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Computational approaches to time scale dynamic equations using the Elzaki transform

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Abstract

Integral transform methods are effective techniques for addressing a range of dynamic equations characterized by initial or boundary value conditions, frequently expressed in the form of integral equations. This article presents the ET on an arbitrary timescale $\mathbb{T}$ as a new integral transform for addressing specific problems. The ET on timescales appears absent from the existing literature. This new approach primarily unifies discrete and continuous analysis, allowing for the treatment of differential, difference, and q-difference equations within a singular framework. This study's results pertain to ordinary differential equations for $\mathbb{T} = \mathbb{R}$, difference equations for $\mathbb{T} = \mathbb{N}_0$, and q-difference equations for $\mathbb{T} = q^{\mathbb{N}_0}$, where $q^{\mathbb{N}_0} = \{q^t | t \in \mathbb{N}_0 \text{ for } q > 1\}$ which hold significant relevance in quantum theory. The proposed transform can be applied to various nonstandard timescales, including $\mathbb{T} = h\mathbb{N}_0$, $\mathbb{T} = \mathbb{N}_0^2$, and $\mathbb{T} = \mathbb{T}_n$, which represent the space of harmonic numbers. Numerous examples and applications illustrate the efficacy of the ET on timescales in addressing dynamic equations.

Keywords: Elzaki transform, Dynamic Equation, Time-scale

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1. Introduction

In the literature, various integral transforms find extensive applications in physics, astronomy, and engineering. Elzaki [13, 14] introduced a novel integral transform, called the Elzaki Transform (ET), defined as follows. For a function $\xi(\phi)$ of exponential order, the ET is the transformed function $\omega = \omega(p)$, given by:

$$\omega = \mathcal{E}[\xi(\phi); p] = p \int_0^\infty e^{-\phi/p} \xi(\phi)\, d\phi, \quad p \in (-\tau_1, \tau_2), \tag{1.1}$$

where $\xi(\phi)$ is the original function defined on $\phi \geqslant 0$, p is the transform parameter belonging to a real interval $(-\tau_1, \tau_2)$ with $\tau_1, \tau_2 > 0$, and $\omega = \omega(p)$ denotes the Elzaki transform of $\xi(\phi)$, i.e., the image of ξ in the transform domain. This transform is derived from the classical Laplace transform by applying the

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substitution $p \to \frac{1}{p}$ and subsequently multiplying by p. The construction originates from the well-known Laplace transform [34, 12, 10], which for a function $\xi(\phi)$ of exponential order is defined as

$$L[\xi(\phi);p] = \int_0^\infty e^{-p\phi}\,\xi(\phi)\,d\phi. \quad \Re(p) > c \tag{1.2}$$

where c is the abscissa of absolute convergence of ξ.

Later, basic properties of the ET were developed in [15, 30, 32]. These properties highlight the unique characteristics of the ET, making it suitable for complex applications in science and engineering. The transform has been extended to degenerate cases and further studied in [24, 31]. Recently, Khalid et al. [25, 35] applied this transform to solve fractional differential equations. For recent developments in fractional integral transforms including the Elzaki transform applied to pathway fractional integrals and extended hypergeometric functions, see [2]. In contrast, in [27, 32], fuzzy fractional and partial differential equations were analytically solved using the *fuzzy ET*.

The application of the modified Sumudu transform, or the ET, to PDEs, ODEs, systems of ODEs and PDEs, and integral equations was demonstrated by Elzaki et al. in [5, 20, 21]. The ET can be utilized efficiently when Sumudu and Laplace transformations fail to solve DEs with variable coefficients [16], and in [17, 22], some engineering and biological applications, where the ET's effectiveness in finding exact solutions is discussed, specifically when mentioning generalized special functions and Mittag-Leffler type results [3], where multivariate special functions and their integral transforms appear in the context of fractional calculus applications [23]. An advantageous feature of the ET is its computational efficiency compared to traditional methods, without compromising numerical accuracy. Notably, the original function and its ET share identical Taylor coefficients, except for a factor of $i!$. Specifically, if $\xi(\phi) = \sum_{i=0}^{\infty} a_i \phi^i$, then $\omega = \sum_{i=0}^{\infty} i! a_i p^i$, see [33]. Furthermore, Dirac delta and Heaviside function of the Laplace and ETs are given by:

$$\mathcal{L}\{\delta(t)\} = 1, \qquad \mathcal{L}\{H(t)\} = \frac{1}{p}, \tag{1.3}$$

and

$$\mathcal{E}\{\delta(t)\} = p, \qquad \mathcal{E}\{H(t)\} = p^2. \tag{1.4}$$

This study introduces the *ET on time scale* and demonstrates its utility in solving systems of dynamic and integral equations. Let $\mathbb{T}$ denote a timescale with $\sup \mathbb{T} = \infty$, and fix $\phi_0 \in \mathbb{T}$. Suppose it $z \in \mathcal{R}$ is a regressive function, then $\ominus z \in \mathcal{R}$, and the exponential function $e_{\ominus z}(\phi, \phi_0)$ is well-defined.

