Math. & Comput. Sci., 7(2) (2026), 106–124
Online: ISSN 2717-2708
Mathematics and Computational Sciences
Journal Homepage: http://mcs.qut.ac.ir/mcs

A new computational strategy for solving a fractional-order smoking epidemic model

Rachid Belgacem[a], Ahmed Bokhari[a], Boualem Sadaoui[b,*]

[a] *Laboratory of Mathematics and its Applications LMA, Hassiba Benbouali University of Chlef, Algeria.*

[b] *Laboratory of Mathematics, computer science and Applications, Djilali Bounaama University, Khemis Miliana, Algeria.*

Abstract

This study aims to investigate a fractional-order mathematical model that describes smoking epidemic behavior, formulated using the Caputo fractional (CF) derivative, which effectively captures long-term memory effects the population is divided into five compartments: potential smokers, current smokers, occasional smokers, permanent quitters, and temporary quitters. The model incorporates several parameters characterizing transition rates between these compartments, allowing for a realistic simulation of smoking dynamics. To obtain efficient approximate solutions, we present a new hybrid approach, both analytical and numerical, which combines the specific general integral transform with the homotopy perturbation method (HPM). Numerical simulations performed in MATLAB for different fractional orders reveal the high precision and numerical performance of the proposed technique. Graphical analyses further highlight the methods effectiveness in capturing the temporal evolution of the model, confirming the reliability of the hybrid approach in representing such complex dynamical systems.

Keywords: Caputo fractional derivative, Smoking model, Jafari integral transform, Homotopy Perturbation Method.

2020 MSC: 26A33, 44A10, 92D30, 65H10.

1. Introduction

Integral transforms represent one of the most powerful analytical tools for the study of differential equations. They enable the transformation of ordinary, partial, and fractional differential equations (ODEs, PDEs, FDEs) into reduced algebraic forms, thereby significantly lowering the complexity of their solution [39, 29, 16]. Over the years, these transforms have played a central role in the development of new solution techniques and in the enhancement of classical methods across various scientific and engineering disciplines. Nevertheless, the inversion process remains a critical step to guarantee the accurate reconstruction of the original solutions in either the time or space domain.

In recent years, considerable developments have been achieved in the field of integral transform techniques, consolidating their central role in both the analytical and numerical treatment of differential equations. Owing to their ability to preserve essential structural properties when transitioning between

*Corresponding author
Email addresses: r.belgacem@univ-chlef.dz (Rachid Belgacem), a.bokhari@univ-chlef.dz (Ahmed Bokhari), b.sadaoui@univ-dbkm.dz (Boualem Sadaoui)

doi: 10.30511/mcs.2026.2069397.1447

Received: 19 August 2025 Accepted: 21 February 2026

different functional domains, these methods have found wide applications in mathematics, physics, biology, and engineering.

Recent studies have shown that combining integral transforms with semi-analytical methods, such as the HPM, provides a particularly effective framework for solving non linear and multiscale problems. This hybrid approach accelerates convergence and expands the applicability of analytical techniques to complex fractional differential systems.

During the last two decades, numerous new integral transforms have been introduced, including the Sumudu transform [44], Natural transform [27], Mohand transform [2], Kamal transform [24], Pourgha transform [8, 9], Aboodh transform [4], Elzaki transform [18], Shehu transform [10, 11, 13, 34], Rishi Transform [31], Anuj transform[30, 21], Sawi transform [19, 33] and the Upadhyaya integral transform[22]. The duality relations associated with the Laplace class of integral transforms hold significant importance in tackling numerous scientific and engineering problems. Comprehensive discussions of these relations can be found in the following references [5, 6, 7].

Hossein Jafari [20] introduced the $\mathcal{T}_{\mathcal{J}}$-transform, a general and versatile integral transform that has attracted significant attention due to its operational flexibility and its effectiveness in solving a wide range of analytical and applied mathematical problems. Since its introduction, the $\mathcal{T}_{\mathcal{J}}$-transform has found numerous applications across various scientific domains, as evidenced by recent studies [3, 12, 15, 17, 23, 25, 26, 35, 37, 38, 40, 41, 42].

Manjarekar,

Motivation and Originality

Although integral transforms and the HPM method have been widely used independently, their synergistic combination using the recently proposed $\mathcal{T}_{\mathcal{J}}$-transform remains underexplored, particularly in the context of epidemic models governed by fractional-order dynamics.

The originality of this work lies in presenting a new analytical-numerical framework called the $\mathcal{T}_{\mathcal{J}}$-HPM method, which combines the strengths of Jafaris transform and the Homotopy Perturbation Method. We apply this approach to a fractional-order smoking epidemic model using the CF derivative, enabling a more accurate representation of the system's memory effects compared to classical models based on integer-order derivatives.

This paper is presented according to the following structure: Section 2 introduces the basic concepts of fractional calculus and reviews the essential main properties of the Jafari integral transform, which form the theoretical basis of the proposed methodology. Section 3 introduces the mathematical modelling of the fractional smoking epidemic model and its properties. Section 4 focuses on deriving analytical solutions for the proposed model using the $\mathcal{T}_{\mathcal{J}}$-HPM approach and provides numerical simulations and graphical comparisons to validate the analytical findings. Finally, we conclude the study with key observations and potential future research directions.

2. Preliminary

Definition 2.1. For a given function $\bar{\mathcal{V}}(t)$ defined on the interval $[0, t]$, the Riemann-Liouville fractional (RLF) Integral of order $\alpha > 0$, denoted by ${}^{RL}I^{\alpha}_{0,t}$, is defined as follows [28]:

$$ {}^{RL}I^{\bar{\beta}}_{0,t}\bar{\mathcal{V}}(t) = \int_0^t \frac{(t-\tau)^{\bar{\beta}-1}}{\Gamma(\bar{\beta})}\bar{\mathcal{V}}(\tau)d\tau = \frac{1}{\Gamma(\bar{\beta})}t^{\bar{\beta}-1}\star\bar{\mathcal{V}}(t), \quad t>0. \tag{2.1}$$

The RLF Derivative ${}^{RL}D^{\bar{\beta}}_{0,t}$ of order $\bar{\beta} > 0$, is expressed as:

$$ {}^{RL}D^{\bar{\beta}}_{0,t}\bar{\mathcal{V}}(t) = \frac{d^j}{dt^j}\left({}^{RL}I^{j-\bar{\beta}}_{0,t}\bar{\mathcal{V}}(t)\right) = \frac{d^j}{dt^j}\int_0^t \frac{\bar{\mathcal{V}}(x)}{\Gamma(j-q)(t-x)^{j-\bar{\beta}-1}}dx, \quad j-1<\bar{\beta}\leqslant j, j\in\mathbb{N}. \tag{2.2}$$

where $\Gamma(.)$ is the gamma function.

Table 1: Special cases of the $\mathcal{T}_{\mathcal{J}}$-Transform corresponding to classical transforms

Transform	Jafari transform
Laplace	$\mathcal{K}(s) = 1$ and $\mathcal{M}(s) = s$
Sumudu	$\mathcal{K}(s) = \mathcal{M}(s) = \frac{1}{s}$
Elzaki	$\mathcal{K}(s) = 1$ and $\mathcal{M}(s) = \frac{1}{s}$
Natural	$\mathcal{K}(s) = u$ and $\mathcal{M}(s) = \frac{s}{u}$
Aboodh	$\mathcal{K}(s) = \frac{1}{s}$ and $\mathcal{M}(s) = 1$
Mohand	$\mathcal{K}(s) = s^2$ and $\mathcal{M}(s) = s$
Sawi	$\mathcal{M}(s) = \frac{1}{s}$ and $\mathcal{K}(s) = \mathcal{M}^2(s)$
Kamel	$\mathcal{K}(s) = 1$ and $\mathcal{M}(s) = \frac{1}{s}$
Shehu	$\mathcal{K}(s) = 1$ and $\mathcal{M}(s) = \frac{s}{u}$

Definition 2.2. The Caputo fractional (CF) derivative of order $\bar{\beta} > 0$ for a given function $\bar{\mathcal{V}}(t)$ is defined as follows [14]:

$$ {}^{C}D_{0,t}^{\bar{\beta}}\bar{\mathcal{V}}(t) = \frac{1}{\Gamma(j-\bar{\beta})}\int_{a}^{t}(t-\tau)^{j-\bar{\beta}-1}\bar{\mathcal{V}}^{(j)}(\tau)\,d\tau, \quad 0 < j-1 < \bar{\beta} < j, j \in \mathbb{N}, \tag{2.3}$$
$$= \frac{d^j}{t^j}\bar{\mathcal{V}}(t), \quad \bar{\beta} = j,$$

Lemma 2.3. *If* $\bar{\mathcal{V}} \in AC^r[a,b]$ *or* $\bar{\mathcal{V}}(t) \in C^r[a,b]$, *then [29]:*

$$\sum_{i=0}^{\lceil\bar{\beta}\rceil-1}\bar{\mathcal{V}}^{(i)}(0)\frac{t^i}{i!} = {}^{RL}I_{0,t}^{\bar{\beta}}\left({}^{C}D_{0,t}^{\bar{\beta}}\bar{\mathcal{V}}(t)\right) - \bar{\mathcal{V}}(t), \quad t > 0. \tag{2.4}$$

where $\lceil\bar{\beta}\rceil$ is the ceiling function.

