



Fixed points of Geraghty Ćirić type contraction maps via graph structures and its applications in machine learning

Padala Harikrishna^a, Kusuma Tummala^{b,*}, Nadakuduti V.V.S. Suryanarayana^c, Rajamahanthi Santhi Kumar^d, Chitturi S.S.N. Murthy^e

^aDepartment of B S and H, Vignana's Institute of Information Technology(A), Visakhapatnam, India

^bDepartment of Mathematics and Management Sciences, Vallurupalli Nageswara Rao Vignana Jyothi Institute of Engineering & Technology, Hyderabad, India.

^cDepartment of Mathematics, N S Raju Institute of Technology, Visakhapatnam, India

^dDepartment of Mathematics, Aditya Institute of Technology and Management, Tekkali, India.

^eDepartment of Mathematics, Aditya University, Surampalem , Kakinada, 533437, India.

Abstract

In this work, we show that the existence of fixed points of $\beta - \psi$ weak generalized Ćirić type rational contraction mappings for a pair of metrics provided by a dIGraph are identical. We provided examples and machine learning applications to back up our conclusions.

Keywords: Fixed Point, Directed Graph, Geraghty Contraction, Metric Space

2020 MSC: 47H10, 54H25

©2026 All rights reserved.

1. Introduction and Preliminaries

Extending the ambient space and the contractive condition, the Banach Contraction Principle (B C P) is one of the fundamental results of fixed point theory can be expanded and developed in a variety of ways. Jachmski [10] replaced the structure of orders in extended fixed point theory in graph structure. Theories of fixed point finds with mappings converge in the different fields. Numerous scholars [1, 2, 5, 6, 7, 14, 13, 9] investigated fixed point outcomes on a range of graph-endowed spaces. Fixed point results are expanded using Geraghty [8, 3, 4] contractions with specific features. Recently, it was shown that there are fixed point theorems of control functions fractional differential equations with applications[15].

Definition 1.1. [10] The Cartesian product of the diagonal of the TXT is denoted by Δ_δ . Let (T, δ) be a metric space[MSP]. The MSP (T, δ) is a digraph $\bar{\alpha}_G = (\alpha_V, \alpha_E)$ if G is a digraph such that the edges α_E , Δ_δ ignoring parallel edges, and the vertices α_V with all the components of T .

*Corresponding author

Email addresses: phk.2003@gmail.com (Padala Harikrishna^a), kusumatummala9@gmail.com (Kusuma Tummala^b), drnvssuryanarayana.sh@nsrit.edu.in (Nadakuduti V.V.S. Suryanarayana^c), skrmahanthi@gmail.com (Rajamahanthi Santhi Kumar^d), chitturimurthy@gmail.com (Chitturi S.S.N. Murthy^e)

doi: [10.30511/mcs.2026.735275](https://doi.org/10.30511/mcs.2026.735275)

Received: 13 November 2025 Accepted: 26 April 2026

Definition 1.2. [7] Assume that $\rho, \kappa : \perp \rightarrow \perp$ are functions and that (\perp, δ) is an MSP with a graph. Consider the following: $\perp(\rho, \kappa) := \{t \in \perp : (\rho t, \kappa t) \in E(G)\}$; $C(\rho, \kappa) := \{t \in \perp, \rho t = \kappa t\}$ Points of correspondence between h and K are represented by $C(\rho, \kappa)$, while common fixed points of ρ and κ are represented by $C_m(\rho, \kappa) := \{t \in \perp : \rho t = \kappa t = t\}$.

Lemma 1 [10] Let (\perp, δ) be a MSP with a diGraph $\bar{\alpha}_G = (\alpha_V, \alpha_E)$, $\mathfrak{h}, \mathfrak{j} : \perp \rightarrow \perp$ be functions. If $C(\mathfrak{h}, \mathfrak{j}) \neq \phi$, then $C_m(\mathfrak{h}, \mathfrak{j}) \neq \phi$.

Definition 3[7] Assume (T, δ) be a MSP with a digraph $\bar{\alpha}_G = (\alpha_V, \alpha_E)$

(i) for a seq $\{w_n\}$ in M such that $(w_n, w_{n+1}) \in M$, for all in N , we have if $w_n \rightarrow r \in M$, then $T w_n \rightarrow T r$, then a mapping $F : T \rightarrow T$ is G -continuous at r in T . More over, When T is G -continuous at at each point in T , it is termed G -continuous.

(2) $\alpha_E(G)$ is transitive if, for every $\alpha, \beta, \gamma \in T$, $(\alpha, \beta), (\beta, \gamma) \in \alpha_E(G)$, then $(\alpha, \gamma) \in \alpha_E(G)$.

(3) for every sequence $\{r_n\}$ in M $r_n \rightarrow r \in T$ and $(r_n, r_{n+1}) \in \alpha_E(G)$ for each $n \in N$, it is true that $(r_n, y) \in E(G)$ for all $n \in N$, the triple (M, δ, \bar{G}) is said to have the condition A.

Definition 4 [11] Let (ℓ, d) be MSP, and let $\mathfrak{h}, \mathfrak{k} : \ell \rightarrow \ell$ be functions. ℓ is d - compatible whenever $\lim_{n \rightarrow \infty} d(\mathfrak{h}\mathfrak{k}t_n, \mathfrak{k}\mathfrak{h}t_n) = 0$ with $\lim_{n \rightarrow \infty} \mathfrak{h}\mathfrak{k}t_n = \lim_{n \rightarrow \infty} \mathfrak{k}\mathfrak{h}t_n$, for all $\{t_n\}$ in ℓ

Definition 5 [7] Let (M, δ) and (M', δ') be MSPs, and let $h : M \rightarrow M'$ and $K : M' \rightarrow M$ be functions. Then h is said to be S - Cauchy on M , for any sequence $\{t_n\}$ in M with $\{Kt_n\}$ Cauchy in (M, d) , then sequence $\{ht_n\}$ also Cauchy in (K, δ') .

Consider $A(\tau')$, MSP with δ' . Let $h' : \tau' X \tau' \rightarrow [0, 1]$, if $\lim_{n \rightarrow \infty} h'(u_n, v_n) = 1$ then $\lim_{n \rightarrow \infty} \delta'(u_n, v_n) = 0$ where the sequences $\{u_n\}, \{v_n\}$ in τ .

Ben Wongasaiji [15], Phakdi Charoensawan, Teeranush Suebcharoen, and Watchareepan Atiponrat developed and demonstrated common fixed point theorems in 2021 that have two metrics and a diGraph.

Definition 6 [15] Let (M'', δ) be a MSP with a diGraph $\bar{\alpha}_G = (\alpha_V, \alpha_E)$, and let $h, K : M' \rightarrow M''$ be functions. The pair h, K is an $h - \varphi$ -contraction w.r.to δ if the conditions hold:

(1) With regard to G , h is K -edge preserving;

(2) there are two functions, h in $A(X)$, and φ in Φ , such that for any s, t in M'' with (hs, Kt) in $E(G)$, $\varphi(\delta(Ks, Kt) \leq h(hs, Kt)\varphi(R(Ks, Kt))$,

where $R : XXX \rightarrow [0, \infty)$ is a function such that, for any $x, y \in X$,

$$R(hs, Kt) = \max\left\{\frac{\delta(hs, Ks)\delta(Kt, ht)}{\delta(hs, ht)}, \delta(Ss, St), \delta(Ss, Ts), \delta(St, Tt), \frac{\delta(hs, Kt) + \delta(Kt, hs)}{2}\right\}.$$

Theorem 1.3. [15] Let (M'', δ') be a CMSP with a diGraph $\bar{\alpha}_G = (\alpha_V, \alpha_E)$, let δ'' be another metric on X , and let $\mathfrak{h}', \mathfrak{k}'' : \tau'' \rightarrow \tau''$ be functions. Suppose that $(\mathfrak{h}', \mathfrak{k})$ is an $h' - \varphi$ -contraction w.r. to δ , Further, assume:

(i) $\mathfrak{h}' : (\tau'', \delta') \rightarrow \tau'', \delta')$ is a continuous and $\mathfrak{h}'(\tau'')$ is δ' - closed;

(ii) $\mathfrak{k}(M'') \subset \mathfrak{h}(M'')$;

(iii) $E(G)$ is transitive;

(iv) If $\delta < \delta'$, $T : (M'', \delta) \rightarrow (M'', \delta)$ is G - Cauchy on X ;

(v) $(M'', \delta) \rightarrow (M'', \delta')$ is G' - continuous, and \mathfrak{k} and h' are δ' -compatible.