The Laplace transform on a timescale for a function $\xi : \mathbb{T} \to \mathbb{R}$ was defined by,

$$\mathcal{L}\{\xi\}(z) := \int_{\phi_0}^{\infty} \xi(\phi) e^{\sigma}_{\ominus z}(\phi, \phi_0)\Delta\phi, \quad z \in \Omega(\xi), \tag{1.5}$$

where $\Omega(\xi)$ is the set of all complex numbers $z \in \mathbb{R}$ for which the improper integral exists (see [4, 7, 8, 9, 11, 18]).

Motivated by the efficiency of the ET in continuous domains and its extensions to fractional and distributional settings, this paper proposes the definition and properties of a new integral transform, the *ET on timescales*. This framework aims to unify discrete, continuous, and quantum calculus-based systems, enhancing the toolkit for solving linear and nonlinear dynamic equations [36, 28] across arbitrary time domains.

The ET on timescales offers several concrete advantages over the Laplace transform on timescales. First, the ET produces simpler algebraic expressions when incorporating initial conditions, since initial values appear multiplied by positive powers of the transform parameter p, facilitating partial fraction decomposition in the inversion step. Second, the ET converts Volterra-type integral equations on timescales directly into algebraic equations, as demonstrated in Example 3.4, reducing computational complexity compared to the Laplace-based approach. Third, the ET unifies computations across continuous, discrete, and quantum timescales within a single consistent framework. Fourth, the original function and its ET share identical Taylor coefficients up to a factor of i! [33], a property not shared by the Laplace transform, enabling series-based computation without evaluating improper integrals. These advantages collectively justify the development of the ET on timescales as a complementary and, in several respects, superior computational tool to the Laplace transform on timescales.

The present work substantially extends that contribution by:

1. establishing rigorous convergence conditions via the Hilger circle framework (Theorem 2.5);
2. proving the convolution theorem on arbitrary timescales (Theorem 2.19);
3. developing the complete differentiation and integration operational calculus (Theorems 2.11 and 2.12);
4. solving systems of dynamic equations and integral equations on timescales (Section 3); and
5. treating generalized exponential and trigonometric function transforms under constant graininess (Section 3.3). The Laplace transform on timescales treated in [19] provides the foundational motivation for the ET on timescales as a computationally superior alternative, as detailed in the advantages outlined above.

2. The Principal Themes

Definition 2.1. [11] *Exponential type I* refers to a function $\xi : \mathbb{T} \to \mathbb{R}$ with positive constants M and c such that,

$$|\xi(\phi)| \leqslant Me^{c\phi}, \quad \text{for all } \phi \in \mathbb{T}. \tag{2.1}$$

similarly, it ξ is said to be of *exponential type II* if there exist constants $M, c > 0$ for which

$$|\xi(\phi)| \leqslant Me_{c(\phi,\phi_0)}, \quad \text{for all } \phi \in \mathbb{T}, \tag{2.2}$$

there $\phi_0 \in \mathbb{T}$ is a fixed initial point.

Definition 2.2 (Elzaki Transform on Timescales)**.** Let $\xi : \mathbb{T}_0 \to \mathbb{R}$ be an rd-continuous function of exponential type II with exponential constant $c > 0$, i.e., there exists $M > 0$ such that

$$|\xi(\phi)| \leqslant M\, e_c(\phi, \phi_0), \quad \text{for all } \phi \in \mathbb{T}.$$

The Elzaki transform of ξ is defined by

$$\mathcal{E}\{\xi\}(p) := p \int_{\phi_0}^{\infty} \xi(\phi)\, e^{\sigma}_{\ominus \frac{1}{p}}(\phi, \phi_0)\, \Delta\phi, \tag{2.3}$$

for all $p \in D\{\xi\}$, where the domain of convergence is

$$D\{\xi\} = \left\{ p \in \mathbb{C} \setminus \{0\} : \ominus\frac{1}{p} \in \mathcal{R}^+ \text{ and } \mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(c) \text{ for all } \phi \in \mathbb{T} \right\},$$

and $\mathcal{R}^+$ denotes the set of positively regressive functions on $\mathbb{T}$.

Theorem 2.3 (Linearity Property)**.** *Let $\mathbb{T}$ be a timescale with $\sup \mathbb{T} = \infty$, and let $\xi, \tau : \mathbb{T} \to \mathbb{R}$ be rd-continuous functions of exponential type II with exponential constants $c_1, c_2 > 0$, respectively, so that*

$$|\xi(\phi)| \leqslant M_1\, e_{c_1}(\phi, \phi_0), \quad |\tau(\phi)| \leqslant M_2\, e_{c_2}(\phi, \phi_0), \quad \textit{for all } \phi \in \mathbb{T},$$

for some $M_1, M_2 > 0$. Suppose that the Elzaki transforms $\mathcal{E}\{\xi\}(p)$ and $\mathcal{E}\{\tau\}(p)$ exist for all $p \in D\{\xi\}$ and $p \in D\{\tau\}$, respectively, where

$$D\{\xi\} = \left\{ p \in \mathbb{C} \setminus \{0\} : \mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(c_1) \textit{ for all } \phi \in \mathbb{T} \right\},$$

$$D\{\tau\} = \left\{ p \in \mathbb{C} \setminus \{0\} : \mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(c_2) \textit{ for all } \phi \in \mathbb{T} \right\}.$$

Let $\alpha, \beta \in \mathbb{R}$ be constants. Then, for all $p \in D\{\xi\} \cap D\{\tau\}$, the following hold:

(i) *The linear combination* $\alpha\xi + \beta\tau$ *is rd-continuous and of exponential type II with exponential constant* $c = \max\{c_1, c_2\}$ *and bound* $M = |\alpha|M_1 + |\beta|M_2$.

