mathbb{M}_\alpha$-admissible pairs of mappings and establishes common fixed point results under generalized contraction conditions, enabling the analysis of coupled integral systems with anisotropic growth behavior.

The aims of this paper are as follows:

• Defining $\mathbb{M}_\alpha$-admissible self-map pairs and extending theorems and results regarding the existence of a common fixed point (CFP) for pairs of α-admissible self-maps in the literature by introducing a new generalized contraction condition within extended vector-valued metric spaces.

These metric spaces are characterized by incorporating the product of two matrices as multiples of m-vectors on the right-hand side of the triangle inequality, referred to in this study as "Perov-type generalized double-controlled vector-valued metric spaces (PDVMS)". The proposed double-controlled vector-valued metric structures introduce inherent symmetry, which influences contraction mappings and ensures the existence of common fixed points. Furthermore, the integral equation systems analyzed later in the paper reveal symmetrical relationships between the components of solution pairs, further aligning the study with the principles of symmetry. By harnessing these intrinsic symmetry properties, the fixed-point approach becomes more versatile and applicable to a wider range of integral systems.

• Proving the existence of solutions for some integral equation systems (IES) by employing the fixed-point method to confirm the validity of the derived results.

As an example, consider the IES as bellow:

$$\begin{cases} \hat{v}(t) = \hat{f}(t) + \left(\int_{t_0}^{t} \mathcal{K}_{\hat{v}_1}(t,\zeta)\hat{v}_1(\zeta)d\zeta, \int_{t_0}^{t} \mathcal{K}_{\hat{v}_2}(t,\zeta)\hat{v}_2(\zeta)d\zeta, \cdots, \int_{t_0}^{t} \mathcal{K}_{\hat{v}_n}(t,\zeta)\hat{v}_n(\zeta)d\zeta \right) \\ \hat{w}(t) = \hat{f}(t) + \left(\int_{t_0}^{t} \mathcal{K}_{\hat{w}_1}(t,\zeta)\hat{w}_1(\zeta)d\zeta, \int_{t_0}^{t} \mathcal{K}_{\hat{w}_2}(t,\zeta)\hat{w}_2(\zeta)d\zeta, \cdots, \int_{t_0}^{t} \mathcal{K}_{\hat{w}_n}(t,\zeta)\hat{w}_n(\zeta)d\zeta \right) \end{cases} \quad (\forall t, \zeta \in J), \qquad (1.1)$$

where $J = [t_0, t_1] \subseteq \mathbb{R}$, $\hat{v}, \hat{w}, \hat{f} : J \to \mathbb{R}^n$ are continuous functions and $\mathcal{K}_{\hat{v}_j}, \mathcal{K}_{\hat{w}_j} : J^2 \to \mathbb{R}^2$ are kernel of integrals. The existence of a solution for the integral equation system is established based on the derived results. Following this, a numerical example is provided to validate the effectiveness of the theoretical findings. The structure of this research is organized as follows: Section 2 introduces the definitions and fundamental implications of PDVMS and $\mathbb{M}_\alpha$-admissible pairs of selfmaps. Section 3 presents proofs of key theorems alongside illustrative examples that support the derived results. Section 4 establishes the existence theorems for solutions to the IES (1.1), grounded in the CFP framework for $\mathbb{M}_\alpha$-admissible pairs of selfmaps. Finally, a numerical example is provided to substantiate the theoretical findings.

2. Preliminaries

Let X be an arbitrary nonempty set, and let $\mathbb{R}^n_+$ denote the space equipped with the usual component-wise partial order. The collection of all nonempty closed subsets of X will be denoted by $\mathcal{P}_{cl}(X)$. Consider a mapping $d : X \times X \to \mathbb{R}^n_+$ that satisfies all the standard axioms of a metric. Such a mapping is referred to as a generalized metric in the sense of Perov [17]. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$, where $\mathbf{r} := (\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n)$ and $\mathbf{s} := (\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_n)$. Define $\max(\mathbf{r}, \mathbf{s}) := (\max(\mathbf{r}_1, \mathbf{s}_1), \cdots, \max(\mathbf{r}_n, \mathbf{s}_n))$, and $|\mathbf{r}| := (|\mathbf{r}_1|, |\mathbf{r}_2|, \cdots, |\mathbf{r}_n|)$. Additionally, $\mathbf{r} \preceq \mathbf{s}$ ($\mathbf{r} \prec \mathbf{s}$) indicates that $\mathbf{r}_i \preceq \mathbf{s}_i$ ($\mathbf{r}_i \prec \mathbf{s}_i$), for each $i = 1, 2, \cdots, n$. Similarly, $\mathbf{r} \preceq k$ implies $\mathbf{r}_i \preceq k$, for all $i = 1, 2, \cdots, n$, where $k = (k, k, \cdots, k) \in \mathbb{R}^n$. Consider $\mathcal{M}_{n,n}(\mathbb{R}^+)$ as the set of all $n \times n$ matrices with positive elements, and $\mathbb{I}$ as the identity $n \times n$ matrix.

Definition 2.1. Let $G, H : X \to X$ be two selfmaps and $\mathbb{M}_\alpha : X \times X \to \mathcal{M}_{n,n}(\mathbb{R}^+)$. Then, the pair (G, H) is $\mathbb{M}_\alpha-$admissible if $\mathbb{M}_\alpha(\hat{v}, \hat{w}) \succeq \mathbb{I}$ implies $\mathbb{M}_\alpha(G\hat{v}, H\hat{w}) \succeq \mathbb{I}$ and $\mathbb{M}_\alpha(H\hat{w}, G\hat{v}) \succeq \mathbb{I}$.

This condition ensures that the admissibility encoded by the matrix function $\mathbb{M}_\alpha$ is preserved when the mappings G and H are applied. In simple terms, $\mathbb{M}_\alpha(\cdot, \cdot)$ identifies pairs of points that satisfy a certain compatibility relation, and $\mathbb{M}_\alpha$-admissibility guarantees that this relation is not lost under iteration. This preservation is crucial for keeping the generated sequence inside a controlled region of the space.

Example 2.2. Let $X = \mathbb{R}$. Define $G\hat{v} = \hat{v}$ and $H\hat{v} = 2\hat{v}$, for $\hat{v} \in X$. Consider $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \begin{bmatrix} 2^{-\hat{v}\hat{w}} & 0 \\ 0 & 1 \end{bmatrix}$, if $\hat{v}\hat{w} \leqslant 0$, and $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \mathbb{Z}$ ($\mathbb{Z}$ is zero matrix) otherwise. Obviously, $\mathbb{M}_\alpha(\hat{v}, \hat{w}) \succeq \mathbb{I}$, for all $\hat{v}, \hat{w}$ whenever $\hat{v}\hat{w} \leqslant 0$, and implies that $\mathbb{M}_\alpha(G\hat{v}, H\hat{w}) \succeq \mathbb{I}$ & $\mathbb{M}_\alpha(H\hat{w}, G\hat{v}) \succeq \mathbb{I}$.

If $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \alpha(\hat{v}, \hat{w})\mathbb{I}_{n,n}$, where $\alpha : X \times X \to \mathbb{R}$, then we get the following definition.

Definition 2.3. Let $G, H : X \to X$ be two selfmaps and $\alpha : X \times X \to \mathbb{R}$. Then, the pair (G, H) is $\alpha-$admissible if $\alpha(\hat{v}, \hat{w}) \geqslant 1$ implies $\alpha(G\hat{v}, H\hat{w}) \geqslant 1$ and $\alpha(H\hat{w}, G\hat{v}) \geqslant 1$.

The notion of α-admissibility is the scalar counterpart of $\mathbb{M}_\alpha$-admissibility, using a single auxiliary function α to encode compatibility between points.
To extend the scope of vector-valued metric techniques beyond the traditional Perov framework, it is natural to introduce a more adaptable structure where the control of the generalized triangle inequality is simultaneously managed by two matrix-valued operators. This enhancement not only consolidates various prior approaches but also offers a more robust tool for addressing coupled or asymmetrically interacting data. Inspired by this perspective, the following concept is proposed.

Definition 2.4. Define two functions $\mathbb{O}_1, \mathbb{O}_2 : X \times X \to \mathcal{M}_{n,n}(\mathbb{R}^+)$ such that for $k=1,2$,

$$\mathbb{O}_k(\hat{v},\hat{w}) = \operatorname{diag}(\omega_{kr}(\hat{v},\hat{w}))_{r=1}^{n}; \quad \omega_{kr}(\hat{v},\hat{w}) \geqslant 1, \quad (\forall r=1,2,\cdots,n).$$

The mapping $d_p : X \times X \to \mathbb{R}^n_+$ is called generalized Perov-type double-controlled vector-valued metric if
(i) $\forall \hat{v}, \hat{w} \in X$, $d_p(\hat{v},\hat{w}) \succeq 0$ and $d_p(\hat{v},\hat{w}) = 0$ iff $\hat{v} = \hat{w}$;
(ii) $\forall \hat{v}, \hat{w} \in X$, $d_p(\hat{v},\hat{w}) = d_p(\hat{w},\hat{v})$;
(iii) $\forall \hat{v}, \hat{w}, \hat{u} \in X$, $d_p(\hat{v},\hat{w}) \preceq d_p(\hat{v},\hat{u})\mathbb{O}_1(\hat{v},\hat{u}) + d_p(\hat{u},\hat{w})\mathbb{O}_2(\hat{u},\hat{w})$.

The pair (X, d_p) is called a generalized Perov-type double-controlled vector-valued metric space (in short, d_p-metric space).
Throughout the paper, d_p is defined as a row vector. The notation d_p^t is used for its column representation, and this choice is merely notational and has no effect on the underlying mathematical structure.
Condition (iii) extends the classical Perov inequality by allowing the contribution of the intermediate point $\hat{u}$ to be regulated through two separate diagonal matrices, $\mathbb{O}_1$ and $\mathbb{O}_2$. This structure reflects settings in which distances propagate with different strengths across various directions or components. As a result, d_p-metric spaces are capable of modeling anisotropic, coupled, or direction-dependent behaviors that lie beyond standard vector-valued metric frameworks.

- The condition (iii) of Definition 2.4 can be replaced by
  $(\acute{iii})$ $\forall \hat{v}, \hat{w}, \hat{u} \in X$, $d_p^t(\hat{v},\hat{w}) \preceq \mathbb{O}_1(\hat{v},\hat{u})d_p^t(\hat{v},\hat{u}) + \mathbb{O}_2(\hat{u},\hat{w})d_p^t(\hat{u},\hat{w})$, where $d_p^t(.,.)$ is $d_p(.,.)$ transposed.
- For all $\hat{v}, \hat{w}, \hat{u} \in X$ and for every $r = 1, 2, \ldots, n$, if $\omega_{kr}(\hat{v},\hat{u}) = \omega_{kr}(\hat{u},\hat{w}) = b \geqslant 1$, then the d_p-metric space is referred to as a vector-valued b-metric space. Notably, when $n = 2$, this framework results in double-controlled complex-valued metric spaces and b-metric spaces.

Example 2.5. Let $X = \mathbb{R}$. Assume $d_p : X^2 \to \mathbb{R}^3_+$ is defined by:

$$d_p(\hat{v},\hat{w}) = (|\hat{v}-\hat{w}|, |\hat{v}-\hat{w}|^{\frac{3}{2}}, |\hat{v}-\hat{w}|^2), \quad (\forall \hat{v}, \hat{w} \in X).$$

Clearly, $d_p(\hat{v},\hat{w})$ is a complete d_p-metric space with $\mathbb{O}_1(\hat{v},\hat{w}) = \mathbb{O}_2(\hat{v},\hat{w}) = \operatorname{diag}(1, \sqrt{2}, 2)$.