$X(\mathfrak{k}, h') \neq \varphi$ iff $C(\mathfrak{k}, h') \neq \varphi$.

In 1983 Polish Mathematician Geraghty extended Banach Contraction Principle using class of functions instead of contraction constant as follows. We denote S_n by $\beta_n : R^+ \rightarrow [0, 1)$ for $\{s_n\}$ in R^+ , $\beta_n(s_n) \rightarrow 1 \implies s_n \rightarrow 0$

Definition 7 [8] Let M be a MSP metric d . A self map $T : M \rightarrow M$ is said to be Geraghty contraction if there exist $\beta \in S$ such that $d(Ks, Kt) \leq \beta(d(s, t))d(s, t)$ for all s, t in M .

In 1884, [12] M.S. Khan, M. Swaleh, S. Sessa, defined altering distance function as follows, which we denote $\Psi = \{\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ such that (i) } \psi \text{ is non decreasing, (ii) } \zeta \text{ is continuous (iii) } \zeta(z) = 0 \text{ implies } z = 0\}$.

Definition 8 [9] A selfmap τ on a MSP (τ, D_i) is Ćirić-type Geraghty contraction if there exists $\beta_1 \in S$ such that $d(T'a, T'b) \leq M(a, b)$ for all a, b in τ . where

$$M(a, b) = \max\{\beta_1(D_i(a, b)D_i(a, y)), \beta(D_i(a, Ta)D_i(a, Ta)), \beta(D_i(b, Tb)D_i(b, Tb)), \beta(D_i(a, Tb)D_i(a, Tb)), \beta(D_i(b, Ta)D_i(b, Ta))\}.$$

In this section, using Graph Structure to exhibit the presence fixed points of Ćirić rational type-weak Geraghty contraction maps with altering distance functions.

2. Main Results

Definition 2.1 Let M' be a MSP with metric δ endowed with a digraph $\bar{\alpha}_G = (\alpha_V, \alpha_E)$, and let $\wp', \kappa : M' \rightarrow M'$ be functions. The pair \wp', κ is said to be Ćirić type Geraghty ψ weak rational contraction with respect to d if

- (1) with regard to the graph G , \wp' is κ -edge preserving;
- (2) there exists two functions $\beta_1 \in S$ and $\psi_1 \in \Psi$ such that, for all $s, t \in X$ with $(\kappa s, \kappa t) \in E(G)$, we have $\psi(\delta(\wp's, \wp't)) \leq \psi(P(\kappa s, \kappa t))\psi(P(\kappa s, \kappa t)) + L.N(\kappa s, \kappa t)$, for $L \geq 0$ where $K : MXM \rightarrow [0, \infty)$ is a function such that, for any $s, t \in X$,

$$P(\kappa s, \kappa t) = \max\{\beta_1\left(\frac{\delta(\kappa s, \wp's)\delta(\wp't, \kappa t)}{\delta(\kappa s, \kappa t)}\right) \cdot \left(\frac{\delta(\kappa s, \wp's)\delta(\wp't, \kappa t)}{\delta(\kappa s, \kappa t)}\right), \beta_1\left(\frac{\delta(\kappa t, \wp't)[1+\delta(\kappa s, \wp's)]}{1+\delta(\kappa s, \kappa t)}\right) \cdot \left(\frac{\delta(\kappa t, \wp't)[1+\delta(\kappa s, \wp's)]}{1+\delta(\kappa s, \kappa t)}\right), \beta_1\left(\frac{\delta(\wp's, \kappa s)[1+\delta(\wp't, \kappa t)]}{1+\delta(\kappa s, \kappa t)}\right) \cdot \left(\frac{\delta(\wp's, \kappa s)[1+\delta(\wp't, \kappa t)]}{1+\delta(\kappa s, \kappa t)}\right), \beta_1\left(\frac{\delta(\kappa t, \wp's)[1+\delta(\wp's, \kappa t)]}{1+\delta(\kappa s, \kappa t)}\right) \cdot \left(\frac{\delta(\kappa t, \wp's)[1+\delta(\wp's, \kappa t)]}{1+\delta(\kappa s, \kappa t)}\right), \beta_1(\delta(\kappa s, \kappa t))\delta(\kappa s, \kappa t), \beta\left(\frac{\delta(\kappa s, \wp't)+\delta(\kappa t, \wp's)}{2}\right) \cdot \left(\frac{\delta(\kappa s, \wp't)+\delta(\kappa t, \wp's)}{2}\right)\},$$

$$N(\kappa s, \kappa t) = \min\{\beta_1\left(\frac{\delta(\kappa s, \wp's)\delta(\wp't, \kappa t)}{\delta(\kappa s, \kappa t)}\right) \cdot \left(\frac{\delta(\kappa s, h's)\delta(\wp't, \kappa t)}{\delta(\kappa s, \kappa t)}\right), \beta_1\left(\frac{\delta(\kappa t, h's)[1+\delta(\wp's, \kappa t)]}{1+\delta(\kappa s, \kappa t)}\right) \cdot \left(\frac{\delta(\kappa t, h's)[1+\delta(\wp's, \kappa t)]}{1+\delta(\kappa s, \kappa t)}\right), \beta_1(\delta(\kappa s, \kappa t))\delta(\kappa s, \kappa t), (\beta_1(\delta(\kappa s, \wp't))\delta(\kappa s, \wp't))\}.$$

Theorem 2.1. Let M' be a CMS with metric δ endowed with a directed graph $\bar{\alpha}_{G1} = (\alpha_{V1}, \alpha_{E1})$, let δ'' be another metric on M' , and let $h', \kappa : M' \rightarrow M'$ be functions. Suppose that $(\alpha'_{h'}, \alpha_{\kappa})$ is a Ćirić type Geraghty ψ' weak rational contraction with respect to δ' , Further, assume that the following conditions satisfied:

- (i) $\kappa : (M', \delta') \rightarrow (M', \delta')$ is a continuous function such that $A'(X')$ is δ' - closed;
- (ii) $h'(M') \subset \kappa(M')$;
- (iii) $E_1(G_1)$ is transitive;
- (iv) If $\delta < \delta'$, $A' : (M', \delta') \rightarrow (M', \delta')$ is G_1 - Cauchy's on X ;
- (v) $h' : (M', \delta') \rightarrow (M', \delta')$ is G_1 - continuous, and h' and κ are δ' - compatible, then h' and κ have a coincident points.

Proof. Let $\rho_0 \in M$ be such that $(\kappa\rho_0, \kappa\rho_1) \in E_1(G_1)$.

Since $h(M) \subset \kappa(M')$ and $h'(\rho'_0) \in M'$.

Choose $\rho_1 \in M$ such that $h'(\rho_0) = \kappa(\rho_1)$

For any $n \in \mathbb{N}$ we may create $\{\rho_n\}$ in M' such that $\kappa(\rho_n) = h'(\rho_{n-1})$.

If $\kappa(\rho_{n+1}) = \kappa(\rho_n)$ for some $n \in \mathbb{N}$ i.e., $h'(\rho_n) = \kappa(\rho_n)$, ρ_n is a coincident point of h' and κ .

hence, w l g, $\kappa\rho_n \neq \kappa\rho_{n+1}$ for each $n \geq N$. Since $(\kappa\rho_0, h'\rho_0) = (\kappa\rho_0, h'\rho_1) \in E_1(G_1)$, and the function h' e-edge preserving with respect to G_1 , so that $(h'\rho_0, \kappa\rho_1) = (\kappa\rho_1, \kappa\rho_2) \in E_1(G_1)$.