(ii) *The Elzaki transform of* $\alpha\xi + \beta\tau$ *exists for all* $p \in D\{\xi\} \cap D\{\tau\}$, *and*

$$\mathcal{E}\{\alpha\xi + \beta\tau\}(p) = \alpha\, \mathcal{E}\{\xi\}(p) + \beta\, \mathcal{E}\{\tau\}(p). \tag{2.4}$$

Proof. Part (i). Since ξ and τ are of exponential type II with constants c_1 and c_2, respectively, we have for all $\phi \in \mathbb{T}$:

$$|(\alpha\xi + \beta\tau)(\phi)| \leqslant |\alpha|\, |\xi(\phi)| + |\beta|\, |\tau(\phi)| \leqslant |\alpha| M_1\, e_{c_1}(\phi, \phi_0) + |\beta| M_2\, e_{c_2}(\phi, \phi_0).$$

Since the timescale exponential satisfies $e_{c_i}(\phi, \phi_0) \leqslant e_c(\phi, \phi_0)$ for $c_i \leqslant c = \max\{c_1, c_2\}$ (cf. [7]), it follows that

$$|(\alpha\xi + \beta\tau)(\phi)| \leqslant M\, e_c(\phi, \phi_0), \quad M = |\alpha| M_1 + |\beta| M_2 > 0,$$

confirming that $\alpha\xi + \beta\tau$ is of exponential type II with constant c and bound M.

Part (ii). For all $p \in D\{\xi\} \cap D\{\tau\}$, the condition $\mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(c) = \max\{\mathrm{Re}_\mu(c_1), \mathrm{Re}_\mu(c_2)\}$ ensures, by Definition 2.2, that the integrals

$$\int_{\phi_0}^{\infty} |\xi(\phi)|\, \left| e^{\Delta}_{\ominus \frac{1}{p}}(\phi, \phi_0) \right| \Delta\phi \leqslant M_1 \int_{\phi_0}^{\infty} e_c(\phi, \phi_0)\, \left| e^{\Delta}_{\ominus \frac{1}{p}}(\phi, \phi_0) \right| \Delta\phi < \infty,$$

and similarly for τ, both converge absolutely. Hence, the delta integral is absolutely convergent for $\alpha\xi + \beta\tau$, and the constants α and β may be factored through the integral by linearity of the delta integral see [7] Theorem 1.77

$$\begin{aligned}
\mathcal{E}\{\alpha\xi + \beta\tau\}(p) &= p \int_{\phi_0}^{\infty} \big(\alpha\xi(\phi) + \beta\tau(\phi)\big)\, e^{\Delta}_{\ominus \frac{1}{p}}(\phi, \phi_0)\, \Delta\phi \\
&= \alpha \left(p \int_{\phi_0}^{\infty} \xi(\phi)\, e^{\Delta}_{\ominus \frac{1}{p}}(\phi, \phi_0)\, \Delta\phi \right) + \beta \left(p \int_{\phi_0}^{\infty} \tau(\phi)\, e^{\Delta}_{\ominus \frac{1}{p}}(\phi, \phi_0)\, \Delta\phi \right) \\
&= \alpha\, \mathcal{E}\{\xi\}(p) + \beta\, \mathcal{E}\{\tau\}(p),
\end{aligned}$$

which completes the proof. □

Now, we deduce that if $\mathbb{T}$ is a timescale with bounded graininess, i.e., $\exists\, \mu^* > 0$ such that $0 < \mu_* \leqslant \mu(\phi) \leqslant \mu^*$ for all $\phi \in \mathbb{T}$. Define the *Hilger circle* as follows (cf. [11]):

$$\mathbb{H}_t = \left\{ z \in \mathbb{C} : 0 < \left| z + \frac{1}{\mu(\phi)} \right| < \frac{1}{\mu(\phi)} \right\}. \tag{2.5}$$

With associated minimum and maximum Hilger regions:

$$\mathbb{H}_{\min} = \left\{ z \in \mathbb{C} : 0 < \left| z + \frac{1}{\mu^*} \right| < \frac{1}{\mu^*} \right\}, \quad \mathbb{H}_{\max} = \left\{ z \in \mathbb{C} : 0 < \left| z + \frac{1}{\mu_*} \right| < \frac{1}{\mu_*} \right\}, \tag{2.6}$$

where $\mu_* = \inf_{\phi \in \mathbb{T}} \mu(\phi) > 0$. Clearly, we observe that $\mathbb{H}_{\min} \subset \mathbb{H}_\phi \subset \mathbb{H}_{\max}$ for all $\phi \in \mathbb{T}$.