This example illustrates how the double-controlled structure arises naturally when different components of the distance exhibit different growth rates. The diagonal nature of $\mathbb{O}_1$ and $\mathbb{O}_2$ emphasizes that each coordinate may obey its own control factor, demonstrating the flexibility of the d_p-metric in capturing heterogeneous componentwise dynamics.
In a generalized d_p-metric space, the concepts of Cauchy sequences, convergent sequences, and space completeness are defined analogously to those in standard metric spaces.

Definition 2.6. X possesses the property $(B^\star)$ with respect to $\mathbb{M}_\alpha$ if, for the sequence $\{\hat{v}_n\}_{n\geqslant 1}$ in X, where $\mathbb{M}_\alpha(\hat{v}_n, \hat{v}_{n+1}) \succeq \mathbb{I}$ for all $n \geqslant 1$ and $d_p(\hat{v}_n, \hat{v}) \to 0$, there exists a natural number N such that $\mathbb{M}_\alpha(\hat{v}_n, \hat{v}) \succeq \mathbb{I}$ and $\mathbb{M}_\alpha(\hat{v}, \hat{v}_n) \succeq \mathbb{I}$ for all $n \geqslant N$.

Property $(B^\star)$ acts as a continuity-type condition for the admissibility function $\mathbb{M}_\alpha$. It guarantees that if consecutive terms of a sequence satisfy the admissibility condition, and the sequence converges, then the limit point also satisfies this condition with all sufficiently large terms. This prevents the admissibility structure from breaking down at the limit, which is essential for fixed point arguments based on iteration.
The next example confirms that Property $((B^\star))$ is not merely an abstract assumption.

Example 2.7. Let $X=\mathbb{R}^2$. Consider $\mathbb{M}_\alpha(\hat{v},\hat{w})=\begin{bmatrix} 2 & 0.5 \\ 0.3 & 1.5 \end{bmatrix}$, if $\| \hat{v}-\hat{w} \|_\infty \leqslant 1$, and $\mathbb{M}_\alpha(\hat{v},\hat{w})=\mathbb{Z}$ otherwise. Define $d_p(\hat{v},\hat{w}) = (\|\hat{v}-\hat{w}\|_\infty^{1.5}, \|\hat{v}-\hat{w}\|_\infty)$, for each $\hat{v},\hat{w} \in X$. Clearly, (X, d_p) is a d_p-metric space with $\mathbb{O}_j(\hat{v},\hat{w})=\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$. Let $\{\hat{v}_n\}_{n\geqslant 1} \subseteq X$ such that $\| \hat{v}_n - \hat{v}_{n+1} \|_\infty \leqslant 1$, for $n \geqslant 1$, and $\hat{v}_n \to \hat{v}$ ($d_p(\hat{v}_n, \hat{v}) \to 0$). Then, $\mathbb{M}_\alpha(\hat{v}_n, \hat{v}_{n+1}) \succeq \mathbb{I}$, for $n \geqslant 1$, and there exists a natural number N such that $d_p(\hat{v}_n,\hat{v}) \prec 1$, for $n \geqslant N$. So, $\|\hat{v}_n - \hat{v}\|_\infty < 1$, for $n \geqslant N$, and consequently, $\mathbb{M}_\alpha(\hat{v}_n,\hat{v}) \succeq \mathbb{I}$ and $\mathbb{M}_\alpha(\hat{v},\hat{v}_n) \succeq \mathbb{I}$, for all $n \geqslant N$.

A matrix $\mathbb{U} \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero whenever $\mathbb{U}^k \to 0$. The following theorem, as shown in [18], is used to prove the results presented in this paper.

Theorem 2.8. *Let $\mathbb{U} \in \mathcal{M}_{n,n}(\mathbb{R}^+)$. The following are equivalents:*
(i) $\mathbb{U}^k \to 0$;
(ii) *The eigenvalues of $\mathbb{U}$ are in the open unit disc, i.e. $|\lambda| < 1$, for all $\lambda \in C$ with $\det(\mathbb{U} - \lambda\mathbb{I}) = 0$;*
(iii) *The matrix $\mathbb{I} - \mathbb{U}$ is non-singular and $(\mathbb{I} - \mathbb{U})^{-1} = \mathbb{I} + \mathbb{U} + \cdots + \mathbb{U}^k + \cdots$;*
(iv) *The matrix $\mathbb{I} - \mathbb{U}$ is non-singular and $(\mathbb{I} - \mathbb{U})^{-1}$ has nonnegative elements;*
(v) *$\mathbb{U}^k \mathbf{q} \to 0$ and $\mathbf{q}\mathbb{U}^k \to 0$, for all $\mathbf{q} \in \mathbb{R}^n$.*

3. Main Results

Throughout this paper, the vector-valued metric d_p is presented in its transposed (column) form for notational consistency, a choice that does not alter the mathematical structure. This convention is maintained in all theoretical results, with the non-transposed (row) representation used only in some computational examples for brevity and compact presentation.

Theorem 3.1. *Let (X, d_p) be a complete d_p-metric space and $\mathbb{U} \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ such that $\mathbb{O}_2(\hat{v},\hat{w})\mathbb{U}$ converges to zero, for each $\hat{v},\hat{w} \in X$. Consider $G, H: X \to X$ are continuous $\mathbb{M}_\alpha-$admissible selfmaps on X. Suppose that*
(i) *For each $\hat{v},\hat{w} \in X$, there exists $\mathfrak{L}(\hat{v},\hat{w}) \in \{d_p^t(\hat{v},\hat{w}), d_p^t(\hat{v},G\hat{v}), d_p^t(\hat{w},H\hat{w})\}$ such that*

$$\mathbb{M}_\alpha(\hat{v},\hat{w}) d_p^t(G\hat{v},H\hat{w}) \preceq \mathbb{U}\mathfrak{L}(\hat{v},\hat{w}), \tag{3.1}$$

(ii) *For any arbitrary sequence $\{\hat{v}_\kappa\}_{\kappa\geqslant 0} \subseteq X$ and $\hat{w} \in X$,*

$$\mathbb{O}_1(\hat{v}_{\kappa+1},\hat{v}_{\kappa+2}) \preceq \mathbb{O}_1(\hat{v}_\kappa,\hat{v}_{\kappa+1}), \quad \mathbb{O}_2(\hat{v}_{\kappa+1},\hat{w}) \preceq \mathbb{O}_2(\hat{v}_\kappa,\hat{w}), \quad (\forall \kappa \geqslant 0);$$

(iii) *There exists $\hat{v}_\circ \in X$ such that $\mathbb{M}_\alpha(\hat{v}_\circ, G\hat{v}_\circ) \succeq \mathbb{I}$ and $\mathbb{M}_\alpha(G\hat{v}_\circ,\hat{v}_\circ) \succeq \mathbb{I}$.*
Then G and H have a CFP.

Proof. Take $\hat{v}_\circ \in X$ such that $\mathbb{M}_\alpha(\hat{v}_\circ, G\hat{v}_\circ) \succeq \mathbb{I}$ and $\mathbb{M}_\alpha(G\hat{v}_\circ,\hat{v}_\circ) \succeq \mathbb{I}$. Put $\hat{v}_1 = G\hat{v}_\circ$ and $\hat{v}_2 = H\hat{v}_1$. Then, based on the condition (i), there exists $\mathfrak{L}(\hat{v}_\circ,\hat{v}_1) \in \{d_p^t(\hat{v}_\circ,\hat{v}_1), d_p^t(\hat{v}_\circ,G\hat{v}_\circ), d_p^t(\hat{v}_1,H\hat{v}_1)\}$, such that

$$\mathbb{M}_\alpha(\hat{v}_\circ,\hat{v}_1) d_p^t(G\hat{v}_\circ,H\hat{v}_1) \preceq \mathbb{U}\mathfrak{L}(\hat{v}_\circ,\hat{v}_1).$$

Since $\mathbb{M}_\alpha(\hat{v}_\circ,\hat{v}_1) = \mathbb{M}_\alpha(\hat{v}_\circ,G\hat{v}_\circ) \succeq \mathbb{I}$ and $d_p^t(.,.) \succeq 0$, we have

$$d_p^t(\hat{v}_1,\hat{v}_2) = d_p^t(G\hat{v}_\circ,H\hat{v}_1) \preceq \mathbb{M}_\alpha(\hat{v}_\circ,\hat{v}_1) d_p^t(G\hat{v}_\circ,H\hat{v}_1) \preceq \mathbb{U}\mathfrak{L}(\hat{v}_\circ,\hat{v}_1).$$

If $\hat{v}_\circ = \hat{v}_1$, then $\mathfrak{L}(\hat{v}_\circ,\hat{v}_1)$ is equal to zero or $d_p^t(\hat{v}_1,\hat{v}_2)$, and so

$$\begin{cases} d_p^t(\hat{v}_1,\hat{v}_2) \preceq 0 \Rightarrow \hat{v}_1 = \hat{v}_2; \\ d_p^t(\hat{v}_1,\hat{v}_2) \preceq \mathbb{U} d_p^t(\hat{v}_1,\hat{v}_2) \Rightarrow (\mathbb{I} - \mathbb{U}) d_p^t(\hat{v}_1,\hat{v}_2) \preceq 0 \Rightarrow d_p^t(\hat{v}_1,\hat{v}_2) = 0 \Rightarrow \hat{v}_1 = \hat{v}_2. \end{cases}$$

Thus, $\hat{v}_\circ$ will be a CFP of G and H. Assume that $\hat{v}_\circ \neq \hat{v}_1$ and $\hat{v}_1 \neq \hat{v}_2$. Then, $\mathfrak{L}(\hat{v}_\circ, \hat{v}_1)$ must be equal to $d_p^t(\hat{v}_\circ, \hat{v}_1)$ and we get

$$d_p^t(\hat{v}_1, \hat{v}_2) \preceq \mathbb{U} d_p^t(\hat{v}_\circ, \hat{v}_1).$$

Put $\hat{v}_3 = G\hat{v}_2$. Since (G, H) is an $\mathbb{M}_\alpha$-admissible pair, $\mathbb{M}_\alpha(\hat{v}_1, \hat{v}_2) \succeq \mathbb{I}$ and

$$\mathbb{M}_\alpha(\hat{v}_2, \hat{v}_1) \succeq \mathbb{I}.$$

Hence, by using the condition (i) again, there exists

$$\mathfrak{L}(\hat{v}_2, \hat{v}_1) \in \left\{ d_p^t(\hat{v}_2, \hat{v}_1), d_p^t(\hat{v}_2, G\hat{v}_2), d_p^t(\hat{v}_1, H\hat{v}_1) \right\}$$

such that

$$d_p^t(\hat{v}_3, \hat{v}_2) = d_p^t(G\hat{v}_2, H\hat{v}_1) \preceq \mathbb{M}_\alpha(\hat{v}_2, \hat{v}_1) d_p^t(G\hat{v}_2, H\hat{v}_1) \preceq \mathbb{U}\mathfrak{L}(\hat{v}_2, \hat{v}_1).$$

In the same way, one can check that if $\hat{v}_1 = \hat{v}_2$, then $\hat{v}_2 = \hat{v}_3$, and so $\hat{v}_1$ is a CFP of G and H. Assume that $\hat{v}_1 \neq \hat{v}_2$ and $\hat{v}_2 \neq \hat{v}_3$. Then, $\mathfrak{L}(\hat{v}_2, \hat{v}_1)$ must be equal to $d_p^t(\hat{v}_1, \hat{v}_2)$ and

$$d_p^t(\hat{v}_2, \hat{v}_3) \preceq \mathbb{U} d_p^t(\hat{v}_1, \hat{v}_2) \preceq \mathbb{U}^2 d_p^t(\hat{v}_\circ, \hat{v}_1).$$