By proceeding in this way, we have $(\kappa\rho_{n-1}, \kappa\rho_n) \in E_1(G_1)$ for all $n \in \mathbb{N}$.
 Since (h', κ) is $\beta - \psi$ - rational contraction with δ , for all $n \geq 0$, we have

$$\psi(\delta(\kappa\rho_n, \kappa\rho_{n+1})) = \psi\delta(h'\rho_{n-1}, h'\rho_n) \leq \beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n)))(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n))) + LN(\kappa\rho_{n-1}, \kappa\rho_n) \tag{2.1}$$

$$\begin{aligned} P(\kappa\rho_{n-1}, \kappa\rho_n) &= \max\{\beta_1\left(\frac{\delta(\kappa\rho_{n-1}, h'\rho_{n-1})\delta(h'\rho_n, \kappa\rho_n)}{\delta(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \left(\frac{\delta(\kappa\rho_{n-1}, h'\rho_{n-1})\delta(h'\rho_n, \kappa\rho_n)}{\delta(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \\ &\beta - 1\left(\frac{\delta(\kappa\rho_{n-1}, h'\rho_n)[1+\delta(\kappa\rho_{n-1}, h'\rho_{n-1})]}{1+\delta(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \left(\frac{\delta(\kappa\rho_n, h'\rho_n)[1+\delta(\kappa\rho_{n-1}, h'\rho_{n-1})]}{1+\delta(\kappa\rho_{n-1}, \kappa\rho_n)}\right) \\ &\beta\left(\frac{\delta(h'\rho_{n-1}, \kappa\rho_{n-1})[1+\delta(h'\rho_n, \kappa\rho_n)]}{1+\delta(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \left(\frac{\delta(h'\rho_{n-1}, \kappa\rho_{n-1})[1+\delta(h'\rho_n, \kappa\rho_n)]}{1+\delta(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \\ &\beta\left(\frac{d(\kappa\rho_n, h'\rho_{n-1})[1+d(h'\rho_{n-1}, \kappa\rho_n)]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \left(\frac{d(\kappa\rho_n, h'\rho_{n-1})[1+d(h'\rho_{n-1}, \kappa\rho_n)]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \\ &\beta_1(d(\kappa\rho_{n-1}, \kappa\rho_n))d(\kappa\rho_{n-1}, \kappa\rho_n), \beta_1\left(\frac{d(\kappa\rho_{n-1}, h'\rho_n)+d(\kappa\rho_n, h'\rho_{n-1})}{2}\right), \frac{d(\kappa\rho_{n-1}, h'\rho_n)+d(\kappa\rho_n, h'\rho_{n-1})}{2}\} \end{aligned}$$

$$\begin{aligned} P(\kappa\rho_{n-1}, \kappa\rho_n) &= \max\{d(\kappa\rho_{n+1}, \kappa\rho_n), d(\kappa\rho_{n+1}, \kappa\rho_n), \frac{d(\kappa\rho_n, \kappa\rho_{n-1})[1+d(\kappa\rho_{n+1}, \kappa\rho_n)]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}, \\ &0, d(\kappa\rho_{n-1}, \kappa\rho_n), \frac{d(\kappa\rho_{n-1}, \kappa\rho_{n+1})}{2}\} \\ &\leq \max\{d(\kappa\rho_{n+1}, \kappa\rho_n), \frac{d(\kappa\rho_n, \kappa\rho_{n-1})[1+d(\kappa\rho_{n+1}, \kappa\rho_n)]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}, d(\kappa\rho_{n-1}, \kappa\rho_n), \frac{d(\kappa\rho_{n-1}, \kappa\rho_{n+1})}{2}\} \end{aligned}$$

$$\begin{aligned} N(\kappa\rho_{n-1}, \kappa\rho_n) &= \min\{\beta_1\left(\frac{d(\kappa\rho_{n-1}, h'\rho_{n-1})d(h'\rho_n, \kappa\rho_n)}{d(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \left(\frac{d(\kappa\rho_{n-1}, h'\rho_{n-1})d(h'\rho_n, \kappa\rho_n)}{d(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \\ &\beta\left(\frac{d(\kappa\rho_n, h\rho_{n-1})[1+d(h\rho_{n-1}, \kappa\rho)]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \left(\frac{d(\kappa\rho_n, h'\rho_{n-1})[1+d(h'\rho_{n-1}, \kappa\rho_n)]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}\right), \\ &\beta_1(d(\kappa\rho_{n-1}, \kappa\rho_n))d(\kappa\rho_{n-1}, \kappa\rho_n), (\beta(d(\kappa\rho_{n-1}, h'\rho_n)).d(\kappa\rho_{n-1}, h'\rho_n))\} \end{aligned}$$

$$\begin{aligned} N(\kappa\rho_{n-1}, \kappa\rho_n) &= \min\left\{\frac{d(\kappa\rho_{n-1}, \kappa\rho_n)d(\kappa\rho_{n+1}, \kappa\rho_n)}{d(\kappa\rho_{n-1}, \kappa\rho_n)}, \frac{d(\kappa\rho_n, \kappa\rho_n)[1+d(\kappa\rho_n, \kappa\rho_n)]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}, \right. \\ &d(\kappa\rho_{n-1}, \kappa\rho_n), d(\kappa\rho_n, \kappa\rho_{n+1})\} \\ &= \min\{d(\kappa\rho_{n+1}, \kappa\rho_n), \frac{0[1+0]}{1+d(\kappa\rho_{n-1}, \kappa\rho_n)}, d(\kappa\rho_{n-1}, \kappa\rho_n), d(\kappa\rho_n, \kappa\rho_{n+1})\} \\ &= 0 \end{aligned}$$

If $\max\{d(\kappa\rho_{n+1}, \kappa\rho_n), d(\kappa\rho_{n-1}, \kappa\rho_n)\} = d(\kappa\rho_{n+1}, \kappa\rho_n)$

From (2.1), we have

$$\psi(d(\kappa\rho_n, \kappa\rho_{n+1})) = \psi(d(h'\rho_{n-1}, h'\rho_n)) \leq \beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n)))(\psi(d(\kappa\rho_{n+1}, \kappa\rho_n))) + LN.0$$

Since $\beta_1 \in S$, We get $\psi(d(\kappa\rho_n, \kappa\rho_{n+1})) < \psi(d(\kappa\rho_{n+1}, \kappa\rho_n))$, a contradiction.

hence $\max\{\delta(\kappa\rho_{n+1}, \kappa\rho_n), \delta(\kappa\rho_{n-1}, \kappa\rho_n)\} = d(\kappa\rho_{n-1}, \kappa\rho_n)$
 from (2.1), we have

$$\psi(d(\kappa\rho_n, \kappa\rho_{n+1})) = \psi(\delta(h'\rho_{n-1}, h'\rho_n)) < \psi d(\kappa\rho_{n-1}, \kappa\rho_n) \tag{2.2}$$

So, sequence $\psi(\delta(\kappa\rho_n, \kappa\rho_{n+1}))$ is strictly decreasing and so $\lim_{n \rightarrow \infty} \psi(\delta(\kappa\rho_n, \kappa\rho_{n+1})) = v \geq 0$

We now demonstrate $v = 0$. Assuming $v > 0$ we can infer from (2.1) we have

$$\begin{aligned} \psi(\delta(\kappa\rho_n, \rho_{n+1})) &= \psi\delta(h'\rho_{n-1}, h'\rho_n) \leq \beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n B)))(\psi(P'(\kappa\rho_{n-1}, \kappa\rho_n))) + LN(\kappa\rho_{n-1}, \kappa\rho_n) \\ &\leq \beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n)))(\psi(\delta(\kappa\rho_{n-1}, \kappa\rho_n))) + L.0 \end{aligned}$$

$$\psi(d(\kappa\rho_n, \kappa\rho_{n+1})) \leq \beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n)))(\psi(\delta(\kappa\rho_{n-1}, \kappa\rho_n)))$$

$$\frac{\psi(d(\kappa\rho_n, \kappa\rho_{n+1}))}{\psi(\delta(\kappa\rho_{n-1}, \kappa\rho_n))} \leq \beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n))) < 1 \text{ for each } n \geq 1$$

letting $n \rightarrow \infty$, we get

$$1 = \lim_{n \rightarrow \infty} \frac{\psi(d(\kappa\rho_n, \kappa\rho_{n+1}))}{\psi(\delta(\kappa\rho_{n-1}, \kappa\rho_n))} \leq \lim_{n \rightarrow \infty} \beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n))) \leq 1$$

so that $\beta_1(\psi(P(\kappa\rho_{n-1}, \kappa\rho_n))) \rightarrow 1$ as $n \rightarrow \infty$.