To identify the domain where the ET converges, let $c > 0$ and define:

$$D = \left\{ z \in \mathbb{C} : \mathrm{Re}_\mu\left(\frac{1}{z}\right) > \mathrm{Re}_\mu(c) \text{ for all } \phi \in \mathbb{T} \right\}. \tag{2.7}$$

This set may also be characterized as:

$$D = \left\{ z \in \mathbb{C} : \frac{1}{z} \in \overline{\mathbb{H}}^{c}_{max} \text{ and } \mathrm{Re}_{\mu_*}\left(\frac{1}{z}\right) > \mathrm{Re}_{\mu_*}(c) \text{ for all } \phi \in \mathbb{T} \right\}, \tag{2.8}$$

or equivalently,

$$D = \left\{ z \in \mathbb{C} : \ominus\frac{1}{z} \in \mathbb{H} \text{ and } \operatorname{Re}_{\mu}\left(\frac{1}{z}\right) > \operatorname{Re}_{\mu}(c) \text{ for all } \phi \in \mathbb{T} \right\}. \tag{2.9}$$

Here, $\overline{\mathbb{H}}^{c}_{max}$ refers to the complement of the closure of the largest Hilger circle determined by μ^*. It is worth noting that if $\mu^* = 0$, then D simplifies to a right half-plane in the complex plane, as discussed in [11].

Lemma 2.4. *If* $\ominus\frac{1}{z} \in \mathbb{H}$ *and* $\operatorname{Re}_{\mu}\left(\frac{1}{z}\right) > \operatorname{Re}_{\mu}(c)$ *for all* $\phi \in \mathbb{T}$*, then*

$$\frac{1}{\left|1+\frac{\mu(\phi)}{z}\right|} \leqslant 1 \quad \textit{and} \quad \left(\ominus\frac{1}{z} \oplus c\right) \in \mathbb{H}. \tag{2.10}$$

Proof. Since $\ominus\frac{1}{z} \in \mathbb{H}$, we have

$$\left|\ominus\frac{1}{z} + \frac{1}{\mu(\phi)}\right| < \frac{1}{\mu(\phi)}, \tag{2.11}$$

which implies

$$\frac{1}{\left|1+\frac{\mu(\phi)}{z}\right|} \leqslant 1. \tag{2.12}$$

Also, since $\operatorname{Re}_{\mu}\left(\frac{1}{z}\right) > \operatorname{Re}_{\mu}(c)$ implies

$$\left|1+\frac{\mu(\phi)}{z}\right| > |1+c\mu(\phi)|, \tag{2.13}$$

We observe that

$$\left|\left(\ominus\frac{1}{z} \oplus c\right) + \frac{1}{\mu(\phi)}\right| = \left|\frac{1+c\mu(\phi)}{\mu(\phi)\left(1+\frac{\mu(\phi)}{z}\right)}\right| < \frac{1}{\mu(\phi)}. \tag{2.14}$$

□

Theorem 2.5 (Domain of the Transform). *Let* $\xi : \mathbb{T} \to \mathbb{R}$ *be of exponential type II with constants* $M, \rho > 0$*, so that* $|\xi(\phi)| \leqslant M\, e_{\rho}(\phi, \phi_0)$ *for all* $\phi \in \mathbb{T}$*. Suppose it* $\mathbb{T}$ *has bounded graininess with* $\mu^* = \sup_{\phi\in\mathbb{T}} \mu(\phi) < \infty$*. Then the integral*

$$p\int_{\phi_0}^{\infty} \xi(\phi)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)\, \Delta\phi \tag{2.15}$$

converges absolutely for all p *in the domain*

$$D = \left\{ p \in \mathbb{C}\setminus\{0\} : \ominus\frac{1}{p} \in \mathcal{R}^{+} \textit{ and } \operatorname{Re}_{\mu}\left(\frac{1}{p}\right) > \operatorname{Re}_{\mu}(\rho) \textit{ for all } \phi \in \mathbb{T} \right\}.$$

Moreover, the absolute value of the integral is bounded by

$$\left|p\int_{\phi_0}^{\infty} \xi(\phi)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)\, \Delta\phi\right| \leqslant \frac{M(1+\mu^*)}{\alpha},$$

where

$$\alpha = \frac{1}{\mu^*}\log\left(\frac{1+\rho\mu^*}{1+|p|\mu^*}\right) > 0,$$

and, as a consequence,

$$\lim_{\phi\to\infty} e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)\, \xi(\phi) = 0, \quad \textit{for all } p \in D. \tag{2.16}$$

Proof. Since $\ominus\frac{1}{p} \in \mathcal{R}^+$ and $\mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(\rho)$, Lemma 2.4 gives:

$$\frac{1}{\left|1+\dfrac{\mu(\phi)}{p}\right|} \leqslant 1 \quad \text{and} \quad \left|\left(\ominus\frac{1}{p} \oplus \rho\right) + \frac{1}{\mu(\phi)}\right| < \frac{1}{\mu(\phi)}, \quad \forall \phi \in \mathbb{T}.$$

Using the exponential type II bound and the product identity for timescale exponentials:

$$\begin{aligned}\left|p \int_{\phi_0}^{\infty} \xi(\phi)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\, \Delta\phi\right| &\leqslant M \int_{\phi_0}^{\infty} \left|p\, e_\rho(\phi,\phi_0)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\right| \Delta\phi \\ &= M \int_{\phi_0}^{\infty} \left|1+\frac{\mu(\phi)}{p}\right| \cdot \left|e^{\sigma}_{\ominus\frac{1}{p}\oplus\rho}(\phi,\phi_0)\right| \Delta\phi \\ &\leqslant M(1+\mu^*) \int_{\phi_0}^{\infty} \left|e^{\sigma}_{\ominus\frac{1}{p}\oplus\rho}(\phi,\phi_0)\right| \Delta\phi.\end{aligned}$$