Therefore, by induction, the sequence $\{\hat{v}_\kappa\}_{\kappa \geqslant 0}$ can be structured in X such that

$$\begin{gathered} \hat{v}_{2\kappa+1} = G\hat{v}_{2\kappa}, \quad \hat{v}_{2\kappa+2} = H\hat{v}_{2\kappa+1}, \quad (\forall \kappa \geqslant 0), \\ \mathbb{M}_\alpha(\hat{v}_{2j}, \hat{v}_{2\kappa+1}) d_p^t(G\hat{v}_{2j}, H\hat{v}_{2\kappa+1}) \preceq \mathbb{U}\mathfrak{L}(\hat{v}_{2j}, \hat{v}_{2\kappa+1}), \quad (\forall \kappa \geqslant 0,\ j = \kappa, \kappa+1), \end{gathered} \tag{3.2}$$

where

$$\mathfrak{L}(\hat{v}_{2j}, \hat{v}_{2\kappa+1}) \in \left\{ d_p^t(\hat{v}_{2j}, \hat{v}_{2\kappa+1}), d_p^t(\hat{v}_{2j}, G\hat{v}_{2j}), d_p^t(\hat{v}_{2\kappa+1}, H\hat{v}_{2\kappa+1}) \right\}$$

for all $\kappa \geqslant 0$ and $j = \kappa, \kappa+1$. Obviously,

$$\mathbb{M}_\alpha(\hat{v}_\kappa, \hat{v}_{\kappa+1}) \succeq \mathbb{I} \ \& \ \mathbb{M}_\alpha(\hat{v}_{\kappa+1}, \hat{v}_\kappa) \succeq \mathbb{I}$$

for each $\kappa \geqslant 0$ due to (G, H) is an $\mathbb{M}_\alpha$−admissible pair of selfmaps. If $\hat{v}_{2\kappa} = \hat{v}_{2\kappa+1}$, for some $\kappa \geqslant 0$, then $\hat{v}_{2\kappa+1} = \hat{v}_{2\kappa+2}$, and so $\hat{v}_{2\kappa}$ will be a CFP of G and H. Because, if $\hat{v}_{2\kappa} = \hat{v}_{2\kappa+1}$, then by using Inequality (3.1), we have

$$d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2}) \preceq \mathbb{M}_\alpha(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}) d_p^t(G\hat{v}_{2\kappa}, H\hat{v}_{2\kappa+1}) \preceq \mathbb{U}\mathfrak{L}(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}),$$

where

$$\mathfrak{L}(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}) = 0$$

or

$$\mathfrak{L}(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}) = d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2}).$$

If $\mathfrak{L}(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}) = 0$ then

$$d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2}) \preceq 0$$

and so $\hat{v}_{2\kappa+1} = \hat{v}_{2\kappa+2}$. Elseif $\mathfrak{L}(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}) = d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2})$, then

$$d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2}) \preceq \mathbb{U} d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2}) \Rightarrow (\mathbb{I} - \mathbb{U}) d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2}) \preceq 0 \Rightarrow \hat{v}_{2\kappa+1} = \hat{v}_{2\kappa+2}.$$

Assume that $\hat{v}_\kappa \neq \hat{v}_{\kappa+1}$, $(\forall \kappa \geqslant 0)$. As a result,

$$\mathfrak{L}(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}) = d_p^t(\hat{v}_{2\kappa}, \hat{v}_{2\kappa+1}), \quad \mathfrak{L}(\hat{v}_{2\kappa+2}, \hat{v}_{2\kappa+3}) = d_p^t(\hat{v}_{2\kappa+1}, \hat{v}_{2\kappa+2}).$$

Hence, we have

$$d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+1}) \preceq \mathbb{U}^\kappa d_p^t(\hat{v}_\circ, \hat{v}_1), \quad (\forall \kappa \geqslant 1). \tag{3.3}$$

Using (3.3), for $\kappa \geqslant 0$ and any natural number r, we have

$$\begin{aligned}
d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+r}) &\preceq \mathbb{O}_1(\hat{v}_\kappa, \hat{v}_{\kappa+1}) d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+1}) + \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) d_p^t(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \\
&\preceq \mathbb{O}_1(\hat{v}_\kappa, \hat{v}_{\kappa+1}) d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+1}) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \Big\{ \mathbb{O}_1(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+2}) d_p^t(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+2}) + \mathbb{O}_2(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) d_p^t(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) \Big\} \\
&\preceq \mathbb{O}_1(\hat{v}_\kappa, \hat{v}_{\kappa+1}) d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+1}) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_1(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+2}) d_p^t(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+2}) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) d_p^t(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) \\
&\preceq \mathbb{O}_1(\hat{v}_\kappa, \hat{v}_{\kappa+1}) d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+1}) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_1(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+2}) d_p^t(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+2}) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) \mathbb{O}_1(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+3}) d_p^t(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+3}) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+3}, \hat{v}_{\kappa+r}) d_p^t(\hat{v}_{\kappa+3}, \hat{v}_{\kappa+r}) \\
&\preceq \cdots \\
&\preceq \mathbb{O}_1(\hat{v}_\kappa, \hat{v}_{\kappa+1}) \mathbb{U}^\kappa d_p^t(\hat{v}_\circ, \hat{v}_1) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_1(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+2}) \mathbb{U}^{\kappa+1} d_p^t(\hat{v}_\circ, \hat{v}_1) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) \mathbb{O}_1(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+3}) \mathbb{U}^{\kappa+2} d_p^t(\hat{v}_\circ, \hat{v}_1) \\
&+ \cdots \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) \cdots \mathbb{O}_2(\hat{v}_{\kappa+r-2}, \hat{v}_{\kappa+r}) \mathbb{O}_1(\hat{v}_{\kappa+r-2}, \hat{v}_{\kappa+r-1}) \mathbb{U}^{\kappa+r-2} d_p^t(\hat{v}_\circ, \hat{v}_1) \\
&+ \mathbb{O}_2(\hat{v}_{\kappa+1}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+2}, \hat{v}_{\kappa+r}) \cdots \mathbb{O}_2(\hat{v}_{\kappa+r-2}, \hat{v}_{\kappa+r}) \mathbb{O}_2(\hat{v}_{\kappa+r-1}, \hat{v}_{\kappa+r}) \mathbb{U}^{\kappa+r-1} d_p^t(\hat{v}_\circ, \hat{v}_1) \\
&\preceq \Big\{ \mathbb{O}_1(\hat{v}_\kappa, \hat{v}_{\kappa+1}) + \sum_{j=1}^{r-1} \prod_{\iota=1}^{j} \mathbb{O}_2(\hat{v}_{\kappa+\iota}, \hat{v}_{\kappa+r}) \mathbb{O}_1(\hat{v}_{\kappa+j}, \hat{v}_{\kappa+j+1}) \mathbb{U}^j \Big\} \mathbb{U}^\kappa d_p^t(\hat{v}_\circ, \hat{v}_1).
\end{aligned}$$

Using the condition (ii), we get

$$d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+r}) \preceq \mathbb{O}_1(\hat{v}_\kappa, \hat{v}_{\kappa+1}) \Big\{ \sum_{j=1}^{r-1} \mathbb{O}_2^j(\hat{v}_\kappa, \hat{v}_{\kappa+r}) \mathbb{U}^j \Big\} \mathbb{U}^\kappa d_p^t(\hat{v}_\circ, \hat{v}_1).$$

Since $\mathbb{O}_2(\hat{v}, \hat{w})\mathbb{U}$ converges to zero, for each $\hat{v}, \hat{w} \in X$, Theorem 2.8 implies that $\lim_{\kappa, r \to \infty} d_p(\hat{v}_\kappa, \hat{v}_{\kappa+r}) = \lim_{\kappa, r \to \infty} d_p^t(\hat{v}_\kappa, \hat{v}_{\kappa+r}) = 0$. Hence, $\{\hat{v}_\kappa\}_{\kappa \geqslant 0}$ is a Cauchy sequence. As a result, there exists $\hat{v}_* \in X$ such that $\hat{v}_\kappa \to \hat{v}_*$, due to (X, d_p) is a complete d_p-metric space. Therefore, $\hat{v}_{2\kappa+1} = G\hat{v}_{2\kappa} \to G\hat{v}_*$ and $\hat{v}_{2\kappa+2} = H\hat{v}_{2\kappa+1} \to H\hat{v}_*$, because G and H are continuous mappings. Then, $\hat{v}_* = G\hat{v}_* = H\hat{v}_*$, and so $\hat{v}_*$ is a CFP of G and H. □

Example 3.2. Let $X = \mathbb{C}$ and $\mathbb{U} = \begin{bmatrix} \frac{2}{3} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$. Consider $d_p : X \times X \to \mathbb{C}^+$ is defined as follow:

$$d_p(\hat{v}, \hat{w}) = \big||\hat{v}| - |\hat{w}|\big| + \mathbf{i}\big|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\hat{w})\big|^2, \quad (\forall \hat{v}, \hat{w} \in X).$$

It is clear that (X, d_p) is a complete d_p-metric space with $\mathbb{O}_1(\hat{v}, \hat{w}) = \mathbb{O}_2(\hat{v}, \hat{w}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and the assumption (ii) of Theorem 3.1 holds. Define continuous selfmaps $G, H : X \to X$ by:

$$G\hat{v} = \frac{1}{2}|\hat{v}|e^{\frac{i\theta}{2}}, \qquad H\hat{v} = \begin{cases} \frac{1}{2}|\hat{v}|e^{-\frac{i\theta}{2}} & \mathrm{Re}(\hat{v})\mathrm{Im}(\hat{v}) < 0, \\ \frac{1}{2}|\hat{v}|e^{\frac{i\theta}{2}} & \mathrm{Re}(\hat{v})\mathrm{Im}(\hat{v}) \geqslant 0, \end{cases}$$

where $\theta=\mathrm{Arg}(\hat{v})$. Denote the sets $D_1=\{\hat{v}\in X : \mathrm{Re}(\hat{v})\mathrm{Im}(\hat{v})\geqslant 0\}$ and $D_2=\{\hat{v}\in X : \mathrm{Re}(\hat{v})\mathrm{Im}(\hat{v})<0\}$. Consider $\mathbb{M}_\alpha(\hat{v},\hat{w})=\begin{bmatrix}\frac{4}{3} & 4\\ 0 & 2\end{bmatrix}$, if $(\hat{v},\hat{w})\in D_1\times D_1$, and $\mathbb{M}_\alpha(\hat{v},\hat{w})=\mathbb{Z}$, otherwise ($\mathbb{Z}$ is zero matrix). It is clear that (G,H) is $\mathbb{M}_\alpha-$admissible pair and $\mathbb{M}_\alpha(\hat{v}_\circ,G\hat{v}_\circ)=\mathbb{M}_\alpha(G\hat{v}_\circ,\hat{v}_\circ)\succeq\mathbb{I}$, for each $\hat{v}_\circ\in D_1$. Now, we must show that G and H satisfy the Inequality (3.1) for all $\hat{v},\hat{w}\in X$. Clearly, (3.1) holds for each $\hat{v}=\hat{w}$ and also for all $(\hat{v},\hat{w})\in(D_1\times D_1)^c$. Let $(\hat{v},\hat{w})\in D_1\times D_1$ be distinct. Then,

$$d_p(G\hat{v},H\hat{w})=\frac{1}{2}\Big||\hat{v}|-|\hat{w}|\Big|+\frac{1}{4}\big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2\mathbf{i},$$

$$d_p(\hat{v},\hat{w})=|\hat{v}-\hat{w}|+\mathbf{i}\big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2,\quad d_p(\hat{v},G\hat{v})=\frac{1}{2}|\hat{v}|+\frac{1}{4}\big|\mathrm{Arg}(\hat{v})\big|^2\mathbf{i},\quad d_p(\hat{w},H\hat{w})=\frac{1}{2}|\hat{w}|+\frac{1}{4}\big|\mathrm{Arg}(\hat{w})\big|^2\mathbf{i}.$$

So,

$$\mathbb{M}_\alpha(\hat{v},\hat{w})d_p^t(G\hat{v},H\hat{w})=\begin{bmatrix}\frac{4}{3} & 4\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2}\Big||\hat{v}|-|\hat{w}|\Big|\\ \frac{1}{4}\big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2\end{bmatrix}\preceq\begin{bmatrix}\frac{2}{3}|\hat{v}-\hat{w}|+\big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2\\ \frac{1}{2}\big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2\end{bmatrix},$$

$$\mathbb{U}d_p^t(\hat{v},\hat{w})=\begin{bmatrix}\frac{2}{3} & 1\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}|\hat{v}-\hat{w}|\\ \big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2\end{bmatrix}=\begin{bmatrix}\frac{2}{3}|\hat{v}-\hat{w}|+\big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2\\ \frac{1}{2}\big|\mathrm{Arg}(\hat{v})-\mathrm{Arg}(\hat{w})\big|^2\end{bmatrix}.$$

Hence, $\mathbb{M}_\alpha(\hat{v},\hat{w})d_p^t(G\hat{v},G\hat{w})\preceq\mathbb{U}\mathfrak{L}(\hat{v},\hat{w})$, in which $\mathfrak{L}(\hat{v},\hat{w})=d_p^t(\hat{v},\hat{w})$. Therefore, all conditions of Theorem 3.1 are established, and $\hat{v}^*=(0,0)$ is the CFP of G and H.