Definition 2.4. (see [20]) Let $\bar{\mathcal{V}}(t)$ be a integrable function defined for $t \geqslant 0$, $\mathcal{K}(s) \neq 0$ and $\mathcal{M}(s)$ are positive real functions. The Jafari integral transform $\mathcal{T}_{\mathcal{J}}$ of a $\bar{\mathcal{V}}(t)$ is is defined by

$$\mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t); s\right] = \mathcal{F}(s) = \mathcal{K}(s)\int_0^\infty \bar{\mathcal{V}}(t)e^{-\mathcal{M}(s)t}\,dt, \tag{2.5}$$

provided the integral exists for some $\mathcal{M}(s)$.

A comprehensive discussion of this formulation can be found in [20]. The corresponding inversion formula for equation is e(2.5) expressed as:

$$\mathfrak{F}^{-1}(s) = \mathcal{T}_{\mathcal{J}}^{-1}\left\{\mathcal{K}(s)\int_0^\infty \bar{\mathcal{V}}(t)\,e^{-\mathcal{M}(s)t}dt\right\} = \bar{\mathcal{V}}(t). \tag{2.6}$$

The proposed integral transform exhibits wide applicability, as an appropriate choice of $\mathcal{K}(s)$ and $\mathcal{K}(s)$ enables its use in solving diverse classes of problems.

Theorem 2.5. *Assume that* $\mathcal{K}(s)$ *and* $\mathcal{M}(s)$ *are strictly positive functions. Then [20]:*

$$\mathcal{T}_{\mathcal{J}}\left\{\bar{\mathcal{V}}^{(r)}(t), s\right\} = (\mathcal{M}(s))^r\left[\mathfrak{F}(s) - \mathcal{K}(s)\sum_{i=0}^{r-1}(\mathcal{M}(s))^{-1-i}\bar{\mathcal{V}}^{(i))}(0)\right], \forall r \in \mathbb{N}. \tag{2.7}$$

Table 2: $\mathcal{T}_{\mathcal{J}}$-transform for selected basic functions.

$\bar{\mathcal{V}}(t)$	$\mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t)\right]$
c	$c\dfrac{\mathcal{K}(s)}{\mathcal{M}(s)}, \quad a \in \mathbb{R}$
t	$\dfrac{\mathcal{K}(s)}{\mathcal{M}^2(s)}$
t^n	$n!\dfrac{\mathcal{K}(s)}{\mathcal{M}^{n+1}(s)}$
$t^{\bar{q}}$	$\dfrac{\Gamma(\bar{\beta})\,\mathcal{K}(s)}{\mathcal{M}^{\bar{q}+1}(s)}, \quad \bar{q} > 0$
$\sin t$	$\dfrac{\mathcal{K}(s)}{\mathcal{M}^2(s)}$
e^t	$\dfrac{\mathcal{K}(s)}{\mathcal{M}(s)}$

Theorem 2.6. *Let $\mathfrak{F}_1(s) = \mathcal{T}_{\mathcal{J}}\left\{\bar{\mathcal{V}}_1, s\right\}$ and $\mathfrak{F}_2(s) = \mathcal{T}_{\mathcal{J}}\left\{\bar{\mathcal{V}}_2, s\right\}$, then [20]:*

$$\mathcal{T}_{\mathcal{J}}\left\{\bar{\mathcal{V}}_1 \star \bar{\mathcal{V}}_2, s\right\} = \int_0^{\infty} \bar{\mathcal{V}}_1(t)\,\bar{\mathcal{V}}_2(t-z)\,dz = \frac{1}{\mathcal{K}(s)}\mathfrak{F}_1(s).\mathfrak{F}_2(s). \tag{2.8}$$

The $\mathcal{T}_{\mathcal{J}}$ transform can be conveniently utilized for a given problem by appropriately choosing $\mathcal{K}(s)$ and $\mathcal{M}(s)$. In Table 2, we provide the $\mathcal{T}_{\mathcal{J}}$ transform for several fundamental functions.

Property 2.7. *When $\bar{\mathcal{V}}(t) = t^{x-1}$ in formula 2.5, then:*

$$\mathcal{T}_{\mathcal{J}}\left[t^{x-1}\right] = \frac{\Gamma(x)\,\mathcal{K}(s)}{(\mathcal{M}(s))^{x}}. \tag{2.9}$$

Lemma 2.8. *The $\mathcal{T}_{\mathcal{J}}$ transform of ${}^{RL}I^{\bar{\beta}}_{0,t}$ and ${}^{RL}D^{\bar{\beta}}_{0}$ for a given function $\bar{\mathcal{V}}$, is given as follows:*

$$\mathcal{T}_{\mathcal{J}}\left[\left({}^{RL}I^{\bar{\beta}}_{0}\bar{\mathcal{V}}\right)(t)\right] = \frac{\mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t)\right]}{\mathcal{M}^{\bar{\beta}}(s)}. \tag{2.10}$$

$$\mathcal{T}_{\mathcal{J}}\left\{{}^{RL}D^{\bar{\beta}}_{0,t}\bar{\mathcal{V}}(t), s\right\} = (\mathcal{M}(s))^{\bar{\beta}}\,\mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t)\right] - \mathcal{K}(s)\sum_{i=0}^{m-1}\mathcal{M}^{m-1-i}(s)\left[{}^{RL}D^{\bar{\beta}-i-m}_{0,t}\bar{\mathcal{V}}(t)\right]_{t=0}, \tag{2.11}$$

where $m-1 < \bar{\beta} \leqslant m$.

Theorem 2.9. *If $\bar{\mathcal{V}} \in AC^m(a_1, a_2)$ for any $a_2 > a_1$ and of exponential order, then:*

$$\mathcal{T}_{\mathcal{J}}\left[{}^{C}D^{\bar{\beta}}_{0}\bar{\mathcal{V}}(t)\right] = \mathcal{M}^{\bar{\beta}}(s)\,\mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t)\right] - \mathcal{K}(s)\sum_{i=0}^{m-1}\mathcal{M}^{\bar{\beta}-i-1}(s)\,\bar{\mathcal{V}}^{(i)}(0). \tag{2.12}$$

Proof. Since ${}^{RL}I^{\bar{\beta}}_{0,t}\left({}^{C}D^{\bar{\beta}}_{0,t}\bar{\mathcal{V}}(t)\right) = \bar{\mathcal{V}}(t) - \sum_{i=0}^{m-1}\frac{\bar{\mathcal{V}}^{(i)}(0)}{i!}t^i$. Lemma 2.3, yields:

$$\mathcal{T}_{\mathcal{J}}\left[{}^{RL}I^{\bar{\beta}}_{0,t}\left({}^{C}D^{\bar{\beta}}_{0,t}\bar{\mathcal{V}}(t)\right)\right] = \mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t) - \sum_{i=0}^{m-1}\frac{\bar{\mathcal{V}}^{(i)}(0)}{i!}t^i\right], \tag{2.13}$$

therefore, the Eq 2.13 becomes,

$$\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[{}^{C}D_{0}^{\bar{\beta}}\bar{\mathcal{V}}(t)\right]=\mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t)\right]-\sum_{i=0}^{m-1}\frac{\bar{\mathcal{V}}^{(i)}(0)}{i!}\frac{i!\mathcal{K}(s)}{(\mathcal{M}(s))^{i+1}}.$$

Finally, we obtain:

$$\mathcal{T}_{\mathcal{J}}\left[{}^{C}D_{0,t}^{\bar{\beta}}\bar{\mathcal{V}}(t)\right]=\mathcal{M}^{\bar{\beta}}(s)\mathcal{T}_{\mathcal{J}}\left[\bar{\mathcal{V}}(t)\right]-\mathcal{K}(s)\sum_{i=0}^{m-1}\mathcal{M}^{\bar{\beta}-i-1}(s)\bar{\mathcal{V}}^{(i)}(0).$$

This concludes the proof. □

3. Mathematical Modelling of fractional Smoking epidemic Model

Fractional-order modeling provides a versatile and powerful framework for the analysis of biological phenomena, as it naturally incorporates memory effects, nonlocal interactions, and complex temporal dynamics. In the present study, we examine a fractional smoking epidemic model formulated as a system of five coupled non linear FDEs involving ${}^{C}D_{0,t}^{\bar{\beta}}$, with $0<\bar{\beta}<1$. This model, originally introduced in [1], offers a quantitative description of the propagation of smoking behavior within a population.

$$\begin{cases} {}^{C}D_{0,t}^{\bar{\beta}}(P(t))=\Lambda-\omega P(t)S(t)-\theta P(t),\\ {}^{C}D_{0,t}^{\bar{\beta}}(O(t))=\omega P(t)S(t)-\alpha_1 O(t)-\theta O(t),\\ {}^{C}D_{0,t}^{\bar{\beta}}(S(t))=\alpha_1 O(t)+\alpha_2 S(t)Q(t)-(\theta+\xi)S(t),\\ {}^{C}D_{0,t}^{\bar{\beta}}(Q(t))=-\alpha_2 S(t)Q(t)-\theta Q(t)+\xi(1-\sigma)S(t),\\ {}^{C}D_{0,t}^{\bar{\beta}}(L(t))=\sigma\xi S(t)-\theta L(t), \end{cases} \tag{3.1}$$

subject to:

$$P(0)=b_1, O(0)=b_2, S(0)=b_3, Q(0)=b_4, L(0)=b_5. \tag{3.2}$$

In this model 3.1, we describes the interactions between different compartments of the population over time, where:

- $N(t)$: total population size.
- $P(t)$: potential smokers.
- $S(t)$: current smoker.
- $O(t)$: occasional smokers.
- $Q(t)$: individuals who have temporarily quit smoking.
- $L(t)$: individuals who have permanently quit smoking

The model is formulated using the derivative ${}^{C}D_{0,t}^{\bar{\beta}}$, where $0<\bar{\beta}<1$, to incorporate the systems memory and hereditary properties. The parameters involved in the equations of the model 3.1 are provided in the table below:

The initial conditions 3.2 define the population sizes at the start of the simulation or analysis. In the system (3.1), When $\bar{\beta}=1$, All admissible states of the system lie within the following bounded region:

$$\Omega=\left\{(P,O,S,Q,L)\in\mathbb{R}_{+}^{5}: P+O+S+Q+L\leqslant\frac{\Lambda}{\theta}\right\}. \tag{3.3}$$

Table 3: Model parameters and their descriptions.