since $\beta_1 \in S$, we get $\lim_{n \rightarrow \infty} \psi(P(\kappa\rho_{n-1}, \kappa\rho_n)) = 0$

so that $\lim_{n \rightarrow \infty} \psi(d(\kappa\rho_{n-1}, \kappa\rho_n)) = 0$

i.e., $v = 0$

we now demonstrate the Cauchy's nature of the series $\{\kappa\rho_n\}$. Assume $\{\kappa\rho_n\}$ is not a Cauchy's. for any

$k \in \mathbb{N}$, $\epsilon > 0$ such that, there are sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ and $d(\kappa\rho_m(k), \kappa\rho_n(k)) \geq \epsilon$ and $d(\kappa\rho_m(k) - 1, \kappa\rho_n(k)) < \epsilon$
 $\epsilon \leq d(\kappa\rho_n(k), \kappa\rho_m(k))$

$$\leq \delta(\kappa\rho_n(k), \kappa\rho_m(k) - 1) + \delta(\kappa\rho_m(k) - 1, \kappa\rho_m(k))$$

$$< \epsilon + \delta(\kappa\rho_m(k) - 1, \kappa\rho_m(k))$$

Taking as $k \rightarrow \infty$ and using $\lim_{n \rightarrow \infty} \delta(\kappa\rho_n, \kappa\rho_{(n+1)}) = 0$,

we get $\lim_{n \rightarrow \infty} d(\kappa\rho_m(k), \kappa\rho_{(n(k))}) = \epsilon > 0$.

Since $E(G)$ has transitivity, we get $(\kappa\rho_m(k), \kappa\rho_n(k)) \in E(G)$ for every $k \in \mathbb{N}$, we get

$$\psi(\delta(\kappa\rho_n(k) + 1, \kappa\rho_m(k) + 1)) = \psi(d(h'\rho_n(k), h'\rho_m(k)))$$

$$\leq \beta_1(P(\kappa\rho_n(k), \kappa\rho_m(k)))\psi(P(\kappa\rho_n(k), \kappa\rho_m(k))) + L.N(\kappa\rho_n(k), \kappa\rho_m(k))$$

Now

$$P(\kappa\rho_n(k), \kappa\rho_m(k)) = \max\{\beta_1\left(\frac{\delta(\kappa\rho_n(k), h'\rho_n(k))\delta(h'\rho_m(k), \kappa\rho_m(k))}{\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right), \left(\frac{\delta(\kappa\rho_n(k), h'\rho_n(k))\delta(h'\rho_m(k), \kappa\rho_m(k))}{\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right),$$

$$\beta_1\left(\frac{\delta(\kappa\rho_m(k), h'\rho_m(k))[1+\delta(\kappa\rho_n(k), h'\rho_n(k))]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right), \left(\frac{\delta(\kappa\rho_m(k), h'\rho_m(k))[1+\delta(\kappa\rho_n(k), h'\rho_n(k))]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right)$$

$$\beta_1\left(\frac{\delta(h'\rho_n(k), \kappa\rho_n(k))[1+\delta(h'\rho_m(k), \kappa\rho_m(k))]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right), \left(\frac{\delta(h'\rho_n(k), \kappa\rho_n(k))[1+\delta(h'\rho_m(k), \kappa\rho_m(k))]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right),$$

$$\beta_1\left(\frac{\delta(\kappa\rho_m(k), h'\rho_m(k))[1+\delta(h'\rho_n(k), \kappa\rho_n(k))]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right), \left(\frac{\delta(\kappa\rho_m(k), h'\rho_m(k))[1+\delta(h'\rho_n(k), \kappa\rho_n(k))]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}\right),$$

$$\beta_1(\delta(\kappa\rho_n(k), \kappa\rho_m(k))\delta(\kappa\rho_n(k), \kappa\rho_m(k)),$$

$$\beta_1\left(\frac{\delta(\kappa\rho_n(k), h'\rho_n(k)) + \delta(\kappa\rho_m(k), h'\rho_m(k))}{2}\right), \frac{\delta(\kappa\rho_n(k), h'\rho_n(k)) + \delta(\kappa\rho_m(k), h'\rho_m(k))}{2}\}$$

$$P(\kappa\rho_n(k), \kappa\rho_m(k)) = \max\left\{\frac{\delta(\kappa\rho_n(k), \kappa\rho_{n(k)+1})\delta(\kappa\rho_{m(k)+1}, \kappa\rho_m(k))}{\delta(\kappa\rho_n(k), \kappa\rho_m(k))}, \frac{\delta(\kappa\rho_m(k), \kappa\rho_{m(k)+1})[1+\delta(\kappa\rho_n(k), \kappa\rho_{n(k)+1})]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}, \frac{\delta(\kappa\rho_{n(k)+1}, \kappa\rho_n(k))[1+\delta(\kappa\rho_{m(k)+1}, \kappa\rho_m(k))]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}, \frac{\delta(\kappa\rho_m(k), \kappa\rho_{m(k)+1})[1+\delta(\kappa\rho_{n(k)+1}, \kappa\rho_{n(k)+1})]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}, \delta(\kappa\rho_n(k), \kappa\rho_m(k)), \frac{\delta(\kappa\rho_n(k), \kappa\rho_{m(k)+1}) + \delta(\kappa\rho_m(k), \kappa\rho_{n(k)+1})}{2}\right\}$$

On letting $k \rightarrow \infty$ and using $\lim_{k \rightarrow \infty} d(\kappa\rho_n, \kappa\rho_{n+1}) = 0$

$$\lim_{k \rightarrow \infty} P(\kappa\rho_n(k), \kappa\rho_m(k)) = \lim_{k \rightarrow \infty} \frac{\delta(\kappa\rho_m(k), \kappa\rho_{m(k)+1})[1+\delta(\kappa\rho_{n(k)+1}, \kappa\rho_{n(k)+1})]}{1+\delta(\kappa\rho_n(k), \kappa\rho_m(k))}, \delta(\kappa\rho_n(k), \kappa\rho_m(k)),$$

$$\frac{\delta(\kappa\rho_n(k), \kappa\rho_{m(k)+1}) + \delta(\kappa\rho_m(k), \kappa\rho_{n(k)+1})}{2}\}$$

$$\lim_{k \rightarrow \infty} P(\kappa\rho_n(k), \kappa\rho_m(k)) = \epsilon$$

$$\lim_{k \rightarrow \infty} N(\kappa\rho_n(k), \kappa\rho_m(k)) = 0$$

$$1 = \lim_{k \rightarrow \infty} \frac{\psi(d(\kappa\rho_n(k), \kappa\rho_{n(k)+1}))}{\psi(\delta(\kappa\rho_{n(k)-1}, \kappa\rho_n(k)))} \leq \lim_{n \rightarrow \infty} \beta_1(\psi(P(\kappa\rho_{n(k)-1}, \kappa\rho_n(k)))) \leq 1$$

so that $\beta_1(\psi(P(\kappa\rho_{n(k)-1}, \kappa\rho_n(k)))) \rightarrow 1$ as $k \rightarrow \infty$.

From the property of $\beta_n \in S$, we get $\lim_{k \rightarrow \infty} \psi(P(\kappa\rho_{n(k)-1}, \kappa\rho_n(k))) = 0$

so that $\lim_{k \rightarrow \infty} \psi(\delta(\kappa\rho_{n(k)-1}, \kappa\rho_n(k))) = 0$, contradiction.

hence $\{g_{x_n}\}$ is a Cauchy MSP (X, d) .

Now, we prove $\{\kappa\rho_n\}$ is a Cauchy MSP (M, d') .

When $\delta \geq \delta'$ the proof is trivial.

So consider $\delta < \delta'$.

Let $\epsilon > 0$. Since the sequence $\{\kappa\rho'(n)\}$ is a Cauchy in MSP (M', δ) and the function h' is κ -Cauchy on M . We have $h'(M') \subset K(M')$, we can obtain that $\{h'\rho(n)\}$ is a Cauchy's MSP (M', δ') . So, there a number $n_0 \in \mathbb{N}$ such that $\delta'(\kappa\rho_{m+1}, \kappa\rho_{n+1}) = \delta(h'\rho_m, h'\rho_n) < \epsilon$ for all $m, n \geq N_0$ and hence $\{\kappa\rho_n\}$ is Cauchy in (X, d') .