Theorem 3.3. *Let (X,d_p) be a complete d_p-metric space, (G,H) a pair of $\mathbb{M}_\alpha-$admissible selfmaps on X, and $\mathbb{U}\in\mathcal{M}_{n,n}(\mathbb{R}^+)$ such that $\mathbb{O}_2(\hat{v},\hat{w})\mathbb{U}$ converges to zero, for each $\hat{v},\hat{w}\in X$. Suppose that*
(i) *For each $\hat{u},\hat{w}\in X$, there exists $\mathfrak{L}(\hat{v},\hat{w})\in\big\{d_p^t(\hat{v},\hat{w}),d_p^t(\hat{v},G\hat{v}),d_p^t(\hat{w},H\hat{w})\big\}$ such that*

$$\mathbb{M}_\alpha(\hat{v},\hat{w})d_p^t(G\hat{u},H\hat{w})\preceq\mathbb{U}\mathfrak{L}(\hat{v},\hat{w}),\tag{3.4}$$

(ii) *For any arbitrary sequence $\{\hat{v}_\kappa\}_{\kappa\geqslant 0}\subseteq X$ and $\hat{w}\in X$,*

$$\mathbb{O}_1(\hat{v}_{\kappa+1},\hat{v}_{\kappa+2})\preceq\mathbb{O}_1(\hat{v}_\kappa,\hat{v}_{\kappa+1}),\quad \mathbb{O}_2(\hat{v}_{\kappa+1},\hat{w})\preceq\mathbb{O}_2(\hat{v}_\kappa,\hat{w}),\quad(\forall\kappa\geqslant 0);$$

(iii) *There exists $\hat{v}_\circ\in X$ such that $\mathbb{M}_\alpha(\hat{v}_\circ,G\hat{v}_\circ)\succeq\mathbb{I}$ and $\mathbb{M}_\alpha(G\hat{v}_\circ,\hat{v}_\circ)\succeq\mathbb{I}$.*
If X has the property $(B^\star)$ with respect to $\mathbb{M}_\alpha$, then G and H have a CFP.

Proof. Similar to the procedure of the proof of Theorem 3.1, we construct the convergent sequence $\{\hat{v}_\kappa\}_{\kappa\geqslant 0}$ to $\hat{v}^*$ in X as follows:

$$\hat{v}_{2\kappa+1}=G\hat{v}_{2\kappa},\quad \hat{v}_{2\kappa+2}=H\hat{v}_{2\kappa+1},\quad d_p^t(\hat{v}_\kappa,\hat{v}_{\kappa+1})\preceq\mathbb{U}^\kappa d_p^t(\hat{v}_\circ,\hat{v}_1),\quad(\forall\kappa\geqslant 0).$$

Since X has the property $(B^\star)$ with respect to $\mathbb{M}_\alpha$, then

$$\mathbb{M}_\alpha(\hat{v}_{2\kappa},\hat{v}^*)\succeq\mathbb{I}\ \&\ \mathbb{M}_\alpha(\hat{v}^*,\hat{v}_{2\kappa})\succeq\mathbb{I},\quad \mathbb{M}_\alpha(\hat{v}_{2\kappa+1},\hat{v}^*)\succeq\mathbb{I}\ \&\ \mathbb{M}_\alpha(\hat{v}^*,\hat{v}_{2\kappa+1})\succeq\mathbb{I},$$

for all $\kappa\geqslant 1$. Therefore, there exist $\mathfrak{L}(\hat{v}_{2\kappa},\hat{v}^*)\in\big\{d_p^t(\hat{v}_{2\kappa},\hat{v}^*),d_p^t(\hat{v}_{2\kappa},G\hat{v}_{2\kappa}),d_p^t(\hat{v}^*,H\hat{v}^*)\big\}$ and $\mathfrak{L}(\hat{v}^*,\hat{v}_{2\kappa+1})\in\big\{d_p^t(\hat{v}^*,\hat{v}_{2\kappa+1}),d_p^t(\hat{v}^*,G\hat{v}^*),d_p^t(\hat{v}_{2\kappa+1},H\hat{v}_{2\kappa+1})\big\}$ such that

$$\begin{cases} d_p^t(G\hat{v}_{2\kappa},H\hat{v}^*)\preceq\mathbb{M}_\alpha(\hat{v}_{2\kappa},\hat{v}^*)d_p^t(G\hat{v}_{2\kappa},H\hat{v}^*)\preceq\mathbb{U}\mathfrak{L}(\hat{v}_{2\kappa},\hat{v}^*),\\ d_p^t(G\hat{v}^*,H\hat{v}_{2\kappa+1})\preceq\mathbb{M}_\alpha(\hat{v}^*,\hat{v}_{2\kappa+1})d_p^t(G\hat{v}^*,H\hat{v}_{2\kappa+1})\preceq\mathbb{U}\mathfrak{L}(\hat{v}^*,\hat{v}_{2\kappa+1}).\end{cases}$$

If $\mathfrak{L}(\hat{v}_{2\kappa},\hat{v}^*)=d_p^t(\hat{v}_{2\kappa},\hat{v}^*)$ or $\mathfrak{L}(\hat{v}_{2\kappa},\hat{v}^*)=d_p^t(\hat{v}_{2\kappa},G\hat{v}_{2\kappa})$, then we have $d_p^t(\hat{v}_{2\kappa+1},H\hat{v}^*)\preceq\mathbb{U}d_p^t(\hat{v}_{2\kappa},\hat{v}^*)$ or $d_p^t(\hat{v}_{2\kappa+1},H\hat{v}^*)\preceq\mathbb{U}d_p^t(\hat{v}_{2\kappa},\hat{v}_{2\kappa+1})$. Letting $\kappa\to\infty$ gives $d_p^t(\hat{v}^*,H\hat{v}^*)\preceq 0$, which implies $H\hat{v}^*=\hat{v}^*$. Similarly,

we can obtain $G\hat{v}^* = \hat{v}^*$, if $\mathfrak{L}(\hat{v}^*, \hat{v}_{2\kappa+1}) = d_p^t(\hat{v}^*, \hat{v}_{2\kappa+1})$ or $\mathfrak{L}(\hat{v}^*, \hat{v}_{2\kappa+1}) = d_p^t(\hat{v}_{2\kappa+1}, H\hat{v}_{2\kappa+1})$.
If $\mathfrak{L}(\hat{v}_{2\kappa}, \hat{v}^*) = d_p^t(\hat{v}^*, H\hat{v}^*)$, then $d_p^t(\hat{v}_{\kappa+1}, H\hat{v}^*) \preceq \mathbb{U} d_p^t(\hat{v}^*, H\hat{v}^*)$, and so $d_p^t(\hat{v}^*, H\hat{v}^*) \preceq \mathbb{U} d_p^t(\hat{v}^*, H\hat{v}^*)$, as $\kappa \to \infty$. Thus,

$$(\mathbb{I} - \mathbb{U}) d_p^t(\hat{v}^*, H\hat{u}^*) \preceq 0 \Rightarrow d_p^t(\hat{v}^*, H\hat{v}^*) = (\mathbb{I} - \mathbb{U})^{-1}(\mathbb{I} - \mathbb{U}) d_p^t(\hat{v}^*, H\hat{v}^*) \preceq 0 \Rightarrow H\hat{v}^* = \hat{v}^*.$$

In the same way, it can be proved that $G\hat{v}^* = \hat{v}^*$, if $\mathfrak{L}(\hat{v}^*, \hat{v}_{2\kappa+1}) = d_p^t(\hat{v}^*, G\hat{v}^*)$. Therefore, in all the above cases, we have $H\hat{v}^* = G\hat{v}^* = \hat{v}^*$, and so $\hat{v}^*$ is a CFP of G and H. □

Example 3.4. Let $X = \{\hat{v} \in \mathbb{C} : |\hat{v}| \leqslant 4\}$ and $\mathbb{U} = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$. Consider $d_p : X \times X \to \mathbb{C}^+$ is defined as follow:

$$d_p(\hat{v}, \hat{w}) = \left|\hat{v} - \hat{w}\right|^{\frac{3}{2}} + 4\mathbf{i}\left|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\hat{w})\right|, \quad (\forall \hat{v}, \hat{w} \in X).$$

Clearly, (X, d_p) is a complete d_p-metric space with $\mathbb{O}_1(\hat{v}, \hat{w}) = \mathbb{O}_2(\hat{v}, \hat{w}) = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$ and the condition (ii) of Theorem 3.3 holds. Define selfmaps $G, H : X \to X$ as follows:

$$G\hat{v} = \begin{cases} -2\hat{v} & 1 < |\hat{v}| \leqslant 2, \\ \frac{-1}{2}\hat{v} & 2 < |\hat{v}| \leqslant 4, \\ \hat{v} & |\hat{v}| \leqslant 1, \end{cases} \qquad H\hat{v} = \begin{cases} 2\hat{v} & 1 < |\hat{v}| \leqslant 2, \\ \frac{1}{2}\hat{v} & 2 < |\hat{v}| \leqslant 4, \\ \hat{v} & |\hat{v}| \leqslant 1, \end{cases} \qquad (\forall \hat{v} \in X).$$