Parameter	Description
Λ	Recruitment rate into the potential smoker compartment (P).
ω	Transmission rate between susceptible individuals (S) and potential smokers (P).
θ	Natural death rate across all compartments.
α_1	Rate at which occasional smokers revert to the susceptible group.
α_2	Interaction rate between susceptible individuals and heavy smokers.
ξ	Relapse rate from quitting back into susceptibility.
σ	Fraction of smokers who have permanently quit smoking.

By combining the equations presented in (3.1), and considering the linearity of ${}^{C}D_{0,t}^{\bar{\beta}}$, we obtain:

$$ {}^{C}D_{0,t}^{\bar{\beta}}(N(t)) = \Lambda - \theta N(t). \tag{3.4} $$

When the initial conditions for the model (3.1) are positive, it has a unique positive solution. We also address in the following that system (3.1) has two equilibrium, the Smoking-Free Equilibrium (SFE) and the Smoking-Present Equilibrium (SPE).

Proposition 3.1. *For the giving up model (3.1), a steady-state equilibrium point (SEP) can be found, denoted by* E_0, *with* $E_0 = (\frac{\Lambda}{\theta}, 0, 0, 0, 0)$.

Proof. Employing the generation matrix method detailed in [43], we aim to examine the existence of the point (SFE). Let. Let $\mathcal{Y} = (O, S, Q, L, P)^T$, where the model can be expressed as:

$$ {}^{C}D_{0,t}^{\bar{\beta}}(\mathcal{Y}) = \mathcal{F}(\mathcal{Y}) + \mathcal{H}(\mathcal{Y}), \tag{3.5} $$

with

$$ \mathcal{F}(\mathcal{Y}) = \begin{pmatrix} \omega P(t)S(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{H}(\mathcal{Y}) = \begin{pmatrix} -(\alpha_1+\theta)O \\ \alpha_1 O + \alpha_2 SQ - (\theta+\xi)S \\ -\alpha_2 SQ - \theta Q + \xi(1-\sigma)S \\ \sigma\xi S - \theta L \\ \Lambda - \omega PS - \theta P \end{pmatrix}. \tag{3.6} $$

Substituting $S = 0$ into (3.1) completes the proof. □

Proposition 3.2. *The parameter* $\mathcal{R}_0$, *representing the reproduction number, is expressed as follows:*

$$ \mathcal{R}_0 = \frac{\alpha_1 \omega P_0}{(\theta+\xi)(\theta+\alpha_1)}, \tag{3.7} $$

Proof. Let $D\mathcal{H}(E_0)$ and $D\mathcal{F}(E_0)$ are the Jacobean matrix of $\mathcal{H}(\mathcal{Y})$ and $\mathcal{F}(\mathcal{Y})$ at the E_0 respectively, where:

$$ D\mathcal{F}(E_0) = \left(\begin{array}{c|ccc|c} 0 & 0 & \omega P_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \ D\mathcal{H}(E_0) = \left(\begin{array}{c|ccc|c} 0 & -(\alpha_1+\theta) & 0 & 0 & 0 \\ 0 & \alpha_1 & -(\theta+\xi) & 0 & 0 \\ 0 & 0 & \xi(1-\sigma) & -\theta & 0 \\ \hline 0 & 0 & \xi\sigma & 0 & -\theta \\ -\theta & 0 & -\omega P_0 & 0 & 0 \end{array}\right), \tag{3.8} $$

A simple calculation gives: 3.8(see [43]). So,

$$ V = \ D\bar{\mathcal{V}}(E_0) = \begin{pmatrix} -(\alpha_1+\theta) & 0 & 0 \\ \alpha_1 & -(\theta+\xi) & 0 \\ 0 & \xi(1-\sigma) & -\theta \end{pmatrix}, \ F = D\mathcal{F}(E_0) = \begin{pmatrix} 0 & \omega P_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, $$

$$V^{-1} = \begin{pmatrix} -\dfrac{1}{\theta+\alpha_1} & 0 & 0 \\ -\dfrac{\alpha_1}{(\theta+\xi)(\theta+\alpha_1)} & -\dfrac{1}{(\theta+\xi)} & 0 \\ -\dfrac{\alpha_1\xi(1-\sigma)}{\theta(\theta+\xi)(\theta+\alpha_1)} & -\dfrac{\xi(1-\sigma)}{\theta(\theta+\xi)} & -\dfrac{1}{\theta} \end{pmatrix},$$

and

$$-FV^{-1} = \begin{pmatrix} \frac{\alpha_1\omega P_0}{(\theta+\xi)(\theta+\alpha_1)} & \frac{\omega P_0}{\theta+\xi} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then:

$$\mathcal{R}_0 = \rho(-FV^{-1}) = \frac{\alpha_1\omega P_0}{(\theta+\xi)(\theta+\alpha_1)}.$$

□

Proposition 3.3. *For the model(3.1) there exists E^* the SPE, with:*

$$E^* = \left(\frac{\Lambda}{\omega S^*+\theta}, \frac{\omega\Lambda S^*}{(\omega S^*+\theta)(\alpha_1+\theta)}, S^*, \frac{\xi(1-\sigma)S^*}{\alpha_2 S^*+\theta}, \frac{\sigma\xi S^*}{\theta}\right). \tag{3.9}$$

Proof. Assuming $S^* > 0$, we investigate the conditions under which system (3.1) admits a positive equilibrium E^*, and

$${}^C D^{\bar{\beta}}_{0,t}(P^*) = {}^C D^{\bar{\beta}}_{0,t}(O^*) = {}^C D^{\bar{\beta}}_{0,t}(S^*) = {}^C D^{\bar{\beta}}_{0,t}(Q^*) = {}^C D^{\bar{\beta}}_{0,t}(L^*) = 0.$$

This gives

$$\Lambda - \omega P^* S^* - \theta P^* = 0 \tag{3.10}$$

$$\omega P^* S^* - \alpha_1 O^* - \theta O^* = 0 \tag{3.11}$$

$$\alpha_1 O^* + \alpha_2 S^* Q^* - (\theta+\xi) S^* = 0 \tag{3.12}$$

$$-\alpha_2 S^* Q^* - \theta Q^* + \xi(1-\sigma) S^* = 0 \tag{3.13}$$

$$\sigma\xi S^* - \theta L^* = 0 \tag{3.14}$$

From Equations (3.10), (3.12), (3.13), (3.14), we obtain:

$$P^* = \frac{\Lambda}{\omega S^*+\theta}, O^* = \frac{\omega\Lambda S^*}{(\alpha_1+\theta)(\omega^*+\theta)}, Q^* = \frac{\xi(1-\sigma)S^*}{\alpha_2 S^*+\theta}, L^* = \frac{\xi\sigma S^*}{\theta}.$$

Finally, substituting O^* and Q^* in Equation (3.11) gives:

$$S^*\left[\frac{\alpha_1\omega\Lambda}{(\alpha_1+\theta)(\omega S^*+\theta)} + \frac{\alpha_2\xi(1-\sigma)S^*}{\alpha_2 S^*+\theta} - (\theta+\xi)\right] = 0.$$

Since $S^* \neq 0$, we obtain $AS^{*2} + BS^* + C = 0$, with:

$$A = \omega\alpha_2(\alpha_1+\theta)(\sigma\xi+\theta) > 0,$$

$$C = \theta^2(\theta+\xi)(\alpha_1+\theta) - \alpha_1\omega\lambda\theta = \frac{\theta}{(\theta+\xi)(\alpha_1+\theta)}(1-\mathcal{R}_0),$$

$$B = -\theta\alpha_2\xi(1-\sigma)(\alpha_1+\theta) + \theta\omega(\alpha_1+\theta)(\theta+\xi) + \frac{\alpha_2}{\theta(\alpha_1+\theta)(\theta+\xi)}(1-\mathcal{R}_0).$$

□

Theorem 3.4. 1. *If* $\mathcal{R}_0 = 1$, *and* $\frac{\theta}{\xi} > \alpha_2 - 1$, *there is no current positive* E^*,
2. *if* $\mathcal{R}_0 < 1$, *and* $B > 0$, *there is no current positive equilibrium point* E^*,
3. *if* $\mathcal{R}_0 > 1$, *there exists one current positive* E^*.

Proof. Using the Equation $AS^{*2} + BS^* + C = 0$, this yields:

1. For $\mathcal{R}_0 = 1$, we have $C = 0$, which gives: $S_1^* = 0$ and $S_2^* = -\frac{B}{A}$. Thus, no positive solution exists when $B > 0$ (given that A is always positive).
2. If $\mathcal{R}_0 < 1$ and $B > 0$, then $A, C > 0$. There is no positive solution to the equation.
3. If $\mathcal{R}_0 > 1$, then there $A, C > 0$ (There is one change in the sign of the terms, so there is a positive solution) and $\Delta = B^2 - 4AC > 0$. which gives $S_1 = \frac{-B - \sqrt{\Delta}}{2A}$ and $S_2 = \frac{-B + \sqrt{\Delta}}{2A}$. Note that: $\sqrt{B^2 - 4AC} > -B$. So: $S_2 = \frac{-B + \sqrt{\Delta}}{2A}$ it is a positive solution to the equation.