$\kappa(M')$ is a δ' -closed subset of (M', δ') , and which is complete, then there exist $z = g\rho \in \kappa(M')$ such that

$$\lim_{n \rightarrow \infty} \kappa\rho_n = \lim_{n \rightarrow \infty} h\rho_n = z.$$

We have $h' : (M', \delta') \rightarrow (M', \delta')$ is a G -continuous such that h' and κ are δ' -compatible

$$\lim_{n \rightarrow \infty} \delta(\kappa h' \rho_n, h' \kappa \rho_n) = 0/.$$

Consider $\delta(\kappa u', h' u') \leq \delta'(\kappa u', \kappa h' \rho_n) + \delta'(\kappa h' \rho_n, h' \kappa \rho_n) + d'(h' \kappa \rho_n, h' u')$

taking limit as $n \rightarrow \infty$, we get that $\delta'(h' z', \kappa z') = 0$, and from continuity of κ and that h' is G' -continuous. $\therefore h' z' = \kappa z'$. Which implies that z' is a coincidence point of h' and κ . \square

Theorem 2.2. In addition to the aforementioned theorem 2.1s supposition that h'' and κ' are weakly compatible, h'' and κ' share a fixed point.

Proof. $\{\kappa' t''_n\}$ is a non-decreasing and converges to $\kappa' z'' = h'' z''$, according to the proof of theorem 2.1. The weak compatibility of h'' and κ' results in $h'' \kappa' z'' = \kappa' h'' z''$. Since h'' and κ' are weakly compatible, we have $h'' \kappa' z'' = \kappa' h'' z''$.

Now $h'' z'' = \kappa' z'' = u''$ (say)

Also, $h'' u'' = h'' \kappa' z'' = \kappa' h'' z'' = \kappa' u''$ If $z'' = u''$ then u'' is a common F P of h'' and κ' .

if $z'' \neq u''$, i.e., $\delta(z'', u'') > 0$

$$\psi(d(\kappa' z'', \kappa' u'')) = \psi(\delta(h'' z'', h'' u'')) \leq \beta_1(\psi(P(\kappa' z'', \kappa' u''))) \psi(P(\kappa' z'', \kappa' u'')) + L.N(\kappa' z'', \kappa' u'')$$

$$\begin{aligned} P(\kappa' z'', \kappa' u'') &= \max\{\beta_1\left(\frac{d(\kappa' z'', h'' z'')d(h'' u'', \kappa' u'')}{\delta(\kappa' z'', \kappa' u'')}\right), \left\{\frac{\delta(\kappa' z'', h'' z'')\delta(h'' u'', \kappa' u'')}{\delta(\kappa' z'', \kappa' u'')}\right\}, \\ \beta_1\left(\frac{\delta(\kappa' u'', h'' u'')[1+\delta(\kappa' z'', h'' z'')]}{1+\delta(\kappa' z'', \kappa' u'')}\right), \frac{\delta(\kappa' u'', h'' u'')[1+\delta(\kappa' z'', h'' z'')]}{1+\delta(\kappa' z'', \kappa' u'')}, \\ \beta_1\left(\frac{d(h'' z'', \kappa' z'')[1+\delta(h'' u'', \kappa' u'')]}{1+\delta(\kappa' z'', \kappa' u'')}\right), \frac{\delta(h'' z'', \kappa' z'')[1+\delta(h'' u'', \kappa' u'')]}{1+\delta(\kappa' z'', \kappa' u'')}, \\ \beta_1\left(\frac{\delta(\kappa' u'', h'' z'')[1+\delta(h'' z'', \kappa' u'')]}{1+\delta(\kappa' z'', \kappa' u'')}\right), \frac{\delta(\kappa' u'', h'' z'')[1+\delta(h'' z'', \kappa' u'')]}{1+\delta(\kappa' z'', \kappa' u'')}, \beta_1(d(\kappa' z'', \kappa' u'')).\delta(\kappa' z'', \kappa' u''), \\ \beta_1\left(\frac{\delta(\kappa' z'', h'' u'')+\delta(\kappa' u'', h'' z'')}{2}\right), \left\{\frac{\delta(\kappa' z'', h'' u'')+\delta(\kappa' u'', h'' z'')}{2}\right\}\} \\ P(\kappa' z'', \kappa' u'') &\leq \max\left\{\frac{\delta(u'', u'')\delta(h'' u'', \kappa' u'')}{\delta(\kappa' z'', \kappa' u'')}, \frac{\delta(\kappa' u'', \kappa' u'')[1+\delta(u'', u'')]}{1+\delta(u'', \kappa' u'')}, \frac{\delta(u'', u'')[1+\delta(\kappa' u'', \kappa' u'')]}{1+\delta(u'', \kappa' u'')}, \right. \\ &\left. \frac{\delta(\kappa' u'', u'')[1+\delta(u'', \kappa' u'')]}{1+\delta(u'', \kappa' u'')}, \delta(u'', \kappa' u''), \frac{\delta(u'', \kappa' u'')+\delta(\kappa' u'', u'')}{2}\right\} \end{aligned}$$

$$P(\kappa' z'', \kappa' u'') = \max\{0, 0, 0, \delta(\kappa' u'', u''), \delta(u'', \kappa' u''), \frac{\delta(u'', \kappa' u'')+\delta(\kappa' u'', u'')}{2}\}$$

$$\begin{aligned} P(\kappa' z'', \kappa' u'') &= \delta(\kappa' u'', u'') = \delta(\kappa' z'', \kappa' u'') \\ N(\kappa' z'', \kappa' u'') &= \min\left\{\left\{\beta_1\frac{\delta(\kappa' z'', \kappa' z'')\delta(h'' u'', \kappa' u'')}{\delta(\kappa' z'', \kappa' u'')}\right\}, \left(\frac{\delta(\kappa' z'', \kappa' z'')\delta(h'' u'', \kappa' u'')}{\delta(\kappa' z'', \kappa' u'')}\right), \beta_1\left(\frac{\delta(\kappa' u'', h'' z'')[1+\delta(h'' z'', \kappa' u'')]}{1+\delta(\kappa' z'', \kappa' u'')}\right), \right. \\ &\left. \frac{\delta(\kappa' u'', h'' z'')[1+\delta(h'' z'', \kappa' u'')]}{1+\delta(\kappa' z'', \kappa' u'')}, \beta_1(\delta(\kappa' z'', \kappa' u'')).(\delta(\kappa' z'', \kappa' u'')), \beta_1(\delta(\kappa' u'', h'' u'')).(\delta(\kappa' u'', h'' u''))\right\} \\ N(\kappa' z'', \kappa' u'') &\leq \min\left\{\frac{\delta(u'', u'')\delta(\kappa' u'', \kappa' u'')}{\delta(u'', \kappa' u'')}, \frac{\delta(\kappa' u'', u'')[1+\delta(u'', \kappa' u'')]}{1+\delta(u'', \kappa' u'')}, \delta(\kappa' z'', \kappa' u''), \delta(\kappa' u'', \kappa' u'') = 0\right\} \\ \psi(\delta(\kappa' z'', \kappa' u'')) &= \psi(\delta(h'' z'', h'' u'')) \leq \beta_1(\psi(P(\kappa' z'', \kappa' u''))) \psi(\delta(\kappa' z'', \kappa' u'')) + L.0 \\ 1 &= \frac{\psi(\delta(\kappa' z'', \kappa' u''))}{\psi(\delta(\kappa' z'', \kappa' u''))} \leq \beta_1(\psi(P(\kappa' z'', \kappa' u''))) < 1. \end{aligned}$$

hence $\kappa' z'' = \kappa' u'' \kappa' u'' = h'' u'' = u''$, $\therefore u''$ is a common fixed point of h'' and κ' . \square

Theorem 2.3. In addition to the hypothesis of the above theorem 2.1 For any $s'', t'' \in C(h'', \kappa')$, such that $\kappa' s'' \neq \kappa' t''$, we have $(\kappa' s'', \kappa' t'') \in E_1(G)$. If h'' and κ' are δ' - compatible and $X(h'', \kappa') \neq \emptyset$, then $Cm'(h'', \kappa') \neq \emptyset$.