Denote the sets $D_1^\star = \{\hat{v} \in X : |\hat{v}| \leqslant 1\}$, $D_2^\star = \{\hat{v} \in X : 1 < |\hat{v}| \leqslant 2\}$, $D_3^\star = \{\hat{v} \in Y : 2 < |\hat{v}| \leqslant 4\}$, and $D^\star = \left\{(\hat{v}, \hat{w}) \in X^2 : |\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\hat{w})| = \pi\right\}$. Consider $\lambda_1, \lambda_2, \mu_1, \mu_2 \geqslant 1$, $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \mathbb{I}_{2\times2}$, if $(\hat{v}, \hat{w}) \in D_1^\star \times D_1^\star$ with $\hat{v} = \hat{w}$, $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \begin{bmatrix} 8^{\frac{1}{4}} & \lambda_1 \\ 2^{-\frac{1}{4}} & \mu_1 \end{bmatrix}$, if $(\hat{v}, \hat{w}) \in (D_2^\star \times D_2^\star) \cap D^\star$, $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \begin{bmatrix} 2 & \lambda_2 \\ 2\sqrt{2} & \mu_2 \end{bmatrix}$, if $(\hat{v}, \hat{w}) \in (D_3^\star \times D_3^\star) \cap D^\star$, $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \mathbb{M}_\alpha(\hat{w}, \hat{v}) = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{bmatrix}$, if $(\hat{v}, \hat{w}) \in (D_2^\star \times D_3^\star) \cap D^\star$, and $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \mathbb{Z}$, otherwise. It is clear that (G, H) is $\mathbb{M}_\alpha$−admissible pair and $\mathbb{M}_\alpha(\hat{v}_\circ, G\hat{v}_\circ) = \mathbb{M}_\alpha(G\hat{v}_\circ, \hat{v}_\circ) \succeq \mathbb{I}$, for each $\hat{v}_\circ \in X$. On the other hand, X has the property $(\mathcal{B}^\star)$ with respect to $\mathbb{M}_\alpha$. Now, we must show that G and H satisfy the inequality (3.4) for all $\hat{v}, \hat{w} \in X$. Let $(\hat{v}, \hat{w}) \in (D_2^\star \times D_2^\star) \cap D^\star$. Then, we have $\mathrm{Arg}(-2\hat{v}) = \mathrm{Arg}(2\hat{w}) = \mathrm{Arg}(\hat{w})$, and so

$$\begin{aligned}
&d_p(G\hat{v}, H\hat{w}) = d_p(-2\hat{v}, 2\hat{w}) = (2|\hat{v} + \hat{w}|)^{\frac{3}{2}} + 4\mathbf{i}\left|\mathrm{Arg}(-2\hat{v}) - \mathrm{Arg}(2\hat{w})\right| = (2|\hat{v} + \hat{w}|)^{\frac{3}{2}}, \\
&d_p(\hat{v}, \hat{w}) = |\hat{v} - \hat{w}|^{\frac{3}{2}} + 4\mathbf{i}\left|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\hat{w})\right| = |\hat{v} - \hat{w}|^{\frac{3}{2}} + 4\pi\mathbf{i}, \\
&d_p(\hat{v}, G\hat{v}) = d_p(\hat{v}, -2\hat{v}) = |\hat{v} + 2\hat{v}|^{\frac{3}{2}} + 4\mathbf{i}\left|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(-2\hat{v})\right| = (3|\hat{v}|)^{\frac{3}{2}} + 4\pi\mathbf{i}, \\
&d_p(\hat{w}, H\hat{w}) = d_p(\hat{w}, 2\hat{w}) = |\hat{w} - 2\hat{w}|^{\frac{3}{2}} + 4\mathbf{i}\left|\mathrm{Arg}(\hat{w}) - \mathrm{Arg}(2\hat{w})\right| = |\hat{w}|^{\frac{3}{2}}.
\end{aligned}$$

Since $|\hat{v}|, |\hat{w}| \in (1, 2]$ and $|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\hat{w})| = \pi$, it can be easily prove that $|\hat{v} + \hat{w}| \leqslant \sqrt{2}$ and $|\hat{v} - \hat{w}| \geqslant 2$. Hence,

$$\mathbb{M}_\alpha(\hat{v}, \hat{w}) d_p^t(G\hat{v}, H\hat{w}) = \begin{bmatrix} 8^{\frac{1}{4}} & \lambda_1 \\ 2^{-\frac{1}{4}} & \mu_1 \end{bmatrix} \begin{bmatrix} (2|\hat{v} + \hat{w}|)^{\frac{3}{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \times 2^{\frac{1}{4}} |\hat{v} + \hat{w}|^{\frac{3}{2}} \\ 2 \times 2^{\frac{1}{4}} |\hat{v} + \hat{w}|^{\frac{3}{2}} \end{bmatrix} \preceq \begin{bmatrix} 8 \\ 4 \end{bmatrix},$$

$$\mathbb{U} d_p^t(\hat{v}, \hat{w}) = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} |\hat{v} - \hat{w}|^{\frac{3}{2}} \\ 4\pi \end{bmatrix} = \begin{bmatrix} \frac{2}{3}|\hat{v} - \hat{w}|^{\frac{3}{2}} + 3\pi \\ 2\pi \end{bmatrix} \succeq \begin{bmatrix} \frac{4\sqrt{2}}{3} + 3\pi \\ 2\pi \end{bmatrix} \succeq \begin{bmatrix} 8 \\ 4 \end{bmatrix},$$

$$\mathbb{U} d_p^t(\hat{v}, G\hat{v}) = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} (3|\hat{v}|)^{\frac{3}{2}} \\ 4\pi \end{bmatrix} \succeq \begin{bmatrix} 2\sqrt{3} + 3\pi \\ 2\pi \end{bmatrix} \succeq \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

Now, let $(\hat{v},\hat{w}) \in (D_3^\star \times D_3^\star) \cap D^\star$. Similarly, we have $\mathrm{Arg}(\frac{-\hat{v}}{2}) = \mathrm{Arg}(\frac{\hat{w}}{2}) = \mathrm{Arg}(\hat{w})$, and so

$$\begin{aligned}
&d_p(G\hat{v},H\hat{w}) = d_p(\tfrac{-\hat{v}}{2},\tfrac{\hat{w}}{2}) = \big(\tfrac{|\hat{v}+\hat{w}|}{2}\big)^{\frac{3}{2}} + 4\mathbf{i}\big|\mathrm{Arg}(\tfrac{-\hat{v}}{2}) - \mathrm{Arg}(\tfrac{\hat{w}}{2})\big| = \big(\tfrac{1}{2}|\hat{v}+\hat{w}|\big)^{\frac{3}{2}},\\
&d_p(\hat{v},\hat{w}) = |\hat{v}-\hat{w}|^{\frac{3}{2}} + 4\mathbf{i}\big|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\hat{w})\big| = |\hat{v}-\hat{w}|^{\frac{3}{2}} + 4\pi\mathbf{i},\\
&d_p(\hat{v},G\hat{v}) = d_p(\hat{v},\tfrac{-\hat{v}}{2}) = |\hat{v}+\tfrac{\hat{v}}{2}|^{\frac{3}{2}} + 4\mathbf{i}\big|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\tfrac{\hat{v}}{2})\big| = \big(\tfrac{3|\hat{v}|}{2}\big)^{\frac{3}{2}} + 4\pi\mathbf{i},\\
&d_p(\hat{w},H\hat{w}) = d_p(\hat{w},\tfrac{\hat{w}}{2}) = |\hat{w}-\tfrac{\hat{w}}{2}|^{\frac{3}{2}} + 4\mathbf{i}\big|\mathrm{Arg}(\hat{w}) - \mathrm{Arg}(\tfrac{\hat{w}}{2})\big| = \big(\tfrac{|\hat{w}|}{2}\big)^{\frac{3}{2}}.
\end{aligned}$$

Since $|\hat{v}|,|\hat{w}| \in (2,4]$ and $|\mathrm{Arg}(\hat{v}) - \mathrm{Arg}(\hat{w})| = \pi$, then $|\hat{v}+\hat{w}| \leqslant 2\sqrt{2}$ and $|\hat{v}-\hat{w}| \geqslant 4$. Hence,

$$\mathbb{M}_\alpha(\hat{v},\hat{w})d_p^{\mathbf{t}}(G\hat{v},H\hat{w}) = \begin{bmatrix} 2 & \lambda_2 \\ 2\sqrt{2} & \mu_2 \end{bmatrix}\begin{bmatrix} \big(\frac{|\hat{v}+\hat{w}|}{2}\big)^{\frac{3}{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{|\hat{v}+\hat{w}|^{\frac{3}{2}}}{\sqrt{2}} \\ |\hat{v}+\hat{w}|^{\frac{3}{2}} \end{bmatrix} \preceq \begin{bmatrix} 4 \\ 4\sqrt{2} \end{bmatrix},$$

$$\mathbb{U}d_p^{\mathbf{t}}(\hat{v},\hat{w}) = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} |\hat{v}-\hat{w}|^{\frac{3}{2}} \\ 4\pi \end{bmatrix} = \begin{bmatrix} \frac{2}{3}|\hat{v}-\hat{w}|^{\frac{3}{2}} + 3\pi \\ 2\pi \end{bmatrix} \succeq \begin{bmatrix} 4 \\ 4\sqrt{2} \end{bmatrix},$$

$$\mathbb{U}d_p^{\mathbf{t}}(\hat{v},G\hat{v}) = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \big(\frac{3|\hat{v}|}{2}\big)^{\frac{3}{2}} \\ 4\pi \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{3}{2}}|\hat{v}|^{\frac{3}{2}} + 3\pi \\ 2\pi \end{bmatrix} \succeq \begin{bmatrix} 4 \\ 4\sqrt{2} \end{bmatrix}.$$

Finally, if $(\hat{v},\hat{w}) \in (D_2^\star \times D_3^\star) \cap D^\star$ or $(\hat{v},\hat{w}) \in (D_3^\star \times D_2^\star) \cap D^\star$, then $\mathrm{Arg}(G\hat{v}) = \mathrm{Arg}(H\hat{w}) = \mathrm{Arg}(\hat{w})$, and so

$$d_p(G\hat{v},H\hat{w}) = \begin{cases} d_p(-2\hat{v},\frac{\hat{w}}{2}) = \big(\frac{1}{2}|4\hat{v}+\hat{w}|\big)^{\frac{3}{2}} & (\hat{v},\hat{w}) \in (D_2^\star \times D_3^\star) \cap D^\star, \\ d_p(\frac{-\hat{v}}{2},2\hat{w}) = \big(\frac{1}{2}|\hat{v}+4\hat{w}|\big)^{\frac{3}{2}} & (\hat{v},\hat{w}) \in (D_3^\star \times D_2^\star) \cap D^\star, \end{cases}$$

$$d_p(\hat{v},G\hat{v}) = \begin{cases} \big(3|\hat{v}|\big)^{\frac{3}{2}} + 4\pi\mathbf{i} & 1 < |\hat{v}| \leqslant 2, \\ \big(\frac{3}{2}|\hat{v}|\big)^{\frac{3}{2}} + 4\pi\mathbf{i} & 2 < |\hat{v}| \leqslant 4, \end{cases} \Rightarrow 3\sqrt{3} + 4\pi\mathbf{i} \prec d_p(\hat{v},G\hat{v}) \preceq 6\sqrt{6} + 4\pi\mathbf{i},$$

$$d_p(\hat{w},H\hat{w}) = \begin{cases} |\hat{w}|^{\frac{3}{2}} & 1 < |\hat{w}| \leqslant 2, \\ \big(\frac{1}{2}|\hat{w}|\big)^{\frac{3}{2}} & 2 < |\hat{w}| \leqslant 4, \end{cases} \Rightarrow 1 \prec d_p(\hat{w},H\hat{w}) \preceq 2\sqrt{2}.$$

Hence,

$$\mathbb{M}_\alpha(\hat{v},\hat{w})d_p^{\mathbf{t}}(G\hat{v},H\hat{w}) = \begin{cases} \begin{bmatrix} \frac{1}{2\sqrt{2}}|4\hat{v}+\hat{w}|^{\frac{3}{2}} \\ \frac{1}{6\sqrt{2}}|4\hat{v}+\hat{w}|^{\frac{3}{2}} \end{bmatrix}, \\ \begin{bmatrix} \frac{1}{2\sqrt{2}}|\hat{v}+4\hat{w}|^{\frac{3}{2}} \\ \frac{1}{6\sqrt{2}}|\hat{v}+4\hat{w}|^{\frac{3}{2}} \end{bmatrix}, \end{cases} \Rightarrow \mathbb{M}_\alpha(\hat{v},\hat{w})d_p^{\mathbf{t}}(G\hat{v},H\hat{w}) \preceq \begin{bmatrix} 6\sqrt{3} \\ 2\sqrt{3} \end{bmatrix},$$

$$\mathbb{U}d_p^{\mathbf{t}}(\hat{v},\hat{w}) = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} |\hat{v}-\hat{w}|^{\frac{3}{2}} \\ 4\pi \end{bmatrix} \succeq \begin{bmatrix} \frac{4\sqrt{2}}{3} + 3\pi \\ 2\pi \end{bmatrix} \succeq \begin{bmatrix} 6\sqrt{3} \\ 2\sqrt{3} \end{bmatrix},$$

and

$$\mathbb{U}d_p^{\mathbf{t}}(\hat{v},G\hat{v}) \succ \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3\sqrt{3} \\ 4\pi \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} + 3\pi \\ 2\pi \end{bmatrix} \succeq \begin{bmatrix} 6\sqrt{3} \\ 2\sqrt{3} \end{bmatrix},$$

Consequently, in all above cases, $\mathbb{M}_\alpha(\hat{v},\hat{w})d_p^{\mathbf{t}}(G\hat{v},H\hat{w}) \preceq \mathbb{U}\mathfrak{L}(\hat{v},\hat{w})$, in which $\mathfrak{L}(\hat{v},\hat{w}) = d_p^{\mathbf{t}}(\hat{v},\hat{w})$ or $\mathfrak{L}(\hat{v},\hat{w}) = d_p^{\mathbf{t}}(\hat{v},G\hat{v})$. Obviously, the inequality (3.4) holds for other cases, too. Therefore, all conditions of Theorem 3.3 are established and any $\hat{v}^* \in D_1^\star$ is a CFP of G and H.