□

The stability of the point E_0 is studied in the following theorem, by using the result proven in [36, 32].

Theorem 3.5. *If* $\mathcal{R}_0 < 1$, *the SFE point* E_0 *is locally asymptotically stable, and unstable if* $\mathcal{R}_0 > 1$.

Proof. By evaluating the Jacobian matrix of (3.1) at equilibrium point $E_0(\frac{\lambda}{\theta}, 0, 0, 0, 0)$, and after modifications to it, we obtain:

$$J(E_0) = \begin{pmatrix} -\theta & 0 & -\omega P_0 & 0 & 0 \\ 0 & -\alpha_1 - \theta & \omega P_0 & 0 & 0 \\ 0 & 0 & \omega P_0 - \frac{(\theta+\xi)(\alpha_1+\theta)}{\alpha_1} & 0 & 0 \\ 0 & 0 & \xi(1-\sigma) & -\theta & 0 \\ 0 & 0 & \xi\sigma & 0 & -\theta \end{pmatrix},$$

Therefore, the eigenvalues of $J(E_0)$ are expressed as follows: $\lambda_1 = -\theta$, $\lambda_2 = -\alpha_1 - \theta$, $\lambda_3 = \frac{1}{\alpha_1}(\theta + \xi)(\alpha_1 + \theta)(\mathcal{R}_0 - 1)$. If $\mathcal{R}_0 < 1$, we have $\lambda_i < 0$ for $i = 1, 2, 3$ and $|arg(\lambda_i)| > \bar{\beta}\frac{\pi}{2}$, $i = 1, 2, 3$ is satisfied; then E_0 is locally asymptotically stable (see [36, 32]).

When $\mathcal{R}_0 = 1$, we get $\lambda_3 = 0$, then E_0 is locally stable. When $\mathcal{R}_0 > 1$, we have $\lambda_3 > 0$ and $|arg(\lambda_3)| \leqslant \bar{\beta}\frac{\pi}{2}$. So E_0 is unstable . □

The local stability of the equilibrium point E^*, can be investigated by evaluating the Jacobian matrix J of system (3.1) at E^*, resulting in:

$$J(E^*) = \begin{pmatrix} -\omega S^* - \theta & 0 & -\omega P^* & 0 & 0 \\ \omega S^* & -\alpha_1 - \theta & \omega P^* & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 Q^* - (\theta + \xi) & \alpha_2 S^* & 0 \\ 0 & 0 & -\alpha_2 Q^* + \xi(1-\sigma) & -\alpha_2 S^* - \theta & 0 \\ 0 & 0 & \xi\sigma & 0 & -\theta \end{pmatrix}$$

We note that $\lambda_1 = -\theta$ is eigenvalue then the local stability of E^* of depends of sign of real of eigenvalue of the following matrix:

$$J_1(E^*) = \begin{pmatrix} -\omega S^* - \theta & 0 & -\omega P^* & 0 \\ \omega S^* & -\alpha_1 - \theta & \omega P^* & 0 \\ 0 & \alpha_1 & \alpha_2 Q^* - (\theta + \xi) & \alpha_2 S^* \\ 0 & 0 & -\alpha_2 Q^* + \xi(1-\sigma) & -\alpha_2 S^* - \theta \end{pmatrix},$$

The Jacobian matrix evaluated at equilibrium yields the characteristic polynomial can be written as:

$$P(\lambda) = (\Lambda + (\omega S^* + \theta))(\lambda^3 + A_1\Lambda^2 + A_2\Lambda + A_3).$$

Where

$$\begin{aligned}
A_1 &= \frac{(\omega S^* + \theta)(\alpha_1\theta)}{\omega S^*} - \alpha_2 Q^* + (\theta + \xi) + \alpha_2 S^* + \theta,\\
A_2 &= \frac{(\omega S^* + \theta)(\alpha_1 + \theta)}{\omega S^*}(\alpha_2 Q^* - (\theta + \xi) - \alpha_2 S^* - \theta) + (\alpha_2 Q^* - (\theta + \xi))(-\alpha_2 S^* - \theta) - \frac{\theta\alpha_1\omega P^*}{\omega S^*}\\
&\quad + \alpha_2 S^*(\alpha_2 Q^* - \xi(1-\sigma)),\\
A_3 &= \frac{(\omega S^* + \theta)(+\alpha_1 + \theta)(-\alpha_2 Q^* + (\theta + \xi))(\alpha_2 S^* + \theta) - \theta\alpha_1\omega P^*(\alpha_2 + \theta)}{\omega S^*}\\
&\quad + \frac{\alpha_2 S^*(+\alpha_2 Q^* - \xi(1-\sigma))(\omega S^* + \theta)(+\alpha_1 + \theta)}{\omega S^*}.
\end{aligned}$$

The eigenvalues take the form:

$$\lambda_1 = -(\omega S^* + \theta),$$

and by the roots of the equation:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0.$$

Hence, all eigenvalues of the characteristic equation have negative real parts precisely when:

$$A_1 > 0, \quad A_2 > 0, \quad A_3 > 0, \quad \text{and} \quad A_1A_2 - A_3 > 0.$$

Therefore, we obtain the following result:

Theorem 3.6. *The point E^* of of system (3.1) is considered locally asymptotically stable when the conditions $A_1, A_2, A_3 > 0$ and $A_1A_2 - A_3 > 0$ are satisfied.*

4. $\mathcal{T}_{\mathcal{J}}$–HPM Numerical Scheme for the Fractional Smoking Epidemic Model

In this section, we propose a hybrid semi-analytical method, denoted as $\mathcal{T}_{\mathcal{J}}$–HPM, which combines a general integral transform with HPM Method. This approach is designed to address the challenges involved in solving non-linear FDEs. To demonstrate its effectiveness, the method is applied to the fractional smoking epidemic model in system (3.1).

$$\left({}^{C}D^{\bar{\beta}}_{0,t}Y\right)(t) = \mathcal{R}(t, Y(t)) = \mathcal{G}_i(t, P, O, S, Q, L), i = 1, ..., 5. \tag{4.1}$$

$$Y(0) = Y_0, \tag{4.2}$$

where $Y(t) = (P, O, S, Q, L) = (b_1, b_2, b_3, b_4, b_5)$. Applying $I^{\bar{\beta}}_{a^+}$ to equation 4.1 and using the CF derivative identity yields:

$$Y(t) - Y(0) = I^{\bar{\beta}}_{a^+}(\mathcal{R}(t, Y(t))). \tag{4.3}$$

Applying the $\mathcal{T}_{\mathcal{J}}$–transform to equation (4.3) gives:

$$\mathcal{T}_{\mathcal{J}}[Y(t)] = \mathcal{T}_{\mathcal{J}}[Y(0)] + \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}[\mathcal{R}(t, Y(t))], \tag{4.4}$$

Using the initial conditions yields:

$$\mathcal{T}_{\mathcal{J}}\left[(P(t))\right]=\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_1+\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\Lambda-\omega P(t)S(t)-\theta P(t)\right], \tag{4.5}$$

$$\mathcal{T}_{\mathcal{J}}\left[(O(t))\right]=\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_2+\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\omega P(t)S(t)-\alpha_1 O(t)-\theta O(t)\right], \tag{4.6}$$

$$\mathcal{T}_{\mathcal{J}}\left[(S(t))\right]=\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_3+\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\alpha_1 O(t)+\alpha_2 S(t)Q(t)-(\theta+\xi)S(t)\right], \tag{4.7}$$

$$\mathcal{T}_{\mathcal{J}}\left[(Q(t))\right]=\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_4+\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[-\alpha_2 S(t)Q(t)-\theta Q(t)+\xi(1-\sigma)S(t)\right], \tag{4.8}$$

$$\mathcal{T}_{\mathcal{J}}\left[(L(t))\right]=\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_5+\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\sigma\xi S(t)-\theta L(t)\right]. \tag{4.9}$$

According to the HPM approach, the original non-linear system is reformulated as a homotopy that facilitates the construction of an iterative sequence converging to the solution.

$$\begin{cases}
H(v^1;p) = (1-p)\left[\mathcal{T}_{\mathcal{J}}(v^1)-\mathcal{T}_{\mathcal{J}}(P_0)\right]+p\left[\mathcal{T}_{\mathcal{J}}(v^1)-\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_1-\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\Lambda-\omega v_1v_3-\theta v_1\right]\right]=0,\\
H(v^2;p) = (1-p)\left[\mathcal{T}_{\mathcal{J}}(v^2)-\mathcal{T}_{\mathcal{J}}(O_0)\right]+\\
\qquad p\left[\mathcal{T}_{\mathcal{J}}(v^2)-\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_2-\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\omega v_1v_3-(\alpha_1+\theta)v_2\right]\right]=0,\\
H(v^3;p) = (1-p)\left[\mathcal{T}_{\mathcal{J}}(v^3)-\mathcal{T}_{\mathcal{J}}(S_0)\right]\\
\qquad +p\left[\mathcal{T}_{\mathcal{J}}(v^3)-\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_3-\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\alpha_1v_2+\alpha_2v_3v_4-(\theta+\xi)v_3\right]\right]=0,\\
H(v^4;p) = (1-p)\left[\mathcal{T}_{\mathcal{J}}(v^4)-\mathcal{T}_{\mathcal{J}}(Q_0)\right]+\\
\qquad p\left[\mathcal{T}_{\mathcal{J}}(v^4)-\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_4-\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[-\alpha_2v_3v_4-\theta v_3+\xi(1-\sigma)v_3\right]\right]=0,\\
H(v^5;p) = (1-p)\left[\mathcal{T}_{\mathcal{J}}(v^5)-\mathcal{T}_{\mathcal{J}}(L_0)\right]+p\left[\mathcal{T}_{\mathcal{J}}(v^5)-\frac{\mathcal{K}(s)}{\mathcal{M}(s)}b_1-\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\sigma\xi v_3-\theta v_5\right]\right]=0.
\end{cases}$$