Since $\mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(\rho)$, there exists $\alpha > 0$ such that:

$$\left|e^{\sigma}_{\ominus\frac{1}{p}\oplus\rho}(\phi,\phi_0)\right| \leqslant e^{-\alpha(\phi-\phi_0)}, \quad \forall \phi \geqslant \phi_0,$$

where $\alpha = \frac{1}{\mu^*} \log\left(\frac{1+\rho\mu^*}{1+|p|\mu^*}\right) > 0$. Therefore:

$$M(1+\mu^*) \int_{\phi_0}^{\infty} e^{-\alpha(\phi-\phi_0)}\, d\phi = \frac{M(1+\mu^*)}{\alpha} < \infty,$$

which establishes absolute convergence and the stated bound. The limiting condition (2.16) follows immediately, since for all $\phi \geqslant \phi_0$:

$$\left|e^{\delta}_{\ominus\frac{1}{p}}(\phi,\phi_0)\, \xi(\phi)\right| \leqslant M \left|e^{\delta}_{\ominus\frac{1}{p}\oplus\rho}(\phi,\phi_0)\right| \leqslant M\, e^{-\alpha(\phi-\phi_0)} \to 0 \quad \text{as } \phi \to \infty.$$

This completes the proof. □

Theorem 2.6 (Relation Between the ET and the Laplace Transform). *Let $\xi : \mathbb{T} \to \mathbb{R}$ be an rd-continuous function of exponential type II with constant $\rho > 0$. For all $p \in D\{\xi\}$ satisfying $\mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(\rho)$, both the Elzaki transform $\mathcal{E}\{\xi\}(p)$ and the Laplace transform $\mathcal{L}\{\xi\}\left(\frac{1}{p}\right)$ exist, and*

$$\mathcal{E}\{\xi\}(p) = p\, \mathcal{L}\{\xi\}\left(\frac{1}{p}\right). \tag{2.17}$$

Proof. For $p \in D\{\xi\}$, the absolute convergence of the defining integral of $\mathcal{E}\{\xi\}(p)$ is guaranteed by Theorem 2.5. By Definition 2.2 and the definition of the Laplace transform on timescales (1.5):

$$\mathcal{L}\{\xi\}\left(\frac{1}{p}\right) = \int_{\phi_0}^{\infty} \xi(\phi)\, e^{\delta}_{\ominus\frac{1}{p}}(\phi,\phi_0)\, \Delta\phi = \frac{1}{p}\, \mathcal{E}\{\xi\}(p),$$

□

Theorem 2.7. *If $\mathbb{T} = \mathbb{N}_0$, then*

$$\mathcal{E}\{\xi\}(p) = (p+1)\mathcal{Z}\{\xi\}\left(\frac{1}{p}+1\right) \tag{2.18}$$

where $\mathcal{Z}\{\xi\}$ is the $\mathcal{Z}$ -transform of ξ, which is defined by

$$\mathcal{Z}\{\xi\}(u) = \sum_{\phi=0}^{\infty} \frac{\xi(\phi)}{u^{\phi}} \tag{2.19}$$

An infinite sum converges for complex values u.

Proof: Theorem 2.6 and the following relation lead to the proof.

$$(p+1)\mathcal{L}\{\xi\}(p) = \mathcal{Z}\{\xi\}(p+1). \tag{2.20}$$

Example 2.8. Consider $\xi(\phi) = 1$. We aim to determine its ET. According to Definition 2.2, the ET is given by:

$$\mathcal{E}\{1\}(p) = p\int_{\phi_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\,\Delta\phi. \tag{2.21}$$

Following exponential function properties on timescales, we proceed:

$$\begin{aligned}
\mathcal{E}\{1\}(p) &= p\int_{\phi_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\,\Delta\phi \\
&= p\int_{\phi_0}^{\infty} \left[\mu(\phi)e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0) + e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\right]\Delta\phi \\
&= p\int_{\phi_0}^{\infty} \left[\mu(\phi)\left(\ominus\frac{1}{p}\right)e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0) + e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\right]\Delta\phi \\
&= p\int_{\phi_0}^{\infty} \left[1+\mu(\phi)\left(\ominus\frac{1}{p}\right)\right]e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\,\Delta\phi \\
&= p^2\int_{\phi_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\,\Delta\phi \\
&= p^2\left[-e^{\sigma}_{\ominus\frac{1}{p}}(\phi,\phi_0)\right]_{\phi_0}^{\infty} \\
&= p^2.
\end{aligned}$$

Hence, the ET of the constant function is:

$$\mathcal{E}\{1\}(p) = p^2. \tag{2.22}$$

Example 2.9. Let us now compute the ET of the exponential timescale function $e_\alpha(\phi,\phi_0)$. The result is:

$$\mathcal{E}\{e_\alpha(\phi,\phi_0)\}(p) = \frac{p^2}{1-\alpha p}. \tag{2.23}$$

Proof. Utilizing the relation between the Elzaki and Laplace transforms on timescales, we find:

$$\begin{aligned}
\mathcal{E}\{e_\alpha(\phi,\phi_0)\}(p) &= p\cdot\mathcal{L}\{e_\alpha(\phi,\phi_0)\}\left(\frac{1}{p}\right) \\
&= p\cdot\frac{1}{\frac{1}{p}-\alpha} = \frac{p^2}{1-\alpha p}.
\end{aligned}$$