Let $\mathbb{M}_\alpha(\hat{v},\hat{w}) = \alpha(\hat{v},\hat{w})\mathbb{I}_{n,n}$, for all $\hat{v},\hat{w} \in X$, where $\alpha : X\times X \to [0,+\infty)$. Consequently, the following corollary is obtained.

Corollary 3.5. *Let (X, d_p) be a complete d_p-metric space, (G, H) a pair of $\alpha-$admissible selfmaps on X, and $\mathbb{U} \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ such that $\mathbb{O}_2(\hat{v},\hat{w})\mathbb{U}$ converges to zero, for each $\hat{v},\hat{w}\in X$. Suppose that*
(i) *For each $\hat{v},\hat{w}\in X$, there exists $\mathfrak{L}(\hat{v},\hat{w}) \in \{d_p^t(\hat{v},\hat{w}), d_p^t(\hat{v},G\hat{v}), d_p^t(\hat{w},H\hat{w})\}$ such that*

$$\alpha(\hat{v},\hat{w})d_p^t(G\hat{v},H\hat{w}) \preceq \mathbb{U}\mathfrak{L}(\hat{v},\hat{w}),$$

(ii) *For any arbitrary sequence $\{\hat{v}_\kappa\}_{\kappa\geqslant 0}\subseteq X$ and $\hat{w}\in X$,*

$$\mathbb{O}_1(\hat{v}_{\kappa+1},\hat{v}_{\kappa+2})\preceq\mathbb{O}_1(\hat{v}_\kappa,\hat{v}_{\kappa+1}), \quad \mathbb{O}_2(\hat{v}_{\kappa+1},\hat{w})\preceq\mathbb{O}_2(\hat{v}_\kappa,\hat{w}), \quad (\forall\kappa\geqslant 0);$$

(iii) *There exists $\hat{v}_\circ \in X$ such that $\alpha(\hat{v}_\circ, G\hat{v}_\circ) \geqslant 1$ and $\alpha(G\hat{v}_\circ,\hat{v}_\circ) \geqslant 1$.*
Then G and H have a CFP.

This result holds if X has the property $(B^\star)$ with respect to α instead of $G, H : X \to X$ are continuous.

4. Application

The existence theorem for the solution of an n-dimensional Integral Equation System (IES) can be presented using the results obtained in the previous section. Let $J = [t_0, t_1] \subseteq \mathbb{R}$ and $X = \underbrace{C(J,\mathbb{R})\times\cdots\times C(J,\mathbb{R})}_{n \text{ times}}$, where $C(J,\mathbb{R})$ is the space of all continuous functions from J into $\mathbb{R}$. Consider the following n-dimensional IES:

$$\begin{cases} \hat{v}(t) = \hat{f}(t) + \left(\int_{t_0}^t \mathcal{K}_{\hat{v}_1}(t,\zeta)\hat{v}_1(\zeta)d\zeta, \int_{t_0}^t \mathcal{K}_{\hat{v}_2}(t,\zeta)\hat{v}_2(\zeta)d\zeta, \cdots, \int_{t_0}^t \mathcal{K}_{\hat{v}_n}(t,\zeta)\hat{v}_n(\zeta)d\zeta\right) \\ \hat{w}(t) = \hat{f}(t) + \left(\int_{t_0}^t \mathcal{K}_{\hat{w}_1}(t,\zeta)\hat{w}_1(\zeta)d\zeta, \int_{t_0}^t \mathcal{K}_{\hat{w}_2}(t,\zeta)\hat{w}_2(\zeta)d\zeta, \cdots, \int_{t_0}^t \mathcal{K}_{\hat{w}_n}(t,\zeta)\hat{w}_n(\zeta)d\zeta\right), \end{cases} \quad (\forall t,\zeta \in J), \tag{4.1}$$

where $\hat{v}, \hat{w}, \hat{f}\in X$ and $\mathcal{K}_{\hat{v}_j}, \mathcal{K}_{\hat{w}_j} : J^2\times\mathbb{R}\to\mathbb{R}^2$ are kernels of integrals for $j\geqslant 1$. Assume $d_p : X^2\to\mathbb{R}_+^n$ is defined by:

$$d_p(\hat{v},\hat{w}) = (\|\hat{v}_1 - \hat{w}_1\|_1^{r_1}, \|\hat{v}_2 - \hat{w}_2\|_1^{r_2}, \cdots, \|\hat{v}_n - \hat{w}_n\|_1^{r_n}), \quad (\forall\hat{v},\hat{w}\in X), \tag{4.2}$$

where $r_j \geqslant 1$, $\hat{v} = (\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n)$, $\hat{w} = (\hat{w}_1, \hat{w}_2, \cdots, \hat{w}_n)$, and $\|\hat{v}_j - \hat{w}_j\|_1 = \int_{t_0}^{t_1} |\hat{v}_j(t) - \hat{w}_j(t)|dt$. Clearly, $d_p(\hat{v},\hat{w})$ is a complete d_p-metric space with $\mathbb{O}_1(\hat{v},\hat{w}) = \mathbb{O}_2(\hat{v},\hat{w}) = \begin{bmatrix} 2^{r_1-1} & 0 & \dots & 0 \\ 0 & 2^{r_2-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2^{r_n-1} \end{bmatrix}$. Define $G : X \to X$ by:

$$G\hat{v}(t) = \hat{f}(t) + \left(\int_{t_0}^t \mathcal{K}_{\hat{v}_1}(t,\zeta)\hat{v}_1(\zeta)d\zeta, \int_{t_0}^t \mathcal{K}_{\hat{v}_2}(t,\zeta)\hat{v}_2(\zeta)d\zeta, \cdots, \int_{t_0}^t \mathcal{K}_{\hat{v}_n}(t,\zeta)\hat{v}_n(\zeta)d\zeta\right), \quad (\forall t,\zeta \in J), \tag{4.3}$$

where $\hat{v}, \hat{f} \in X$, and $\mathcal{K}_{\hat{v}_j} : J^2 \times \mathbb{R} \to \mathbb{R}^2$ is a bounded continuous function for $j = 1, 2, \cdots, n$. We will now present a theorem demonstrating the existence of a solution to the aforementioned integral system (4.1) in the two-dimensional case based on Theorem 3.1, which can also be extended to higher dimensions.

Theorem 4.1. *Let $X = C(J,\mathbb{R})\times C(J,\mathbb{R})$, $d_p : X^2\to\mathbb{R}_+^2$ is the metric defined by Eq. (4.2), and $G, H : X\to X$ as defined by (4.3), in the two-dimensional case. Consider $\mathbb{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}^+)$ such that $\mathbb{O}_2(\hat{v},\hat{w})\mathbb{U}$ converges* *to zero, for each* $\hat{v}, \hat{w} \in X$*. Let* $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \begin{bmatrix} m_1(\hat{v}, \hat{w}) & \lambda_1 \\ \lambda_2 & m_2(\hat{v}, \hat{w}) \end{bmatrix}$ *such that* $m_j(\hat{v}, \hat{w}) \geqslant 1$, $\lambda_j \leqslant \min\{1, u_{ji}\}$*, for* $i, j = 1, 2$ *&* $i \neq j$*, and* (G, H) *be an* $\mathbb{M}_\alpha$*-admissible pair and the condition* (iii) *of Theorem 3.1 holds. Suppose that*

$$\left|\mathcal{K}_{\hat{v}_j}(t, \zeta)\hat{v}_j(\zeta) - \mathcal{K}_{\hat{w}_j}(t, \zeta)\hat{w}_j(\zeta)\right|(t - t_0) \leqslant \sqrt[r_j]{\frac{u_{jj}}{m_j(\hat{v}, \hat{w})}}\left|\hat{v}_j(t) - \hat{w}_j(t)\right|, \tag{4.4}$$

for $j = 1, 2$ *and all* $t, \zeta \in J$*. Then* G *and* H *have a* CFP *in* X*.*

Proof. For each $t \in J$ and $t \neq t_0$, we have

$$G\hat{v}(t) - H\hat{w}(t) = \left(\int_{t_0}^{t} \left(\mathcal{K}_{\hat{v}_1}(t, \zeta)\hat{v}_1(\zeta) - \mathcal{K}_{\hat{w}_1}(t, \zeta)\hat{w}_1(\zeta)\right) d\zeta, \int_{t_0}^{t} \left(\mathcal{K}_{\hat{v}_2}(t, \zeta)\hat{v}_2(\zeta) - \mathcal{K}_{\hat{w}_2}(t, \zeta)\hat{w}_2(\zeta)\right) d\zeta\right).$$

Put $(G\hat{v})_j(t) - (H\hat{w})_j(t) = \int_{t_0}^{t} \left(\mathcal{K}_{\hat{v}_j}(t, \zeta)\hat{v}_j(\zeta) - \mathcal{K}_{\hat{w}_j}(t, \zeta)\hat{w}_j(\zeta)\right) d\zeta$, for each $t \in J$. Using inequality (4.4) gives

$$\begin{aligned}
\left|(G\hat{v})_j(t) - (H\hat{w})_j(t)\right| &= \left|\int_{t_0}^{t} \left(\mathcal{K}_{\hat{v}_j}(t, \zeta)\hat{v}_j(\zeta) - \mathcal{K}_{\hat{w}_j}(t, \zeta)\hat{w}_j(\zeta)\right) d\zeta\right| \\
&\leqslant \int_{t_0}^{t} \left|\mathcal{K}_{\hat{v}_j}(t, \zeta)\hat{v}_j(\zeta) - \mathcal{K}_{\hat{w}_j}(t, \zeta)\hat{w}_j(\zeta)\right| d\zeta \\
&\leqslant \int_{t_0}^{t} \sqrt[r_j]{\frac{u_{jj}}{m_j(\hat{v}, \hat{w})}} \frac{\left|\hat{v}_j(t) - \hat{w}_j(t)\right|}{t - t_0} d\zeta = \sqrt[r_j]{\frac{u_{jj}}{m_j(\hat{v}, \hat{w})}}\left|\hat{v}_j(t) - \hat{w}_j(t)\right|.
\end{aligned}$$