$$\begin{cases}
\mathcal{T}_{\mathcal{J}}(v^1)=b_1\frac{\mathcal{K}(s)}{\mathcal{M}(s)}+p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\Lambda-\omega v_1v_3-\theta v_1\right],\\
\mathcal{T}_{\mathcal{J}}(v^2)=b_2\frac{\mathcal{K}(s)}{\mathcal{M}(s)}+p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\omega v_1v_3-(\alpha_1+\theta)v_2\right],\\
\mathcal{T}_{\mathcal{J}}(v^3)=b_3\frac{\mathcal{K}(s)}{\mathcal{M}(s)}+p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\alpha_1v_2+\alpha_2v_3v_4-(\theta+\xi)v_3\right],\\
\mathcal{T}_{\mathcal{J}}(v^4)=b_4\frac{\mathcal{K}(s)}{\mathcal{M}(s)}+p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[-\alpha_2v_3v_4-\theta v_4+\xi(1-\sigma)v_3\right],\\
\mathcal{T}_{\mathcal{J}}(v^5)=b_5\frac{\mathcal{K}(s)}{\mathcal{M}(s)}+p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\sigma\xi v_3-\theta v_5\right],
\end{cases} \tag{4.10}$$

assuming the solution for the compartments P(t) , O(t), S(t), Q(t) and L(t) in an infinite series is given below :

$$v_1=\sum_{i=0}^{\infty}p^iv_i^1,\; v_2=\sum_{i=0}^{\infty}p^iv_i^2,\; v_3=\sum_{i=0}^{\infty}p^iv_i^3,\; v_4=\sum_{i=0}^{\infty}p^iv_i^4,\; v_5=\sum_{i=0}^{\infty}p^iv_i^5, \tag{4.11}$$

and,

$$v^1v^3=\sum_{m=0}^{\infty}p^mX_m(t),\; v^3v^4=\sum_{m=0}^{\infty}p^mZ_m(t), \tag{4.12}$$

with :

$$X_m(t) = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[\sum_{n=0}^{m}\lambda^n P_n(t)\sum_{n=0}^{m}\lambda^n S_n(t)\right]_{\lambda=0},\ Z_m(t) = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[\sum_{n=0}^{m}\lambda^n S_n(t)\sum_{n=0}^{m}\lambda^n Q_n(t)\right]_{\lambda=0},$$

we obtain:

$$\begin{cases} \mathcal{T}_{\mathcal{J}}(\sum_{i=0}^{\infty} p^i v_i^1) = b_1\frac{\mathcal{K}(s)}{\mathcal{M}(s)} + p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\Lambda - \omega\sum_{m=0}^{\infty} p^m X_m(t) - \theta\sum_{i=0}^{\infty} p^i v_i^1\right], \\ \mathcal{T}_{\mathcal{J}}(\sum_{n=0}^{\infty} p^i v_i^2) = b_2\frac{\mathcal{K}(s)}{\mathcal{M}(s)} + p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\omega\sum_{m=0}^{\infty} p^m X_m(t) - (\alpha_1+\theta)(\sum_{n=0}^{\infty} p^i v_i^2)\right], \\ \mathcal{T}_{\mathcal{J}}(\sum_{n=0}^{\infty} p^i v_i^3) = b_3\frac{\mathcal{K}(s)}{\mathcal{M}(s)} + p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\alpha_1\sum_{n=0}^{\infty} p^i v_i^2 + \alpha_2\sum_{m=0}^{\infty} p^m Z_m(t) - (\theta+\xi)\sum_{n=0}^{\infty} p^i v_i^3\right], \\ \mathcal{T}_{\mathcal{J}}(\sum_{n=0}^{\infty} p^i v_i^4) = b_4\frac{\mathcal{K}(s)}{\mathcal{M}(s)} + p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[-\alpha_2\sum_{m=0}^{\infty} p^m Z_m(t) - \theta\sum_{n=0}^{\infty} p^i v_i^4 + \xi(1-\sigma)\sum_{n=0}^{\infty} p^i v_i^3\right], \\ \mathcal{T}_{\mathcal{J}}(\sum_{n=0}^{\infty} p^i v_i^5) = b_5\frac{\mathcal{K}(s)}{\mathcal{M}(s)} + p\frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\mathcal{T}_{\mathcal{J}}\left[\sigma\xi\sum_{n=0}^{\infty} p^i v_i^3 - \theta\sum_{n=0}^{\infty} p^i v_i^5\right]. \end{cases} \tag{4.13}$$

By identifying terms with the same powers of p we obtain:

$$p^0 : \begin{cases} \mathcal{T}_{\mathcal{J}}\left[v_0^1(t)\right] = \frac{\mathcal{K}(s)}{\mathcal{M}(s)}\, b_1, \\ \mathcal{T}_{\mathcal{J}}\left[v_0^2(t)\right] = \frac{\mathcal{K}(s)}{\mathcal{M}(s)}\, b_2, \\ \mathcal{T}_{\mathcal{J}}\left[v_0^3(t)\right] = \frac{\mathcal{K}(s)}{\mathcal{M}(s)}\, b_3, \\ \mathcal{T}_{\mathcal{J}}\left[v_0^4(t)\right] = \frac{\mathcal{K}(s)}{\mathcal{M}(s)}\, b_4, \\ \mathcal{T}_{\mathcal{J}}\left[v_0^5(t)\right] = \frac{\mathcal{K}(s)}{\mathcal{M}(s)}\, b_5. \end{cases}$$

$$p^1 : \begin{cases} \mathcal{T}_{\mathcal{J}}\left[v_1^1(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\Lambda - \omega X_0(t) - \theta v_0^1(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_1^2(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\omega X_0(t) - \alpha_1 v_0^2(t) - \theta v_0^2(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_1^3(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\alpha_1 v_0^2(t) + \alpha_2 Z_0(t) - (\theta+\xi)v_0^3(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_1^4(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\alpha_2 Z_0(t) - \theta v_0^4(t) + \xi(1-\sigma)v_0^3(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_1^5(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\sigma\xi\, v_0^3(t) - \theta v_0^5(t)\right]. \end{cases}$$

$$p^2 : \begin{cases} \mathcal{T}_{\mathcal{J}}\left[v_2^1(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\omega X_1(t) - \theta v_1^1(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_2^2(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\omega X_1(t) - \alpha_1 v_1^2(t) - \theta v_1^2(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_2^3(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\alpha_1 v_1^2(t) + \alpha_2 Z_1(t) - (\theta+\xi)v_1^3(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_2^4(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\alpha_2 Z_1(t) - \theta v_1^4(t) + \xi(1-\sigma)v_1^3(t)\right], \\ \mathcal{T}_{\mathcal{J}}\left[v_2^5(t)\right] = \frac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\sigma\xi\, v_1^3(t) - \theta v_1^5(t)\right]. \end{cases}$$

$$p^3:\begin{cases}\mathcal{T}_{\mathcal{J}}\left[v_3^1(t)\right]=\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\omega X_2(t)-\theta v_2^1(t)\right],\\ \mathcal{T}_{\mathcal{J}}\left[v_3^2(t)\right]=\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\omega X_2(t)-\alpha_1 v_2^2(t)-\theta v_2^2(t)\right],\\ \mathcal{T}_{\mathcal{J}}\left[v_3^3(t)\right]=\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\alpha_1 v_2^2(t)+\alpha_2 Z_2(t)-(\theta+\xi)v_2^3(t)\right],\\ \mathcal{T}_{\mathcal{J}}\left[v_3^4(t)\right]=\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\alpha_2 Z_2(t)-\theta v_2^4(t)+\xi(1-\sigma)v_2^3(t)\right],\\ \mathcal{T}_{\mathcal{J}}\left[v_3^5(t)\right]=\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\sigma\xi\, v_2^3(t)-\theta v_2^5(t)\right].\end{cases}$$

Moreover , using the inverse $\mathcal{T}_{\mathcal{J}}$, we have :

$$p^0: v_0^1(t)=b_1, v_0^2(t)=b_2, v_0^3(t)=b_3, v_0^4(t)=b_4, v_0^5(t)=b_5.$$

$$p^1:\begin{cases}v_1^1(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\Lambda-\omega X_0(t)-\theta v_0^1(t)\right]\right],\\ v_1^2(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\omega X_0(t)-\alpha_1 v_0^2(t)-\theta v_0^2(t)\right]\right],\\ v_1^3(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\alpha_1 v_0^2(t)+\alpha_2 Z_0(t)-(\theta+\xi)v_0^3(t)\right]\right],\\ v_1^4(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\alpha_2 Z_0(t)-\theta v_0^4(t)+\xi(1-\sigma)v_0^3(t)\right]\right],\\ v_1^5(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\sigma\xi\, v_0^3(t)-\theta v_0^5(t)\right]\right].\end{cases}$$

$$p^2:\begin{cases}v_2^1(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\omega X_1(t)-\theta v_1^1(t)\right]\right],\\ v_2^2(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\omega X_1(t)-\alpha_1 v_1^2(t)-\theta v_1^2(t)\right]\right],\\ v_2^3(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\alpha_1 v_1^2(t)+\alpha_2 Z_1(t)-(\theta+\xi)v_1^3(t)\right]\right],\\ v_2^4(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\alpha_2 Z_1(t)-\theta v_1^4(t)+\xi(1-\sigma)v_1^3(t)\right]\right],\\ v_2^5(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\sigma\xi\, v_1^3(t)-\theta v_1^5(t)\right]\right].\end{cases}$$

$$p^3:\begin{cases}v_3^1(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\omega X_2(t)-\theta v_2^1(t)\right]\right],\\ v_3^2(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\omega X_2(t)-\alpha_1 v_2^2(t)-\theta v_2^2(t)\right]\right],\\ v_3^3(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\alpha_1 v_2^2(t)+\alpha_2 Z_2(t)-(\theta+\xi)v_2^3(t)\right]\right],\\ v_3^4(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[-\alpha_2 Z_2(t)-\theta v_2^4(t)+\xi(1-\sigma)v_2^3(t)\right]\right],\\ v_3^5(t)=\mathcal{T}_{\mathcal{J}}^{-1}\left[\dfrac{1}{\mathcal{M}^{\bar{\beta}}(s)}\,\mathcal{T}_{\mathcal{J}}\left[\sigma\xi\, v_2^3(t)-\theta v_2^5(t)\right]\right].\end{cases}$$