From this result, we can derive further ETs of hyperbolic and trigonometric functions on timescales:

$$\mathcal{E}\{\cosh_\alpha(\phi,\phi_0)\}(p) = \frac{\alpha p^2}{1-\alpha^2p^2}, \quad \mathcal{E}\{\sinh_\alpha(\phi,\phi_0)\}(p) = \frac{\alpha p^3}{1-\alpha^2p^2}, \tag{2.24}$$

$$\mathcal{E}\{\cos_\alpha(\phi,\phi_0)\}(p) = \frac{p^2}{1+\alpha^2p^2}, \quad \mathcal{E}\{\sin_\alpha(\phi,\phi_0)\}(p) = \frac{\alpha u^3}{1+\alpha^2p^2}. \tag{2.25}$$

These identities are useful when analyzing dynamic systems defined over arbitrary timescales.

Example 2.10. Consider the function $\xi(\phi) = \alpha^{\phi}$ defined on the timescale $\mathbb{T} = \mathbb{N}_0$. To compute its ET, we utilize the timescale exponential function and the graininess function.

Given:

$$e^{\sigma}_{-\frac{1}{p}}(\phi, 0) = \left(1 - \frac{1}{p}\right)^{\phi}, \quad \text{with} \quad \mu(\phi) = 1. \tag{2.26}$$

The ET is

$$\mathcal{E}\{\alpha^{\phi}\}(p) = p \sum_{\phi=0}^{\infty} \alpha^{\phi} \left(1 - \frac{1}{p}\right)^{\phi} = p \sum_{\phi=0}^{\infty} \left[\alpha\left(1 - \frac{1}{p}\right)\right]^{\phi}. \tag{2.27}$$

Define $r = \alpha\left(1 - \frac{1}{p}\right)$. Provided $|r| < 1$, the geometric series converges

$$\sum_{\phi=0}^{\infty} r^{\phi} = \frac{1}{1-r}, \tag{2.28}$$

and hence

$$\mathcal{E}\{\alpha^{\phi}\}(p) = \frac{p}{1-r} = \frac{p}{1 - \alpha + \frac{\alpha}{p}}. \tag{2.29}$$

Multiplying the numerator and denominator by p, we simplify to

$$\mathcal{E}\{\alpha^{\phi}\}(p) = \frac{p^2}{(1-\alpha)p + \alpha}. \tag{2.30}$$

Theorem 2.11 (Elzaki Transform of the n-th Δ-Derivative)**.** *Let $\xi : \mathbb{T}_0 \to \mathbb{R}$ be rd-continuous of exponential type II, and suppose it ξ^{Δ^k} exists and is of exponential type II for $k = 0, 1, \ldots, n$. Then for $p \in D\{\xi\}$:*

(i) ***First Δ-derivative:***

$$\mathcal{E}\left\{\xi^{\Delta}\right\}(p) = \frac{\mathcal{E}\{\xi\}(p)}{p} - p\,\xi(\phi_0)$$

(ii) ***Second Δ-derivative:***

$$\mathcal{E}\left\{\xi^{\Delta^2}\right\}(p) = \frac{\mathcal{E}\{\xi\}(p)}{p^2} - \xi(\phi_0) - p\,\xi^{\Delta}(\phi_0)$$

(iii) ***n-th Δ-derivative:***

$$\mathcal{E}\left\{\xi^{\Delta^n}\right\}(p) = \frac{\mathcal{E}\{\xi\}(p)}{p^n} - \sum_{k=0}^{n-1} p^{\,2-n+k}\,\xi^{\Delta^k}(\phi_0)$$

Theorem 2.12. *Let $h : \mathbb{T} \to \mathbb{C}$ be an rd-continuous function of exponential type II with constant $\rho > 0$, and define*

$$H(\phi) := \int_{\phi_0}^{\phi} h(\tau)\,\Delta\tau, \quad \phi \in \mathbb{T}. \tag{2.31}$$

Then H is also of exponential type II with constant ρ, and for all $p \in D\{h\}$ satisfying $\mathrm{Re}_{\mu}\left(\frac{1}{p}\right) > \mathrm{Re}_{\mu}(\rho)$, the vanishing condition

$$\lim_{\phi\to\infty} H(\phi)\,e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0) = 0 \tag{2.32}$$

holds, and the ET of $H(\phi)$ satisfies

$$\mathcal{E}\{H(\phi)\}(p) = \frac{1}{p}\,\mathcal{E}\{h(\phi)\}(p). \tag{2.33}$$

Proof. Since h is of exponential type II with constant $\rho > 0$, there exists $M_h > 0$ such that $|h(\phi)| \leqslant M_h\, e_\rho(\phi, \phi_0)$. For all $\phi \geqslant \phi_0$:

$$|H(\phi)| \leqslant \int_{\phi_0}^{\phi} |h(\tau)|\,\Delta\tau \leqslant M_h \int_{\phi_0}^{\phi} e_\rho(\tau, \phi_0)\,\Delta\tau \leqslant \frac{M_h}{\rho}\, e_\rho(\phi, \phi_0),$$

so H is of exponential type II with constant ρ and bound $\frac{M_h}{\rho}$. The vanishing condition (2.32) then follows from Theorem 2.5 applied to H.