Proof. From the theorem (2.1), there exists a coincidence point $s'' \in X''$, i.e., $h'' s'' = \kappa' s''$.

Suppose that there exists another coincidence point $t'' \in X''$ such that $h'' t'' = \kappa' t''$,

Assume that $\kappa' s'' \neq \kappa' t''$.

From the assumption $(\kappa' s'', \kappa' t'') \in E_1(G)$

$$\psi(\delta(h'' s'', h'' t'')) \leq \beta_1(\psi(P(\kappa' s'', \kappa' t''))) \psi(P(\kappa' s'', \kappa' t'')) + L.N(\kappa' s'', \kappa' t''), \text{ for } L \geq 0$$

where $P : M' \times M' \rightarrow [0, \infty)$ is a function such that, for any $s'', t'' \in X'$,

$$\begin{aligned} P(\kappa' s'', \kappa' t'') &= \max\left\{\beta_1\left(\frac{\delta(\kappa' s'', h'' s'')d(h'' t'', \kappa' t'')}{\delta(\kappa' s'', \kappa' t'')}\right), \left\{\frac{\delta(\kappa' s'', h'' s'')d(h'' t'', \kappa' t'')}{\delta(\kappa' s'', \kappa' t'')}\right\}, \right. \\ \beta_1\left(\frac{\delta(\kappa' t'', h'' t'')[1+\delta(\kappa' s'', h'' s'')]}{1+\delta(\kappa' s'', \kappa' t'')}\right), \frac{\delta(\kappa' t'', h'' t'')[1+\delta(\kappa' s'', h'' s'')]}{1+\delta(\kappa' s'', \kappa' t'')}, \\ \beta_1\left(\frac{\delta(h'' s'', \kappa' s'')[1+\delta(h'' t'', \kappa' t'')]}{1+\delta(\kappa' s'', \kappa' t'')}\right), \frac{\delta(h'' s'', \kappa' s'')[1+\delta(h'' t'', \kappa' t'')]}{1+\delta(\kappa' s'', \kappa' t'')}, \\ \beta_1\left(\frac{\delta(\kappa' t'', h'' s'')[1+\delta(h'' s'', \kappa' t'')]}{1+\delta(h'' s'', \kappa' t'')}\right), \frac{\delta(\kappa' t'', h'' s'')[1+\delta(h'' s'', \kappa' t'')]}{1+\delta(h'' s'', \kappa' t'')}, \end{aligned}$$

$$\beta_1(\delta(\kappa's'', \kappa't''))\delta(\kappa's'', \kappa't''), \beta_1\left(\frac{\delta(\kappa's'', h''t'')+\delta(\kappa't'', h''s'')}{2}\right)\frac{\delta(\kappa's'', h''t'')+\delta(\kappa't'', h''s'')}{2}\}$$

$$P(h''s'', \kappa't'') \leq \max\left\{\frac{\delta(\kappa's'', h''s'')\delta(\kappa't'', \kappa't'')}{\delta(h''s'', \kappa't'')}, \frac{\delta(\kappa't'', \kappa't'')[1+\delta(h''s'', h''s'')]}{1+\delta(h''s'', \kappa't'')}, \frac{\delta(h''s'', h''s'')[1+\delta(\kappa't'', \kappa't'')]}{1+\delta(h''s'', \kappa't'')}, \frac{\delta(\kappa't'', \kappa's'')[1+\delta(\kappa's'', \kappa't'')]}{1+\delta(\kappa's'', \kappa't'')}, \delta(\kappa's'', \kappa't''), \frac{\delta(\kappa's'', \kappa't'')+\delta(\kappa't'', \kappa's'')}{2}\right\} = \delta(\kappa's'', \kappa't'')$$

$$N(\kappa's'', \kappa't'') = 0$$

$\psi(\delta(h''s'', h''t'')) \leq \beta_1(\psi(P(\kappa's'', \kappa't'')))\psi(P(\kappa's'', \kappa't'')) + L.N(\kappa's'', \kappa't'') < \psi(\delta(h''s'', h''t''))$, contradiction.

$\kappa's'' = \kappa't''$.

Let $t''_0 = t''$, and define the sequence $\{t''_n\}$ by $\kappa't''_n = h''t''_{n-1}$ for each $n \in \mathbb{N}$.

Since t'' is a coincidence point of h'' and κ' , we have $\kappa't''_n = h''t''_n$ for each $n \in \mathbb{N}$.

Now let $w'' = \kappa't''$

therefore $\kappa'w'' = \kappa'\kappa't'' = \kappa'h''t''$

By the definition of the sequence $\{t''_n\}$, $\kappa't''_n = h''t''_n = h''t''_{n-1}$ for each $n \in \mathbb{N}$.

$$\lim_{n \rightarrow \infty} h''t''_n = \lim_{n \rightarrow \infty} \kappa't''_n = h''t''.$$

Since κ' and h'' are δ' -compatible, $\lim_{n \rightarrow \infty} \delta'(\kappa'h''t''_n, h''\kappa't''_n) = 0$

i.e., $\kappa'h''t'' = h''\kappa't''$.

therefore, we have $\kappa'w'' = \kappa'h''t'' = h''\kappa't'' = h''w''$

and hence w'' is another coincidence point of h and κ' .

Now $h''w'' = \kappa'w'' = \kappa't'' = w''$

$\therefore w''$ is a common fixed point of h and κ' . i.e., $Cm'(h'', \kappa') \neq \emptyset$. □

3. Corollaries and Examples

When $\psi(t)$ is identity then it follows.

Definition 3.1 Let (M', δ) be a MSP with a directed Graph $G_1 = (V_1(G_1), E_1(G_1))$, and let $h', \kappa : M' \rightarrow M'$ be functions. h', κ is said to be β_2 -rational contraction with respect to δ if

(1) h' is κ -edge preserving with respect to G ;

(2) there exists two functions $\beta_2 \in S$ and $\psi \in \Psi$ such that, for all $x'', y'' \in X'$ with $(\kappa x', \kappa y') \in E_1(G_1)$, we have

$$(\delta(h'x', h'y')) \leq \beta(P(\kappa x', \kappa y'))(P(\kappa x', \kappa y') + L.N(\kappa x', \kappa y')), \text{ for } L \geq 0$$

where $P : X \times X \rightarrow [0, \infty)$ is a function such that, for any $x', y' \in X$,

$$P(\kappa x', \kappa y') = \max\left\{\beta_2\left(\frac{d(\kappa x', h'x')d(h'y', \kappa y')}{\delta(\kappa x', \kappa y')}\right)\left(\frac{\delta(\kappa x', h'x')\delta(h'y', \kappa y')}{\delta(\kappa x', \kappa y')}\right), \beta_2\left(\frac{\delta(\kappa y', h'y')[1+\delta(\kappa x', h'x')]}{1+\delta(\kappa x', \kappa y')}\right)\left(\frac{\delta(\kappa y', h'y')[1+\delta(\kappa x', h'x')]}{1+\delta(\kappa x', \kappa y')}\right), \beta_2\left(\frac{\delta(h'x', \kappa x')[1+\delta(h'y', \kappa y')]}{1+\delta(\kappa x', \kappa y')}\right)\left(\frac{\delta(h'x', \kappa x')[1+\delta(h'y', \kappa y')]}{1+\delta(\kappa x', \kappa y')}\right), \beta_2\left(\frac{\delta(\kappa y', h'x')[1+\delta(h'x', \kappa y')]}{1+\delta(\kappa x', \kappa y')}\right)\left(\frac{\delta(\kappa y', h'x')[1+\delta(h'x', \kappa y')]}{1+\delta(\kappa x', \kappa y')}\right), \beta_2(\delta(\kappa x', \kappa y')).(\delta(\kappa x', \kappa y'))\right\}$$

$$\beta_2\left(\frac{\delta(\kappa x', h'y')+\delta(\kappa y', \kappa x')}{2}\right)\left(\frac{\delta(\kappa x', h'y')+\delta(\kappa y', \kappa x')}{2}\right)\}$$

$$N(\kappa x', \kappa y') = \min\left\{\beta_2\left(\frac{\delta(\kappa x', h'x')\delta(h'x', \kappa y')}{\delta(\kappa x', \kappa y')}\right)\left(\frac{\delta(\kappa x', h'x')\delta(h'x', \kappa y')}{\delta(\kappa x', \kappa y')}\right), \beta\left(\frac{\delta(\kappa y', h'x')[1+\delta(h'x', \kappa y')]}{1+\delta(\kappa x', \kappa y')}\right)\left(\frac{d(\kappa y', h'x')[1+d(h'x', \kappa y')]}{1+d(\kappa x', \kappa y')}\right), \beta_2(\delta(\kappa x', \kappa y')).(\delta(\kappa x', \kappa y'))\right\}$$

$$\beta_2(\delta(\kappa y', h'y')).(\delta(\kappa y', h'y'))\}.$$

Corollary 3.1. Let (M'', δ') be a C M S a directed graph $G_1 = (V_1(G_1), E_1(G_1))$, let δ_1 be additional metric on M , and let $h'', \kappa' : M'' \rightarrow M''$ be functions. Assume (h'', κ') is an β_2 -weak rational contraction with respect to d , Further, assume that the following conditions satisfied:

(i) $M'' : (M'', \delta'_1) \rightarrow (M'', \delta_1)$ is a continuous $S(X)$ is δ_1 -closed;

(ii) $h''(X) \subset \kappa'(X)$;

(iii) $E_2(G_2)$ is transitive ;

- (iv) If $\delta \not\geq \delta_1$, $\alpha : (M'', \delta) \rightarrow (M, \delta_1)$ is G -Cauchy on M ;
- (v) $f : (M'', \delta) \rightarrow (M'', \delta_1)$ is G -continuous, and h'' and κ' are δ_1 -compatible, then h'' and κ' have points of correspondence.

When $L = 0$, we get

Definition 3.2 Let (M'', δ_1) be a MSP with a diGraph $G_1 = (V_1(G_1), E_1(G_1))$, and let $h'', \kappa' : M'' \rightarrow M''$ be functions. h'', κ' is said to be $\beta_2 - \psi_2$ -rational contraction with respect to d if

- (1) h'' is κ' -edge preserving with respect to G_2 ;
- (2) there exists two functions $\beta_2 \in S$ and $\psi \in \Psi$ such that, for all $x'', y'' \in X$ with $(\kappa'x'', \kappa'y'') \in E(G)$, we have

$$\begin{aligned} & \psi(d(\kappa'x'', \kappa'y'')) \leq \beta_2(\psi_2(P(\kappa'x'', \kappa'y''))\psi_2(P(\kappa'x'', \kappa'y''))), \\ & \text{where } \kappa' : M'' \times M'' \rightarrow [0, \infty) \text{ is a function such that, for any } x'', y'' \in M, \\ & P(\kappa'x'', \kappa'y'') = \max\left\{\beta_2\left(\frac{\delta_1(\kappa'x'', h''x'')d(h''y'', \kappa'y'')}{\delta_1(\kappa'x'', \kappa'y'')}\right) \cdot \left(\frac{\delta_1(\kappa'x'', h''x'')\delta_1(h''y'', \kappa'y'')}{\delta_1(\kappa'x'', \kappa'y'')}\right), \right. \\ & \beta_2\left(\frac{\delta_1(\kappa'y'', h''y'')[1+\delta_1(\kappa'x'', h''x'')]}{1+\delta_1(\kappa'x'', \kappa'y'')}\right) \cdot \left(\frac{\delta_1(\kappa'y'', h''y'')[1+\delta_1(\kappa'x'', h''x'')]}{1+\delta_1(\kappa'x'', \kappa'y'')}\right), \\ & \beta_2\left(\frac{\delta_1(h''x'', \kappa'x'')[1+\delta_1(h''y'', \kappa'y'')]}{1+\delta_1(\kappa'x'', \kappa'y'')}\right) \cdot \left(\frac{\delta_1(h''x'', \kappa'x'')[1+\delta_1(h''y'', \kappa'y'')]}{1+d(\kappa'x'', \kappa'y'')}\right), \\ & \beta_2\left(\frac{\delta_1(\kappa'y'', h''y'')[1+\delta_1(h''x'', \kappa'x'')]}{1+\delta_1(\kappa'x'', \kappa'y'')}\right) \cdot \left(\frac{\delta_1(\kappa'y'', h''y'')[1+\delta_1(h''x'', \kappa'x'')]}{1+\delta_1(\kappa'x'', \kappa'y'')}\right), \beta_2(\delta_1(\kappa'x'', \kappa'y'')) \cdot (\delta_1(\kappa'x'', \kappa'y'')), \\ & \left. \beta_2\alpha\left(\frac{\delta_1(\kappa'x'', h''y'')+\delta_1(\kappa'y'', h''x'')}{2}\right) \cdot \left(\frac{\delta_1(\kappa'x'', h''y'')+\delta_1(\kappa'y'', h''x'')}{2}\right)\right\} \end{aligned}$$

Corollary 3.2. Let (X'', δ') be a CMS a directed graph $G_1 = (V_1(G_1), E_1(G_1))$, let δ'' be additional metric on X , and let $h'', \kappa' : X' \rightarrow X'$ be functions. Assume (F', G') is an $\beta_2 - \psi_2$ -rational contraction with respect to d . Further, assume that the following conditions satisfied:

- (i) $K : (M', \delta'') \rightarrow (M', \delta')$ is a continuous, $h''(X)$ is δ'' -closed;
- (ii) $h''(X) \subset \kappa'(X)$;
- (iii) $E_1(G_1)$ is transitive;
- (iv) If $\delta' \not\geq \delta''$, $h : (M', \delta') \rightarrow (M', \delta'')$ is G -Cauchy on m ;
- (v) $h' : (M', \delta') \rightarrow (M', \delta'')$ is G -continuous, and h'' and κ' are δ'' -compatible, then h'' and κ' have a points of correspondence.

Now we present an example

Example 3.3. Let $M' : [0, 1]$ and define the metrics $\delta', \delta'' : M \rightarrow M$ defined by $d(s, t) = |s - t|, d'(s, t) = l|s - t| \forall s, t \in M$, where $l > 1$. Clearly $d < d'$.

Now we define $E(G) = \{(s, t)/s = t \text{ or } s, t \in [0, 1] \text{ with } s \leq t\}$ Define the mappings $h : M \rightarrow M$ and $K : M \rightarrow M$ by $h(s) = \frac{s}{2}$ and $K(s) = s \forall s \in M$.

Define $\psi(t) = \frac{t}{2}$, for all t , and define $\beta : [0, \infty) \rightarrow [0, 1)$ by $\beta(t) = \frac{1+t}{1+2t}, t \geq 0$

Let $(Ks, Kt) \in E(G)$, if $s = t$ then $(hs, ht) \in E(G)$

If $(Ks, Kt) \in E(G)$ with $Ks \leq Kt$, then we get $Ks = s \leq Kt = t$

$hs = \frac{s}{2} \leq Kt \leq \frac{t}{2}$ and $hs, ht \in [0, 1]$ that implies $hs, ht \in E(G)$

Let s, t be arbitrary in M and $(Ks, Kt) \in E(G)$

If $Ks = Kt$ then we get $s = t$, inequality in Theorem 2.1 holds.