Let : $X_0 = b_1 b_3$, $Z_0 = b_3 b_4$, so:

$$p^1 : \begin{cases} v_1^1(t) = \frac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)} \left[\Lambda - \omega b_1 b_3 - \theta b_1\right], \\ v_1^2(t) = \frac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)} \left[\omega b_1 b_3 - \theta b_2 - \alpha_1 b_2\right], \\ v_1^3(t) = \frac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)} \left[\alpha_1 b_2 + \alpha_2 b_3 b_4 - (\theta+\xi) b_3\right], \\ v_1^4(t) = \frac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)} \left[-\alpha_2 b_3 b_4 - \theta b_4 + \xi(1-\mu) b_3\right], \\ v_1^5(t) = \frac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)} \left[\mu\xi b_3 - \theta b_5\right]. \end{cases}$$

we pose:

$$\begin{aligned} w_P &= \Lambda - \omega b_1 b_3 - \theta b_1, \\ w_O &= \omega b_1 b_3 - \theta b_2 - \alpha_1 b_2, \\ w_S &= \alpha_1 b_2 + \alpha_2 b_3 b_4 - (\theta+\xi) b_3, \\ w_Q &= -\alpha_2 b_3 b_4 - \theta b_4 + \xi(1-\mu) b_3, \\ w_L &= \mu\xi b_3 - \theta b_5. \end{aligned}$$

Then:

$$X_1 = v_1^1 v_0^3 + v_1^3 v_0^1 = (b_3 w_P + b_1 w_S) \frac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)},$$

$$Z_1 = v_1^4 v_0^3 + v_0^4 v_1^3 = \left(b_3 w_Q + b_4 w_S\right) \frac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)},$$

and

$$p^2 : \begin{cases} v_2^1(t) = \frac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)} \left[-\omega\, w_S b_1 - (\omega b_3 + \theta)\, w_P\right], \\ v_2^2(t) = \frac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)} \left[\omega\, w_S b_1 + \omega\, w_P b_3 - (\theta+\alpha_1) w_O\right], \\ v_2^3(t) = \frac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)} \left[\alpha_1 w_O + \alpha_2 w_Q b_3 + \alpha_2 w_S b_4 - (\theta+\xi) w_S\right], \\ v_2^4(t) = \frac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)} \left[-\alpha_2\left(w_Q b_3 + b_4 w_S\right) - \theta w_Q + \xi(1-\mu) w_S\right], \\ v_2^5(t) = \frac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)} \left[\mu\xi\, w_S - \theta w_L\right]. \end{cases}$$

$w_{PP} = -\omega\, w_S b_1 - (\omega b_3 + \theta)\, w_P$, $\quad w_{OO} = \omega\, w_S b_1 + \omega\, w_P b_3 - (\theta+\alpha_1) w_O$,

$w_{SS} = \alpha_1 w_O + \alpha_2 w_Q b_3 + \alpha_2 w_S b_4 - (\theta+\xi) w_S$, $\quad w_{QQ} = -\alpha_2 (w_Q b_3 + b_4 w_S) - \theta w_Q + \xi(1-\sigma) w_S$,

$w_{LL} = \mu\xi\, w_S - \theta w_L$.

$$X_2(t) = v_2^1 v_0^3 + v_2^3 v_0^1 + v_1^1 v_1^3, \quad Z_2(t) = v_2^4 v_0^3 + v_2^3 v_0^4 + v_1^3 v_1^4.$$

$$p^3 : \begin{cases} v_3^1(t) = \dfrac{t^{3\bar{\beta}}}{\Gamma(3\bar{\beta}+1)}\left[-\omega(w_{PP}b_3 + w_{SS}b_1) - \theta w_{PP} - \omega\, w_P w_S \dfrac{\Gamma(2\bar{\beta}+1)}{\Gamma(\bar{\beta}+1)^2}\right], \\ v_3^2(t) = \dfrac{t^{3\bar{\beta}}}{\Gamma(3\bar{\beta}+1)}\left[\omega b_3 w_{PP} + \omega b_1 w_{SS} - (\theta+\alpha_1)w_{OO} + \omega\, w_P w_S \dfrac{\Gamma(2\bar{\beta}+1)}{\Gamma(\bar{\beta}+1)^2}\right], \\ v_3^3(t) = \dfrac{t^{3\bar{\beta}}}{\Gamma(3\bar{\beta}+1)}\left[\alpha_1 w_{OO} + \alpha_2 w_{QQ} b_3 + \alpha_2 w_{SS} b_4 - (\theta+\xi)w_{SS} + \alpha_2 w_Q w_S \dfrac{\Gamma(2\bar{\beta}+1)}{\Gamma(\bar{\beta}+1)^2}\right], \\ v_3^4(t) = \dfrac{t^{3\bar{\beta}}}{\Gamma(3\bar{\beta}+1)}\left[-\alpha_2 w_{QQ} b_3 - \alpha_2 w_{SS} b_4 - \theta w_{QQ} + \xi(1-\sigma)w_{SS} - \alpha_2 w_Q w_S \dfrac{\Gamma(2\bar{\beta}+1)}{\Gamma(\bar{\beta}+1)^2}\right], \\ v_3^5(t) = \dfrac{t^{3\bar{\beta}}}{\Gamma(3\bar{\beta}+1)}\left[\sigma\xi w_{SS} - \theta w_{LL}\right]. \end{cases}$$

To solve the above system of equations, we apply the initial conditions:

$$b_1 = 40. \quad b_2 = 10. \quad b_3 = 20. \quad b_4 = 10. \quad b_5 = 5.$$

$$p^1 : \begin{cases} v_1^1(t) = \dfrac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)}\left[\Lambda - 800\omega - 40\theta\right], \\ v_1^2(t) = \dfrac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)}\left[800\omega - 10\theta - 10\alpha_1\right], \\ v_1^3(t) = \dfrac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)}\left[10\alpha_1 + 200\alpha_2 - 20\,(\theta+\xi)\right], \\ v_1^4(t) = \dfrac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)}\left[-200\alpha_2 - 10\theta + 20\xi(1-\mu)\right], \\ v_1^5(t) = \dfrac{t^{\bar{\beta}}}{\Gamma(\bar{\beta}+1)}\left[20\mu\xi - 5\theta\right]. \end{cases}$$

We introduce the following notations:

$$\begin{aligned} w_P &= \Lambda - 800\,\omega - 40\,\theta, \\ w_O &= 800\,\omega - 10\,\theta - 10\,\alpha_1, \\ w_S &= 10\,\alpha_1 + 200\,\alpha_2 - 20\,(\theta+\xi)\,, \\ w_Q &= -200\,\alpha_2 - 10\,\theta + 20\,\xi\,(1-\mu), \\ w_L &= 20\,\mu\,\xi - 5\,\theta. \end{aligned}$$

Thus, for p^2 we have:

$$\begin{cases} v_2^1(t) = \dfrac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)}\left[-40\,\omega\, w_S - (20\,\omega + \theta)\, w_P\right], \\ v_2^2(t) = \dfrac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)}\left[40\,\omega\, w_S + 20\,\omega\, w_P - (\theta+\alpha_1)w_O\right], \\ v_2^3(t) = \dfrac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)}\left[\alpha_1 w_O + 20\,\alpha_2 w_Q + 10\,\alpha_2 w_S - (\theta+\xi)w_S\right], \\ v_2^4(t) = \dfrac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)}\left[-10\,\alpha_2 + \xi(1-\mu)w_S + (-20\,\alpha_2 - \theta)w_Q\right], \\ v_2^5(t) = \dfrac{t^{2\bar{\beta}}}{\Gamma(2\bar{\beta}+1)}\left[\mu\,\xi\, w_S - \theta\, w_L\right]. \end{cases}$$

We define:

$$\begin{aligned}
w_{PP} &= -\omega\, w_S b_1 - (\omega b_3 + \theta)\, w_P,\\
w_{OO} &= \omega\, w_S b_1 + \omega\, w_P b_3 - (\theta + \alpha_1) w_O,\\
w_{SS} &= \alpha_1 w_O + \alpha_2 w_Q b_3 + \alpha_2 w_S b_4 - (\theta + \xi) w_S,\\
w_{QQ} &= -\alpha_2 \left(w_Q b_3 + b_4 w_S \right) - \theta w_Q + \xi(1-\sigma) w_S,\\
w_{LL} &= \mu\, \xi\, w_S - \theta\, w_L.
\end{aligned}$$