By Definition 2.2 and integration by parts on timescales, setting $f(\phi) = -e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)$ so that $f^{\Delta}(\phi) = \frac{1}{p}\, e^{\delta}_{\ominus\frac{1}{p}}(\phi, \phi_0)$, and using $H^{\Delta}(f) = h(\phi)$, $H(\phi_0) = 0$, and (2.32):

$$\begin{aligned}\mathcal{E}\{H(\phi)\}(p) &= p\int_{\phi_0}^{\infty} H(\phi)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)\,\Delta\phi \\ &= p\left[-\frac{1}{p} H(\phi)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)\right]_{\phi_0}^{\infty} + \int_{\phi_0}^{\infty} h(\phi)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)\,\Delta\phi \\ &= p\cdot 0 + \int_{\phi_0}^{\infty} h(\phi)\, e^{\sigma}_{\ominus\frac{1}{p}}(\phi, \phi_0)\,\Delta\phi = \frac{1}{p}\,\mathcal{E}\{h(\phi)\}(p),\end{aligned}$$

which completes the proof. □

Remark 2.13. The result can alternatively be obtained by utilising the connection between the Elzaki and Laplace transforms on timescales, given by:

$$\mathcal{E}\{\xi(\phi)\}(p) = p\cdot\mathcal{L}\{\xi(\phi)\}\left(\frac{1}{p}\right). \tag{2.34}$$

Remark 2.14.

$$\mathcal{E}\left\{\int_0^{\phi} h(\tau)\,d\tau\right\}(p) = \frac{1}{p}\mathcal{E}\{h(\phi)\}(p). \tag{2.35}$$

Problem 1: Choose $h(\phi) = 1$. Then $H(\phi) = \int_0^{\phi} 1\,d\tau = \phi$. Applying the Elzaki rule:

$$\mathcal{E}\{\phi\}(p) = \frac{1}{p}\cdot\mathcal{E}\{1\}(p) = \frac{1}{p}\cdot p = 1. \tag{2.36}$$

Problem 2: Take $h(\phi) = \phi$. Then $H(\phi) = \int_0^{\phi} \tau\,d\tau = \frac{\phi^2}{2}$. Therefore,

$$\mathcal{E}\left\{\frac{\phi^2}{2}\right\}(p) = \frac{1}{p}\cdot\mathcal{E}\{\phi\}(p) = \frac{1}{p}\cdot 1 = \frac{1}{p}. \tag{2.37}$$

Multiplying both sides by 2 yields,

$$\mathcal{E}\{\phi^2\}(p) = \frac{2}{p}. \tag{2.38}$$

Using this recursive method:

$$\mathcal{E}\{\phi\}(p) = 1, \quad \mathcal{E}\{\phi^2\}(p) = \frac{2}{p}. \tag{2.39}$$

Theorem 2.15. *Let it $\xi : \mathbb{T} \to \mathbb{C}$ be rd-continuous and suppose its Laplace transform exists and is given by,*

$$\mathcal{L}\{\xi(\phi)\}(p) = F_L(p). \tag{2.40}$$

Then, for any $\alpha \in \mathbb{C}$, the following identity holds

$$\mathcal{L}\left\{e^{\sigma}_{\ominus\alpha}(\phi, \phi_0)\xi(\phi)\right\}(p) = F_L(p \oplus \alpha). \tag{2.41}$$

Proof: By the definition of the Laplace transform on a timescale$\mathbb{T}$, we have

$$\mathcal{L}\left\{e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\right\}(p)=\int_{\phi_0}^{\infty} e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\, e^{\sigma}_{\ominus p}(\phi,\phi_0)\,\Delta\phi. \tag{2.42}$$

Using the identity for exponential functions on timescales,

$$e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\cdot e^{\sigma}_{\ominus p}(\phi,\phi_0)=e^{\sigma}_{\ominus(p\oplus\alpha)}(\phi,\phi_0), \tag{2.43}$$

The above becomes,

$$\mathcal{L}\left\{e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\right\}(p)=\int_{\phi_0}^{\infty} \xi(\phi)\, e^{\sigma}_{\ominus(p\oplus\alpha)}(\phi,\phi_0)\,\Delta\phi = F_L(p\oplus\alpha). \tag{2.44}$$

From equations (2.17) and (2.41), additional properties of Laplace and ETs on timescales can be derived.

Theorem 2.16. *Let $\xi:\mathbb{T}\to\mathbb{C}$ be an rd-continuous function, and suppose that the ET $\xi(\phi)$ is related to its Laplace transform by*

$$\mathcal{E}\{\xi(\phi)\}(p)=p\cdot F_L\left(\frac{1}{p}\right), \tag{2.45}$$

where $\mathcal{E}$ and $\mathcal{L}$ denote the Elzaki and Laplace transforms on timescales, respectively, and $F_L(p)$ is the Laplace transform of $\xi(\phi)$. Then, the ET of the product involving the delta-exponential function satisfies

$$\mathcal{E}\left\{e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\right\}(p)=p\cdot F_L\left(\frac{1}{p}\oplus\alpha\right). \tag{2.46}$$

Proof: Given the relationship between the Elzaki and Laplace transforms, we begin with

$$\mathcal{E}\{\xi(\phi)\}(p)=p\cdot F_L\left(\frac{1}{p}\right). \tag{2.47}$$

Now consider the ET of the product $e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)$. By the definition of the ET in terms of the Laplace transform, we write,

$$\mathcal{E}\left\{e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\right\}(p)=p\cdot\mathcal{L}\left\{e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\right\}\left(\frac{1}{p}\right). \tag{2.48}$$