Then, $\|(G\hat{v})_j - (H\hat{w})_j\|_1^{r_j} \leqslant \frac{u_{jj}}{m_j(\hat{v},\hat{w})}\|\hat{v}_j - \hat{w}_j\|_1^{r_j}$, for $j=1, 2$. Thus, we will get

$$\begin{aligned}
m_1(\hat{v}, \hat{w})\|(G\hat{v})_1 - (H\hat{w})_1\|_1^{r_1} + \lambda_1\|(G\hat{v})_2 - (H\hat{w})_2\|_1^{r_2} &\leqslant u_{11}\|\hat{v}_1 - \hat{w}_1\|_1^{r_1} + \tfrac{\lambda_1 u_{22}}{m_2(\hat{v},\hat{w})}\|\hat{v}_2 - \hat{w}_2\|_1^{r_2} \\
&\leqslant u_{11}\|\hat{v}_1 - \hat{w}_1\|_1^{r_1} + u_{12}\|\hat{v}_2 - \hat{w}_2\|_1^{r_2}, \\
\lambda_2\|(G\hat{v})_1 - (H\hat{w})_1\|_1^{r_1} + m_2(\hat{v}, \hat{w})\|(G\hat{v})_2 - (H\hat{w})_2\|_1^{r_2} &\leqslant \tfrac{\lambda_2 u_{11}}{m_1(\hat{v},\hat{w})}\|\hat{v}_1 - \hat{w}_1\|_1^{r_1} + u_{22}\|\hat{v}_2 - \hat{w}_2\|_1^{r_2} \\
&\leqslant u_{21}\|\hat{v}_1 - \hat{w}_1\|_1^{r_1} + u_{22}\|\hat{v}_2 - \hat{w}_2\|_1^{r_2}.
\end{aligned}$$

Hence, $\mathbb{M}_\alpha(\hat{v}, \hat{w})d_p^t(G\hat{v}, H\hat{w}) \preceq \mathbb{U}d_p^t(\hat{v}, \hat{w})$, and so all conditions of Theorem 3.1 hold. Consequently, G and H have a CFP in X. In other words, two-dimensional IES (4.1) has a solution. □

Remark 4.2. In the special case of definition of d_p in (4.2) for two-dimensional case, if $r_1 = r_2 = r \geqslant 1$, then (X, d_p) is a complete d_p-metric space with $\mathbb{O}_1(\hat{v}, \hat{w}) = \mathbb{O}_2(\hat{v}, \hat{w}) = 2^{r-1}\mathbb{I}_{2\times 2}$, and we can obtain the next result.

Theorem 4.3. *Let* $X = C^2(J, \mathbb{R})$, $d_p : X^2 \to \mathbb{R}_+^2$ *is the metric mentioned in Remark 4.2 and* $G, H : X \to X$ *as defined by (4.3), in the two-dimensional case. Consider* $2^{r-1}\mathbb{U} = 2^{r-1}\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}^+)$ *converges to zero, and* $\mathbb{M}_\alpha(\hat{v}, \hat{w}) = \begin{bmatrix} m_1(\hat{v}, \hat{w}) & 0 \\ 0 & m_2(\hat{v}, \hat{w}) \end{bmatrix}$*, such that* $m_j(\hat{v}, \hat{w}) \geqslant 1$, (G, H) *be an* $\mathbb{M}_\alpha$*-admissible pair and the condition* (iii) *of Theorem 3.1 holds. Suppose that*

$$\left|\mathcal{K}_{\hat{v}_j}(t, \zeta)\hat{v}_j(\zeta) - \mathcal{K}_{\hat{w}_j}(t, \zeta)\hat{w}_j(\zeta)\right|(t - t_0) \leqslant \frac{\sqrt[r]{u_{j1}}\left|\hat{v}_1(t) - \hat{w}_1(t)\right| + \sqrt[r]{u_{j2}}\left|\hat{v}_2(t) - \hat{w}_2(t)\right|}{2^{r-1}\sqrt[r]{m_j(\hat{v}, \hat{w})}},$$

for $j = 1, 2$ *and all* $t, \zeta \in J$*. Then* G *and* H *have a* CFP *in* X*.*

Proof. Similar to the proof of the previous theorem, we can prove that

$$\left|(G\hat{v})_j(t)-(H\hat{w})_j(t)\right| \leqslant \frac{\sqrt[r]{u_{j1}}\left|\hat{v}_1(t)-\hat{w}_1(t)\right|+\sqrt[r]{u_{j2}}\left|\hat{v}_2(t)-\hat{w}_2(t)\right|}{2^{r-1}\sqrt[r]{m_j(\hat{v},\hat{w})}}, \quad (\forall t\in J,\ j=1,2).$$

Then, $m_j(\hat{v},\hat{w})\|(G\hat{v})_j-(H\hat{w})_j\|_1^r \leqslant u_{j1}\|\hat{v}_1-\hat{w}_1\|_1^r+u_{j2}\|\hat{v}_2-\hat{w}_2\|_1^r$, and so, $\mathbb{M}_\alpha(\hat{v},\hat{w})d_p^t(G\hat{v},H\hat{w})\preceq \mathbb{U}d_p^t(\hat{v},\hat{w})$. As a result, G and H have a CFP, and two-dimensional IES (4.1) has a solution. □

Example 4.4. Let $X=C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ and $d_p(\hat{v},\hat{w})=(\|\hat{v}_1-\hat{w}_1\|_1^{1.1},\|\hat{v}_2-\hat{w}_2\|_1^{1.1})$, for each $\hat{v},\hat{w}\in X$. Consider the following two-dimensional IES:

$$\begin{cases} \hat{v}(t)=\hat{f}(t)+\left(\int_0^t \mathcal{K}_{\hat{v}_1}(t,\zeta)\hat{v}_1(\zeta)d\zeta, \int_0^t \mathcal{K}_{\hat{v}_2}(t,\zeta)\hat{v}_2(\zeta)d\zeta\right), \\ \hat{w}(t)=\hat{f}(t)+\left(\int_0^t \mathcal{K}_{\hat{w}_1}(t,\zeta)\hat{w}_1(\zeta)d\zeta, \int_0^t \mathcal{K}_{\hat{w}_2}(t,\zeta)\hat{w}_2(\zeta)d\zeta\right), \end{cases} \quad (\forall t\in[0,1]); \tag{4.5}$$

$$\mathcal{K}_{\hat{v}_1}(t,\zeta)=\frac{\sinh(\zeta)}{\cosh t-1}, \quad \mathcal{K}_{\hat{v}_2}(t,\zeta)=\frac{\sin(\zeta)}{1-\cos t}, \quad \mathcal{K}_{\hat{w}_1}(t,\zeta)=\frac{\cosh\zeta}{\sinh t}, \quad \mathcal{K}_{\hat{w}_2}(t,\zeta)=\frac{\cos\zeta}{\sin t},$$

for all $t,\zeta\in[0,1]$, $t\neq 0$. It can be easily verified that the equations' solutions in the integral system (4.5) are

$$\hat{v}(t)=\left(\frac{5e^t(\cosh t-1)}{16},\frac{5e^t(1-\cos t)}{12}\right), \quad \hat{w}(t)=\left(\frac{5e^t\sinh t}{16},\frac{5e^t\sin t}{12}\right).$$

Let $G\hat{v}(t)=H\hat{v}(t)=\hat{f}(t)+\left(\int_0^t \mathcal{K}_{\hat{v}_1}(t,\zeta)\hat{v}_1(\zeta)d\zeta, \int_0^t \mathcal{K}_{\hat{v}_2}(t,\zeta)\hat{v}_2(\zeta)d\zeta\right)$, $\mathbb{U}=\begin{bmatrix}0.84 & 0.36\\ 0 & 0.84\end{bmatrix}$, $\mathbb{M}_\alpha(\hat{v}(t),\hat{w}(t))=\begin{bmatrix}0.72 & 0\\ 0 & 0.42\end{bmatrix}$, whenever $t\in[0,0.6)$, $\mathbb{M}_\alpha(\hat{v}(t),\hat{w}(t))=\begin{bmatrix}1.2 & 0\\ 0 & 1\end{bmatrix}$, whenever $t\in[0.6,1]$, and $\mathbb{M}_\alpha(\hat{v}(t),\hat{w}(t))=\mathbb{Z}$, otherwise. Obviously, $2^{0.1}\mathbb{U}\in\mathcal{M}_{2,2}(\mathbb{R}^+)$ is a converges matrix to zero, (G,H) is a continuous $\mathbb{M}_\alpha-$admissible pair, and $\mathbb{M}_\alpha(\hat{v},G\hat{v})\succeq\mathbb{I}$, for all $\hat{v}\in X$ whenever $t\in[0.6,1]$. Further,

$$\begin{aligned} t\left|\mathcal{K}_{\hat{v}_1}(t,\zeta)\hat{v}_1(\zeta)-\mathcal{K}_{\hat{w}_1}(t,\zeta)\hat{w}_1(\zeta)\right| &= t\left|\frac{\sinh(\zeta)}{\cosh t-1}\frac{5e^t(\cosh t-1)}{16}-\frac{\cosh\zeta}{\sinh t}\frac{5e^t\sinh t}{16}\right|=\frac{5}{16}te^t\left|\sinh\zeta-\cosh\zeta\right| \\ &\leqslant \frac{\sqrt[1.1]{0.84}\left|\hat{v}_1(t)-\hat{w}_1(t)\right|+\sqrt[1.1]{0.36}\left|\hat{v}_2(t)-\hat{w}_2(t)\right|}{2^{0.1}\times\sqrt[1.1]{m_1(\hat{v},\hat{w})}}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} t\left|\mathcal{K}_{\hat{v}_2}(t,\zeta)\hat{v}_2(\zeta)-\mathcal{K}_{\hat{w}_2}(t,\zeta)\hat{w}_2(\zeta)\right| &= t\left|\frac{\sin(\zeta)}{1-\cos t}\frac{5e^t(1-\cos t)}{12}\right)-\frac{\cos\zeta}{\sin t}\frac{5e^t\sin t}{12}\Big)\Big| \\ &= \frac{5}{12}te^t\left|\sin\zeta-\cos\zeta\right|\leqslant\frac{\sqrt[1.1]{0.84}\left|\hat{v}_2(t)-\hat{w}_2(t)\right|}{2^{0.1}\times\sqrt[1.1]{m_2(\hat{v},\hat{w})}}. \end{aligned} \tag{4.7}$$

Numerical results of Inequalities (4.6) and (4.7) are given in Tables 1 and 2, for $\zeta=0,0.1,0.2,\cdots,0.6$, and Tables 3 and 4, for $\zeta=0.6,0.7,0.8,0.9,1$, respectively. Moreover, figures 1 and 2 show the Inequalities (4.6) and (4.7) verifying for all values of $t,\zeta\in[0,1]$. According to the obtained results, all conditions of Theorem 4.1 hold. It can be easily checked that $t=0$ is a CFP of $\hat{v}_1(t)$, $\hat{v}_2(t)$, $\hat{w}_1(t)$, and $\hat{w}_2(t)$. Consequently, the two-dimensional IES (4.4) admits the solution $t=0$.

Remark 4.5. Although the kernels in Example 4.4 contain apparent singularities at $t=0$, these singularities are removable since the corresponding solution components cancel them pointwise. Hence, the integral operators are well-defined on J.