So:

$$p^3: \begin{cases}
v_3^1(t) = \left[-\omega\,(20 w_{PP} + 40 w_{SS}) - \theta\, w_{PP} - \dfrac{\omega\, w_P w_S \Gamma(2\bar\beta+1)}{(\Gamma(\bar\beta+1))^2} \right] \dfrac{t^{3\bar\beta}}{\Gamma(3\bar\beta+1)},\\
v_3^2(t) = \left[20\omega\, w_{PP} + 40\omega\, w_{SS} - (\theta+\alpha_1) w_{OO} + \dfrac{\omega\, w_P w_S \Gamma(2\bar\beta+1)}{(\Gamma(\bar\beta+1))^2} \right] \dfrac{t^{3\bar\beta}}{\Gamma(3\bar\beta+1)},\\
v_3^3(t) = \left[\alpha_1 w_{OO} + 20\alpha_2 w_{QQ} + 10\alpha_2 w_{SS} - (\theta+\xi) w_{SS} + \dfrac{\alpha_2 w_Q w_S \Gamma(2\bar\beta+1)}{(\Gamma(\bar\beta+1))^2} \right] \dfrac{t^{3\bar\beta}}{\Gamma(3\bar\beta+1)},\\
v_3^4(t) = \left[-20\alpha_2 w_{QQ} - 10\alpha_2 w_{SS} - \theta\, w_{QQ} + \xi(1-\sigma) w_{SS} - \dfrac{\alpha_2 w_Q w_S \Gamma(2\bar\beta+1)}{(\Gamma(\bar\beta+1))^2} \right] \dfrac{t^{3\bar\beta}}{\Gamma(3\bar\beta+1)},\\
v_3^5(t) = \left[\sigma\xi\, w_{SS} - \theta\, w_{QQ} \right] \dfrac{t^{3\bar\beta}}{\Gamma(3\bar\beta+1)}.
\end{cases}$$

Thus, the solution can be represented as:

$$\begin{aligned}
P(t) = 40 &+ \frac{1}{\Gamma(\bar\beta+1)} [\Lambda - 800\omega - 40\theta]\, t^{\bar\beta}\\
&+ \Lambda \frac{1}{\Gamma(2\bar\beta+1)} [-40\omega\, w_S - (\omega w_S + \theta)\, w_P]\, t^{2\bar\beta}\\
&+ \frac{1}{\Gamma(3\bar\beta+1)} \left[-\omega\,(20 w_{PP} + 40 w_{SS}) - \theta w_{PP} - \frac{\omega\, w_P w_S \Gamma(2\bar\beta+1)}{\left(\Gamma(\bar\beta+1)\right)^2} \right] t^{3\bar\beta} + \dots,
\end{aligned}$$

$$\begin{aligned}
O(t) = 10 &+ \frac{1}{\Gamma(\bar\beta+1)} [800\omega - 10\theta - 10\alpha_1]\, t^{\bar\beta}\\
&+ \frac{1}{\Gamma(2\bar\beta+1)} [40\omega\, w_S + 20\omega\, w_P - (\theta+\alpha_1) w_O]\, t^{2\bar\beta}\\
&+ \frac{1}{\Gamma(3\bar\beta+1)} \left[20\omega w_{PP} + 40\omega w_{SS} - (\theta+\alpha_1) w_{OO} + \frac{\omega\, w_P w_S \Gamma(2\bar\beta+1)}{\left(\Gamma(\bar\beta+1)\right)^2} \right] t^{3\bar\beta} + \dots,
\end{aligned}$$

$$\begin{aligned}
S(t) = 20 &+ \frac{1}{\Gamma(\bar\beta+1)} [10\alpha_1 + 200\alpha_2 - 20(\theta+\xi)]\, t^{\bar\beta}\\
&+ \frac{1}{\Gamma(2\bar\beta+1)} \left[\alpha_1 w_O + 20\alpha_2 w_Q + 10\alpha_2 w_S - (\theta+\xi) w_S \right] t^{2\bar\beta}\\
&+ \frac{1}{\Gamma(3\bar\beta+1)} \left[\alpha_1 w_{OO} + 20\alpha_2 w_{QQ} + 10\alpha_2 w_{SS} - (\theta+\xi) w_{SS} + \frac{\alpha_2 w_Q w_S \Gamma(2\bar\beta+1)}{(\Gamma(\bar\beta+1))^2} \right] t^{3\bar\beta} + \dots,
\end{aligned}$$

Table 4: Physical parameter values for the fractional-order smoking model

Parameter	Λ	ω	θ	α_1	α_2	ξ	σ
Value	1	0.14	0.05	0.002	0.0025	0.8	0.1

$$Q(t) = 10 + \frac{1}{\Gamma(\bar{\beta}+1)}\left[-200\alpha_2 - 10\theta + 20\xi(1-\mu)\right] t^{\bar{\beta}} + \frac{1}{\Gamma(2\bar{\beta}+1)}\left[(-10\alpha_2 + \xi(1-\mu))w_S + (-20\alpha_2 - \theta)w_Q\right] t^{2\bar{\beta}} + \frac{1}{\Gamma(3\bar{\beta}+1)}\left[-20\alpha_2 w_{QQ} - 10\alpha_2 w_{SS} - \theta w_{QQ} + \xi(1-\sigma)w_{SS} - \frac{\alpha_2 w_Q w_S \Gamma(2\bar{\beta}+1)}{(\Gamma(\bar{\beta}+1))^2}\right] t^{3\bar{\beta}} + \dots,$$

$$L(t) = 5 + \frac{1}{\Gamma(\bar{\beta}+1)}\left[20\mu\xi - 5\theta\right] t^{\bar{\beta}} + \frac{1}{\Gamma(2\bar{\beta}+1)}\left[\mu\xi w_S - \theta w_L\right] t^{2\bar{\beta}} + \frac{1}{\Gamma(3\bar{\beta}+1)}\left[\sigma\xi w_{SS} - \theta w_{LL}\right] t^{3\bar{\beta}} + \dots$$

4.1. Numerical Results

In order to evaluate the accuracy and efficiency of the proposed numerical scheme, approximate analytical solutions were analyzed by means of numerical simulations and graphical representations carried out using MATLAB.

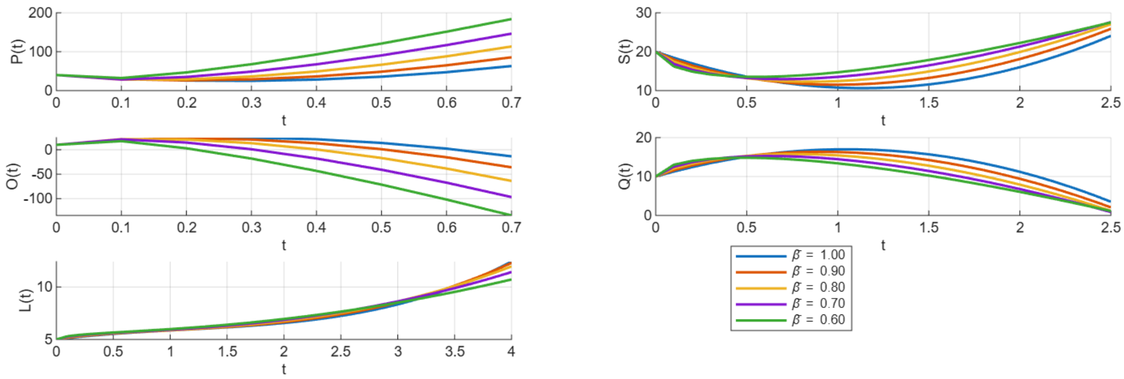

Figure 1: Plots of $P(t)$, $O(t)$, $Q(t)$, $S(t)$, and $L(t)$ for $\bar{\beta} = 1, 0.9, 0.8, 0.7, 0.6$.

Figure 1 illustrates the time evolution of the five state variables: $P(t)$, $S(t)$, $O(t)$, $L(t)$, and $Q(t)$, providing clear insights into how smoking behavior changes across different population groups over time.

The results demonstrate the models ability to effectively capture the dynamics of smoking behavior. They also highlight the influence of individuals' past habits on current states, thus reflecting the memory effect embedded in fractional-order systems. This property allows the model to account for the long-term effects of previous smoking behaviors, enabling a realistic simulation of the gradual transition between smoking phasesfrom initial exposure to temporary and permanent quitting.

The accompanying plots illustrate how the system's dynamic variables respond to varying values of the fractional order $\bar{\beta}$. At lower orders (e.g., $\bar{\beta} = 0.70$), the growth of certain variables is faster and more sensitive, whereas at higher orders (e.g., $\bar{\beta} = 0.9$), the dynamics become more stable and slower, indicating the role of memory depth in influencing system behavior. The plots clearly trace the temporal trajectories of each group: beginning with the susceptible class, progressing through initial smoking stages toward habitual behavior, and eventually moving toward both temporary and permanent cessation.

This smooth flow between compartments highlights the models capability in simulating complex behavioral transitions, and confirms the robustness of the proposed hybrid method in tracking long-term behavioral evolution. The simulation results reinforce the reliability of the model as a predictive tool for analyzing smoking dynamics within heterogeneous populations.

Overall, the numerical results highlight the critical role of the fractional-order parameter in shaping the dynamics of smoking behavior. Lower values of $\bar{\beta}$ lead to faster transitions between states, while higher values produce smoother and more gradual evolutions, reflecting stronger memory effects. This demonstrates that the model can realistically capture long-term behavioral trends and the influence of past habits on current smoking patterns, confirming the effectiveness and reliability of the proposed numerical approach.