Suppose $Ks = s, Kt = t$ in $[0, 1]$.

$$\psi(\delta'(hs, ht)) = \psi(\delta'(\frac{s}{2}, \frac{t}{2})) = \psi(|\frac{s}{2} - \frac{t}{2}|) = |\frac{s-t}{4}|$$

$$\begin{aligned} P(Ks, Kt) = \max\left\{ \beta\left(\frac{\delta'(s, \frac{s}{2})\delta'(\frac{t}{2}, t)}{\delta'(s, t)}\right) \cdot \frac{\delta'(s, \frac{s}{2})\delta'(\frac{t}{2}, t)}{\delta'(s, t)}, \beta\left(\frac{\delta'(t, \frac{t}{2})[1+\delta'(s, \frac{s}{2})]}{1+\delta'(s, t)}\right) \cdot \frac{\delta'(t, \frac{t}{2})[1+\delta'(s, \frac{s}{2})]}{1+\delta'(s, t)}, \right. \\ \left. \beta\left(\frac{\delta'(\frac{s}{2}, s)[1+\delta'(\frac{t}{2}, t)]}{1+\delta'(s, t)}\right) \cdot \frac{\delta'(\frac{s}{2}, s)[1+\delta'(\frac{t}{2}, t)]}{1+\delta'(s, t)}, \beta\left(\frac{\delta'(t, \frac{t}{2})[1+\delta'(\frac{s}{2}, t)]}{1+\delta'(s, t)}\right) \cdot \frac{\delta'(t, \frac{t}{2})[1+\delta'(\frac{s}{2}, t)]}{1+\delta'(s, t)}, \delta'(s, t), \frac{\delta'(s, \frac{s}{2})+\delta'(t, \frac{t}{2})}{2} \right\} \end{aligned}$$

$$P(Ks, Kt) \leq \max\left\{ \frac{\frac{s}{2} \cdot \frac{t}{2}}{\|s-t\|}, \frac{\frac{t}{2} \cdot [1+\frac{s}{2}]}{1+|s-t|}, \frac{\frac{s}{2}[1+\frac{t}{2}]}{1+|s-t|}, \frac{|t-\frac{s}{2}||1+|\frac{s}{2}-t||}{1+|s-t|}, |s-t|, \frac{|s-\frac{t}{2}|+|t-\frac{s}{2}|}{2} \right\}$$

$$P(Ks, Kt) = |s - t|$$

$$N(Ks, Ky) = \min\left\{\beta\left(\frac{\delta'(s, \frac{s}{2})\delta'(\frac{t}{2}, t)}{\delta'(s, t)}\right), \left(\frac{\delta'(s, \frac{s}{2})\delta'(\frac{t}{2}, t)}{\delta'(s, t)}\right), \beta\left(\frac{\delta'(t, \frac{s}{2})[1+\delta'(\frac{s}{2}, t)]}{1+\delta'(s, t)}\right), \left(\frac{\delta'(t, \frac{s}{2})[1+\delta'(\frac{s}{2}, t)]}{1+\delta'(s, t)}\right), \beta(\delta'(s, t)).(\delta'(s, t)), \beta(\delta'(t, \frac{t}{2})).(\delta'(t, \frac{t}{2}))\right\}$$

$$N(Ks, Kt) \leq \min\left\{\frac{\frac{s}{2}, \frac{t}{2}}{|s-t|}, \frac{|t-\frac{s}{2}|[1+|\frac{s}{2}-t|]}{1+|s-t|}, |s-t|, \frac{t}{2}\right\}$$

$$N(Ks, Kt) = \frac{|\frac{2t-s}{2}|[1+|\frac{s-2t}{2}|]}{1+|s-t|}$$

$$\psi(\delta'(Ks, Kt)) \leq \beta(\psi(P(Ks, Kt)))\psi(P(Ks, Kt)) + L.N(Ks, Kt)$$

$$|\frac{s-t}{4}| \leq \beta(\psi(|s-t|))\psi(|s-t|) + L \cdot \frac{|\frac{2t-s}{2}|[1+|\frac{s-2t}{2}|]}{1+|s-t|}$$

$$|\frac{s-t}{4}| \leq \left(\frac{1+|\frac{s-t}{2}|}{1+2\frac{|s-t|}{2}}\right)\frac{|s-t|}{2} + L \cdot \frac{|\frac{2t-s}{2}|[1+|\frac{s-2t}{2}|]}{1+|s-t|} \text{ holds for some } L \geq 0.$$

4. Applications in Machine Learning

Applications of Graph Structures for Attractions with Changing Distance Functions in Machine Learning In this paper, we investigate whether Geraghty generalized rational Ćirić-type weak contraction mappings have common fixed points in the outline of metric spaces with graph structures and changing distance functions. We extend classical fixed point solutions to a more suitable and broader context by utilizing the elasticity of changing distance functions and the connectivity encoded by graph structures. The broad contractions taken into consideration here provide a more comprehensive framework that is beneficial for a larger class of non-linear mappings.

These theoretical developments are not just theoretical; they have practical applications in deep learning and machine learning, particularly in determining equilibrium points in high-dimensional optimization problems, verifying stability in dynamic systems, and carefully understanding the convergence behavior of iterative algorithms. Deeper analysis and the development of more reliable learning models are made possible by the relationship between fixed point theory and computational intelligence. These findings have potential uses in deep learning and machine learning, primarily in understanding the behavior of intricate algorithms and producing stable points in iterative procedures.

5. Conclusion

In this paper, we have defined the new concept of Ćirić type Geraghty ψ weak rational contraction with respect to d also, we develop using Graph Structure to exhibit the presence fixed points of ciric rational type-weak Geraghty contraction maps with altering distance functions, provided some corollaries and examples.

References

- [1] Alfuraidan, M. R. (2016). The contraction principle for multivalued mappings on a modular metric space with a graph. *Canadian Mathematical Bulletin*, 59(1), 3-12. [1](#)
- [2] Alfuraidan, M. R. (2015). Remarks on monotone multivalued mappings on a metric space with a graph. *Journal of Inequalities and Applications*, 2015(1), 202. [1](#)
- [3] Babu, G. V. R., Sarma, K. K. M., Krishna, P. H. (2014). Fixed points of ψ -weak Geraghty contractions in partially ordered metric spaces. *J. Adv. Res. Pure Math*, 6, 9-23. [1](#)
- [4] Babu, G. V. R., Sarma, K. K. M., Kirshna, P. H., Kumari, V. A., Satyanarayana, G., Kumar, P. S. (2017). Common fixed points of a pair of selfmaps satisfying certain weakly contractive inequality involving rational type expressions via two auxiliary functions in partially ordered metric spaces. *Facta Universitatis, Series: Mathematics and Informatics*, 329-352. [1](#)
- [5] Beg, I., Butt, A. R., Radojevi, S. (2010). The contraction principle for set valued mappings on a metric space with a graph. *Computers and Mathematics with Applications*, 60(5), 1214-1219. [1](#)

- [6] Bojor, F. (2012). Fixed point theorems for Reich type contractions on metric spaces with a graph. *Nonlinear Analysis: Theory, Methods Applications*, 75(9), 3895-3901. [1](#)
- [7] Charoensawan, P., Atiponrat, W. (2017). Common fixed point and coupled coincidence point theorems for Geraghtys type contraction mapping with two metrics endowed with a directed graph. *Journal of Mathematics*, 2017(1), 5746704. [1](#), [1.2](#), [1](#)
- [8] Gordji, M. E., Ramezani, M., Cho, Y. J., Pirbavafa, S. (2012). A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations. *Fixed point theory and Applications*, 2012(1), 74. [1](#), [1](#)
- [9] Harikrishna, P., Tummala, K., Murthy, A. S., Jayababu, Y., Nageswara Rao, T., (2025). *Communications on Applied Nonlinear Analysis*, 32(4). [1](#), [1](#)
- [10] Jachymski, J. (2008). The contraction principle for mappings on a metric space with a graph. *Proceedings of the American Mathematical Society*, 136(4), 1359-1373. [1](#), [1.1](#), [1](#)
- [11] Jungck, G. (1986). Compatible mappings and common fixed points. *International journal of mathematics and mathematical sciences*, 9(4), 771-779. [1](#)
- [12] Khan, M. S., Swaleh, M., Sessa, S. (1984). Fixed point theorems by altering distances between the points. *Bulletin of the Australian Mathematical Society*, 30(1), 1-9. [1](#)
- [13] Tummala, K., Murthy, A. S., Ravindranath, V., Harikrishna, P., Suryanarayana, N. V. V. S. (2022). Common Fixed points of Geraghty generalized rational type weak contraction maps with altering distance functions via Geaph Structures, *Journal of Northereaterian University*, 25 (4). [1](#)
- [14] Tummala, K., Murthy, A. S., Ravindranath, V., Harikrishna, P., Suryanarayana, N. V. V. S. (2021, December). Fixed Points of Weak-Generalized Rational Type Contraction via Graph Structure. In *International Conference on Energy Systems, Drives and Automations* (pp. 559-566). Singapore: Springer Nature Singapore. [1](#)
- [15] Wongsaijai, B., Charoensawan, P., Suebcharoen, T., Atiponrat, W. (2021). Common fixed point theorems for auxiliary functions with applications in fractional differential equation. *Advances in Difference Equations*, 2021(1), 503. [1](#), [1](#), [1.3](#)