Table 1: Approximate results of $\hat{v}_1(t)$, $\hat{w}_1(t)$, and two hands of Ineq. (4.6), for $\zeta \in \{0, 0.1, 0.2, \cdots, 0.6\}$.

t	$\hat{v}_1(t)$	$\hat{w}_1(t)$	Left hand of Ineq. (4.6) for							Right hand of Ineq. (4.6)
			$\zeta = 0$	$\zeta = 0.1$	$\zeta = 0.2$	$\zeta = 0.3$	$\zeta = 0.4$	$\zeta = 0.5$	$\zeta = 0.6$	
0.00	0	0	0	0	0	0	0	0	0	0
0.05	0.0004	0.0164	0.0164	0.0149	0.0134	0.0122	0.0110	0.0100	0.0090	0.0278
0.10	0.0017	0.0346	0.0345	0.0313	0.0283	0.0256	0.0232	0.0209	0.0190	0.0570
0.15	0.0041	0.0547	0.0545	0.0493	0.0446	0.0403	0.0365	0.0330	0.0299	0.0875
0.20	0.0077	0.0768	0.0763	0.0691	0.0625	0.0566	0.0512	0.0463	0.0419	0.1195
0.25	0.0126	0.1014	0.1003	0.0908	0.0821	0.0743	0.0672	0.0608	0.0551	0.1528
0.30	0.0191	0.1285	0.1265	0.1145	0.1036	0.0938	0.0848	0.0768	0.0695	0.1875
0.35	0.0274	0.1584	0.1552	0.1404	0.1271	0.1150	0.1040	0.0941	0.0852	0.2235
0.40	0.0378	0.1915	0.1865	0.1687	0.1527	0.1381	0.1250	0.1131	0.1023	0.2609
0.45	0.0505	0.2281	0.2205	0.1996	0.1806	0.1634	0.1478	0.1338	0.1210	0.2995
0.50	0.0658	0.2685	0.2576	0.2331	0.2109	0.1908	0.1727	0.1563	0.1414	0.3395
0.55	0.0840	0.3132	0.2979	0.2696	0.2439	0.2207	0.1997	0.1807	0.1635	0.3806
0.60	0.1056	0.3625	0.3416	0.3091	0.2797	0.2531	0.2290	0.2072	0.1875	0.4229

Table 2: Approximate results of $\hat{v}_2(t)$, $\hat{w}_2(t)$, and two hands of Ineq. (4.7), for $\zeta \in \{0, 0.1, 0.2, \cdots, 0.6\}$.

t	$\hat{v}_2(t)$	$\hat{w}_2(t)$	Left hand of Ineq. (4.7) for							Right hand of Ineq. (4.7)
			$\zeta = 0$	$\zeta = 0.1$	$\zeta = 0.2$	$\zeta = 0.3$	$\zeta = 0.4$	$\zeta = 0.5$	$\zeta = 0.6$	
0.00	0	0	0	0	0	0	0	0	0	0
0.05	0.0005	0.0219	0.0219	0.0196	0.0171	0.0145	0.0116	0.0087	0.0057	0.0374
0.10	0.0023	0.0460	0.0460	0.0412	0.0360	0.0304	0.0245	0.0183	0.0120	0.0765
0.15	0.0054	0.0723	0.0726	0.0650	0.0567	0.0479	0.0386	0.0289	0.0189	0.1172
0.20	0.0101	0.1011	0.1018	0.0911	0.0795	0.0672	0.0541	0.0405	0.0265	0.1594
0.25	0.0166	0.1324	0.1338	0.1197	0.1045	0.0883	0.0711	0.0533	0.0349	0.2028
0.30	0.0251	0.1662	0.1687	0.1510	0.1318	0.1113	0.0897	0.0672	0.0440	0.2472
0.35	0.0358	0.2027	0.2069	0.1853	0.1617	0.1365	0.1100	0.0824	0.0539	0.2924
0.40	0.0491	0.2421	0.2486	0.2226	0.1943	0.1641	0.1322	0.0990	0.0648	0.3381
0.45	0.0651	0.2842	0.2941	0.2632	0.2298	0.1940	0.1563	0.1171	0.0767	0.3840
0.50	0.0841	0.3293	0.3435	0.3075	0.2684	0.2266	0.1826	0.1368	0.0895	0.4297
0.55	0.1065	0.3775	0.3972	0.3556	0.3104	0.2621	0.2112	0.1581	0.1035	0.4748
0.60	0.1326	0.4287	0.4555	0.4078	0.3559	0.3006	0.2422	0.1814	0.1188	0.5188

Table 3: Approximate results of $\hat{v}_1(t)$, $\hat{w}_1(t)$, and two hands of Ineq. (4.6), whenever $\zeta \in \{0.6, 0.7, \cdots, 1\}$.

t	$\hat{v}_1(t)$	$\hat{w}_1(t)$	Left hand of Ineq. (4.6) for					Right hand of Ineq. (4.6)
			$\zeta = 0.6$	$\zeta = 0.7$	$\zeta = 0.8$	$\zeta = 0.9$	$\zeta = 1$	
0.60	0.1056	0.3625	0.1875	0.1697	0.1535	0.1389	0.1257	0.2658
0.65	0.1310	0.4171	0.2135	0.1932	0.1748	0.1582	0.1431	0.2930
0.70	0.1606	0.4774	0.2418	0.2188	0.1979	0.1791	0.1621	0.3209
0.75	0.1950	0.5440	0.2723	0.2464	0.2229	0.2017	0.1825	0.3494
0.80	0.2347	0.6177	0.3054	0.2763	0.2500	0.2262	0.2047	0.3783
0.85	0.2804	0.6991	0.3411	0.3086	0.2792	0.2527	0.2286	0.4076
0.90	0.3329	0.7890	0.3796	0.3435	0.3108	0.2813	0.2545	0.4373
0.95	0.3929	0.8884	0.4213	0.3812	0.3449	0.3121	0.2824	0.4672
1.00	0.4613	0.9983	0.4662	0.4218	0.3817	0.3454	0.3125	0.4973

Table 4: Approximate results of $\hat{v}_2(t)$, $\hat{w}_2(t)$, and two hands of Ineq. (4.7), whenever $\zeta \in \{0.6, 0.7, \cdots, 1\}$.

t	$\hat{v}_2(t)$	$\hat{w}_2(t)$	Left hand of Ineq. (4.7) for					Right hand of Ineq. (4.7)
			$\zeta = 0.6$	$\zeta = 0.7$	$\zeta = 0.8$	$\zeta = 0.9$	$\zeta = 1$	
0.60	0.1326	0.4287	0.1188	0.0549	0.0094	0.0737	0.1372	0.2358
0.65	0.1628	0.4830	0.1352	0.0626	0.0107	0.0839	0.1562	0.2550
0.70	0.1973	0.5405	0.1531	0.0708	0.0121	0.0950	0.1769	0.2733
0.75	0.2367	0.6013	0.1725	0.0798	0.0137	0.1070	0.1992	0.2903
0.80	0.2812	0.6652	0.1934	0.0895	0.0153	0.1200	0.2234	0.3057
0.85	0.3315	0.7324	0.2160	0.1000	0.0171	0.1340	0.2496	0.3192
0.90	0.3878	0.8028	0.2405	0.1113	0.0190	0.1492	0.2778	0.3304
0.95	0.4507	0.8764	0.2668	0.1235	0.0211	0.1655	0.3082	0.3389
1.00	0.5207	0.9531	0.2953	0.1366	0.0234	0.1832	0.3411	0.3443

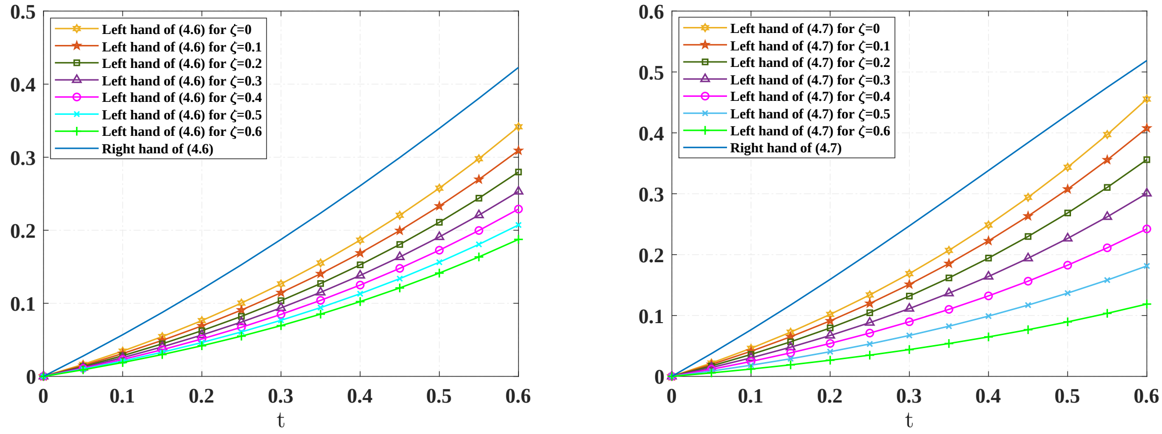

Figure 1: Graphical representation of (4.6) and (4.7) for system (4.5), whenever $t \in [0, 0.6]$ and $\zeta \in \{0, 0.1, 0.2, \cdots, 0.6\}$.

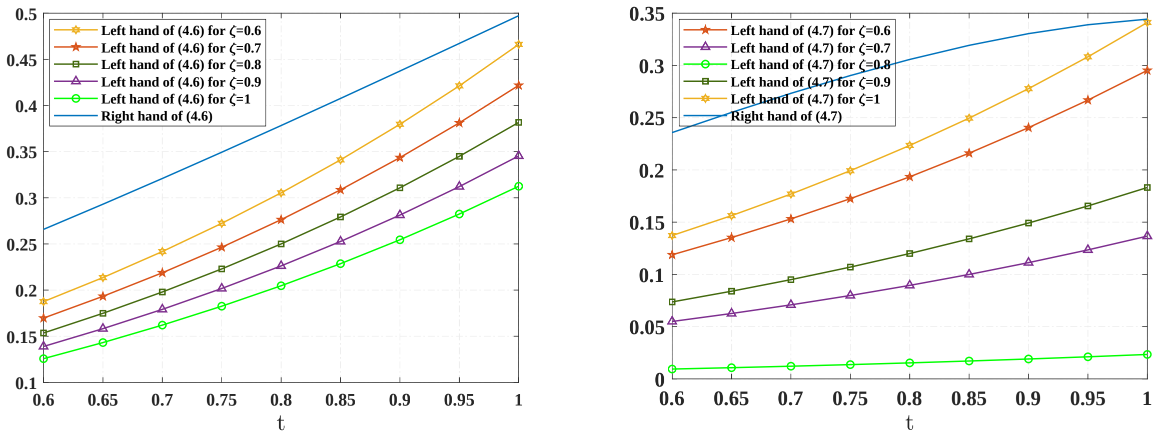

Figure 2: Graphical representation of (4.6) and (4.7) for system (4.5), whenever $t \in [0.6, 1]$ and $\zeta \in \{0.6, 0.7, \cdots, 1\}$.

5. Conclusion

This research establishes several theorems and results regarding the existence of common fixed points for $\mathbb{M}_\alpha$-admissible pairs of self-maps under a newly introduced generalized contraction condition. These findings are derived within the framework of a novel generalized vector-valued metric space, called the Perov-type generalized vector-valued double-controlled metric space (d_p-metric space). To support the theoretical results, various illustrative examples are provided. Additionally, the obtained results are applied to demonstrate the existence of solutions for certain systems of integral equations. A numerical example is also presented to validate the theoretical outcomes of the study. Potential generalizations of these conclusions include:

- Investigating common fixed point of $\mathbb{M}_\alpha-$admissible pairs of selfmaps in d_p-metric space;
- Applying the common fixed point approach for $\mathbb{M}_\alpha-$admissible pairs of selfmaps to establish the existence of solutions for some integral equation systems in d_p-metric spaces.

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