5. Conclusion

This work introduces an innovative methodology for the comprehensive analysis of a fractional-order smoking epidemic model using the $\mathcal{T}$J-HPM scheme. The model is formulated as a system of FDEs involving the CF derivative, which is particularly well-suited to capturing memory effects and long-term temporal dependencies characteristic of smoking behavior. A hybrid numerical methodology, $\mathcal{T}$J-HPM, combining a specific integral transform (Jafari transform) with the Homotopy Perturbation Method (HPM), is employed to solve the model. Due to their rapid convergence, yield highly accurate results with low computational effort.

The obtained numerical and graphical results demonstrate the significant impact of the fractional order and model parameters on the systems dynamics and stability. These findings confirm the relevance of fractional-order models for describing biological processes with memory characteristics and pronounced temporal dependence.

In summary, this research highlights the key advantages of fractional-order models in the analysis of behavioral epidemics, particularly when using the CF derivative. It encourages further investigation into other types of fractional operators or the extension of the model to incorporate additional social and psychological factors. The results clearly show that memory effects, as represented by the fractional order, play a significant role in shaping smoking behavioreither by delaying its onset or supporting cessation.

Moreover, the study demonstrates the effectiveness and robustness of the hybrid $\mathcal{T}_{\mathcal{J}}$-HPM approach in solving non linear fractional systems arising in epidemiological modelling. Future research could extend this method to more complex models involving time delays, optimal control strategies, or stochastic influences, thereby broadening the scope of fractional calculus applications in public health.

Acknowledgment

The authors sincerely thank the anonymous reviewers for their careful reading of the manuscript and for their insightful comments and constructive suggestions, which significantly contributed to improving the quality and clarity of this work.

References

[1] Abdullah, M., Ahmad, A., Raza, N., Farman, M., Ahmad, M.O. (2018). Approximate solution and analysis of smoking epidemic model with Caputo fractional derivatives. *Int. J. Appl. Comput. Math.*, 4(112), 1–16. 3

[2] Abdelrahim Mahgoub, M.M. (2017). The new integral transform Mohand transform. Adv. Theor. Appl. Math., 12(2), 113–120. 1

[3] Abdulshareef Hussein, M. (2022). A review on integral transforms of the fractional derivatives of CaputoFabrizio and AtanganaBaleanu. Eurasian J. Media Commun., 7, 17–23. 1

[4] Aboodh, K.S. (2013) The new integral transform. Glob. J. Pure Appl. Math., 9(1), 35–43. 1

[5] Aggarwal, S., Bhatnagar, K. (2019). Dualities between Laplace transform and some useful integral transforms. International Journal of Engineering and Advanced Technology, **9**(1), 936–941. 1

[6] Aggarwal, S. and Gupta, A.R. (2019). Dualities between Mohand transform and some useful integral transforms. International Journal of Recent Technology and Engineering, 8(3), 843-847. 1

[7] Aggarwal, S., Sharma, N., Chauhan, R. (2020). Duality relations of Kamal transform with Laplace, Laplace–Carson, Aboodh, Sumudu, Elzaki, Mohand and Sawi transforms. SN Applied Sciences, **2**, 135. 1

[8] Ahmadi, S.A.P., Hosseinzadeh, H., Cherat, A.Y. (2019). A new integral transform for solving higher order linear ordinary differential equations. Nonlinear Dyn. Syst. Theory, 19(2), 243–252. 1

[9] Ahmadi, S.A.P., Hosseinzadeh, H., Cherat, A.Y. (2019). A new integral transform for solving higher order linear ordinary Laguerre and Hermite differential equations. Int. J. Appl. Comput. Math., 5, 142. 1

[10] Belgacem, R., Baleanu, D., Bokhari, A. (2019). Shehu transform and applications to Caputo-fractional differential equations. Int. J. Anal. Appl., 17, 917–927. 1

[11] Belgacem, R., Bokhari, A., Sadaoui, B. (2021). Shehu transform of HilferPrabhakar fractional derivatives and applications on some Cauchy type problems. Adv. Theory Nonlinear Anal. Appl., 5(2), 203–214. 1

[12] Belgacem, R., Bokhari, A., Baleanu, D., Djilali, S. (2024). New generalized integral transform via Dzherbashian-Nersesian fractional operator. IJOCTA, 14(2), 90–98. 1

[13] Bokhari, A., Baleanu, D., Belgacem, R. (2020). Application of Shehu transform to AtanganaBaleanu derivatives. J. Math. Comput. Sci., 20(2), 101–107. 1

[14] Caputo, M., Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl., 73, 1–13. 2.2

[15] Costa, F.S., Soares, J.C.A., Jarosz, S., Sousa, J. Vanterler da C. (2022). Integral transforms of the Hilfer-type fractional derivatives. Asian Res. J. Math., 18(12), 57–74. 1

[16] Debnath, L., Bhatta, D. Integral Transforms and Their Applications. Chapman & Hall/CRC, Boca Raton, 2007. 1

[17] El-Mesady, A.I., Hamed, Y.S., Alsharif, A.M. (2021). Jafari transformation for solving a system of ordinary differential equations with medical application. Fractal Fract., 5, 130. 1

[18] Elzaki, T.M. (2011). The new integral transform Elzaki transform. Glob. J. Pure Appl. Math., 7, 57–64. 1

[19] Higazy, M., Aggarwal, S. (2021). Sawi transformation for system of ordinary differential equations with application. Ain Shams Engineering Journal, **12**(3), 3173–3182. 1

[20] Jafari, H. (2020) A new general integral transform for solving integral equations. J. Adv. Res. 1, 2.4, 2, 2.5, 2.6

[21] Jafari, H., Aggarwal, S., Kumar, A., et al. (2025). Anuj integral transform to solving Abel's integral equation of classical mechanics. National Academy Science Letters, in press. 1

[22] Jafari, H., Aggarwal, S. (2024). Upadhyaya integral transform: a tool for solving non-linear Volterra integral equations. Mathematics and Computational Sciences, **5**(2), 63–71. 1

[23] Jafari, H., Manjarekar, S. (2022). A modification on the new general integral transform. Advanced Mathematical Models & Applications, **7**(3), 253–263. 1

[24] Kamal, H., Sedeeg, A. (2016). The new integral transform Kamal transform. Adv. Theor. Appl. Math., 11(4), 451–458. 1

[25] Khalid, M., Alha, S. (2023). New generalized integral transform on HilferPrabhakar fractional derivatives and its applications. arXiv preprint, arXiv:2301.06797. 1

[26] Khalouta, A. (2023). A new decomposition transform method for solving nonlinear fractional logistic differential equation. J. Supercomput., 80, 8179–8201. 1

[27] Khan, M., Salahuddin, T., Malik, M.Y., Alqarni, M.S., Alkahtani, A.M. (2020). Numerical modeling and analysis of bioconvection on MHD flow due to an upper paraboloid surface of revolution. Physica A, 553, 124231. 1

[28] Samko, S.G., Kilbas, A.A., Marichev, O.I. (1993). Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon. 2.1

[29] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. (2006). Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam. 1, 2.3

[30] Kumar, A., Bansal, S., Aggarwal, S. (2022). A new novel integral transform 'Anuj transform' with application. Journal of Scientific Research, **14**(2), 521–532. 1

[31] Kumar, R., Chandel, J., Aggarwal, S. (2022). A new integral transform "Rishi transform" with application. Journal of Scientific Research, **14**(2), 521–532. 1

[32] Li, C., Ho, M., Muhammadhaji, A., Zhang, L., Teng, Z. (2018). Stability analysis of a fractional-order predatorprey model incorporating a constant prey refuge and feedback control. Adv. Differ. Equ., 325. 3, 3

[33] Mahgoub, M.A., Mohand, M. (2019) The new integral transform Sawi Transform. Adv. Theor. Appl. Math., 14(1), 81–87. 1

[34] Maitama, S., Zhao, W. (2019). New integral transform: Shehu transforma generalization of Sumudu and Laplace transform for solving differential equations. Int. J. Anal. Appl., 17(2), 167–190. 1

[35] Mansour, E.A., Meftin, N.K. (2021). Mathematical modeling for cryptography using Jafari transformation method. Period. Eng. Nat. Sci., 9(4), 892–897. 1

[36] Matignon, D. (1996). Stability results for fractional differential equations with applications to control processing. Comput. Eng. Syst. Appl., 2, 963. 3, 3

[37] Meddahi, M., Jafari, H., Ncube, M.M. (2021). New general integral transform via AtanganaBaleanu derivatives. Adv. Differ. Equ., 2021, 385. 1

[38] Meddahi, M., Jafari, H., Yang, X. J. (2021). Towards new general double integral transform and its applications to differential equations. Math. Methods Appl. Sci., 1–18. 1

[39] Podlubny, I. Fractional Differential Equations. Academic Press, San Diego, 1999. 1

[40] Rashid, S., Sultana, S., Ashraf, R., Kaabar, M.K.A. (2021). On comparative analysis for the BlackScholes model in the generalized fractional derivatives sense via Jafari transform. J. Funct. Spaces, vol. 2021. 1

[41] Rashid, S., Ashraf, R., Jarad, F. (2022) Strong interaction of Jafari decomposition method with nonlinear fractional-order partial differential equations arising in plasma via the singular and nonsingular kernels. AIMS Math., 7(5), 7936–7963. 1

[42] Rashid, S., Ashraf, R., Bonyah, E. (2022). On analytical solution of time-fractional biological population model by means of generalized integral transform with uniqueness and convergence analysis. J. Funct. Spaces, vol. 2022. 1

[43] Van den Driessche, P., Watmough, J. (2002). Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Biosci., 180, 29–48. 3, 3

[44] Watugala, G.K. (1993). Sumudu transform: A new integral transform to solve differential equations and control engineering problems. Int. J. Math. Educ. Sci. Technol., 24, 35–43. 1