From Theorem 2.15, we know that,

$$\mathcal{L}\left\{e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\right\}\left(\frac{1}{p}\right)=F_L\left(\frac{1}{p}\oplus\alpha\right). \tag{2.49}$$

Substituting this into the earlier equation gives,

$$\mathcal{E}\left\{e^{\sigma}_{\ominus\alpha}(\phi,\phi_0)\xi(\phi)\right\}(p)=p\cdot F_L\left(\frac{1}{p}\oplus\alpha\right). \tag{2.50}$$

Definition 2.17. According to [11], the convolution of two functions ξ and ρ on a timescale is given by

$$(\xi*\rho)(\phi)=\int_{\phi_0}^{\phi}\hat{\xi}(\phi,\sigma(s))\,\rho(s)\,\Delta s,\quad \text{for } \phi\in\mathbb{T}_0, \tag{2.51}$$

where $\hat{\xi}(\phi,\sigma(s))$ denotes the forward shift of ξ, and the Laplace transform of the convolution satisfies the identity,

$$\mathcal{L}\left\{\xi*\rho\right\}(p)=\mathcal{L}\{\xi\}(p)\cdot\mathcal{L}\{\rho\}(p). \tag{2.52}$$

Remark 2.18. In the special case when $\mathbb{T} = \mathbb{R}$, the above definition reduces to the classical convolution integral:

$$(\xi * \rho)(\phi) = \int_{\phi_0}^{\phi} \xi(\phi - s)\, \rho(s)\, ds. \tag{2.53}$$

Theorem 2.19 (Convolution Theorem for the ET)**.** *[26] Let* ξ *and* ρ *be regulated functions on the timescale* $\mathbb{T}$*, both of exponential type II with constants* $c_1, c_2 > 0$*, respectively. Set* $c = \max\{c_1, c_2\}$*. Then, for all* $p \in D\{\xi\} \cap D\{\rho\}$ *satisfying* $\mathrm{Re}_\mu\left(\frac{1}{p}\right) > \mathrm{Re}_\mu(c)$*, the convolution* $\xi * \rho$ *is well-defined, of exponential type II with constant* c*, and the ET of the convolution satisfies:*

$$\mathcal{E}\{\xi * \rho\}(p) = p \cdot \mathcal{E}\{\xi\}(p) \cdot \mathcal{E}\{\rho\}(p). \tag{2.54}$$

Proof. For $p \in D\{\xi\} \cap D\{\rho\}$, the absolute convergence of $\mathcal{E}\{\xi\}(p)$ and $\mathcal{E}\{\rho\}(p)$ is guaranteed by Theorem 2.5. Using the relation between the Elzaki and Laplace transforms (Theorem 2.6):

$$\mathcal{E}\{\xi * \rho\}(p) = p \cdot \mathcal{L}\{\xi * \rho\}\left(\frac{1}{p}\right).$$

By the convolution property of the Laplace transform on timescales (Definition 2.17):

$$\mathcal{L}\{\xi * \rho\}\left(\frac{1}{p}\right) = \mathcal{L}\{\xi\}\left(\frac{1}{p}\right) \cdot \mathcal{L}\{\rho\}\left(\frac{1}{p}\right) = \frac{\mathcal{E}\{\xi\}(p)}{p} \cdot \frac{\mathcal{E}\{\rho\}(p)}{p}.$$

Substituting back:

$$\mathcal{E}\{\xi * \rho\}(p) = p \cdot \frac{\mathcal{E}\{\xi\}(p)}{p} \cdot \frac{\mathcal{E}\{\rho\}(p)}{p} = p \cdot \mathcal{E}\{\xi\}(p) \cdot \mathcal{E}\{\rho\}(p),$$

which completes the proof. □

3. Applications

3.1. Solution of Homogeneous Dynamic Equations with Constant Coefficients:

Consider the homogeneous linear dynamic equation of order n with constant coefficients[16],

$$\sum_{k=0}^{n} a_k \eta^{\Delta^k}(\phi) = 0. \tag{3.1}$$

Applying the ET on both sides and using Theorem 2.11 we obtain,

$$\sum_{k=0}^{n} a_k \mathcal{E}\{\eta^{\Delta^k}(\phi)\}(\sigma) = 0. \tag{3.2}$$

By the operational property of the ET for higher-order delta derivatives, we have,

$$\mathcal{E}\{\eta^{\Delta^k}(\phi)\}(\sigma) = \frac{\sigma^2}{T^k} \mathcal{E}\{\eta(\phi)\}(\sigma) - \sum_{j=0}^{k-1} \frac{\sigma^{k-j+1}}{T^{k-j}} \eta^{\Delta^j}(0). \tag{3.3}$$

Substituting this into the above equation gives,

$$\sum_{k=0}^{n} a_k \left(\frac{\sigma^2}{T^k} \mathcal{E}\{\eta(\phi)\}(\sigma) - \sum_{j=0}^{k-1} \frac{\sigma^{k-j+1}}{T^{k-j}} \eta^{\Delta^j}(0) \right) = 0. \tag{3.4}$$

Grouping terms, we rewrite the equation as,

$$\left(\sum_{k=0}^{n} \frac{a_k \sigma^2}{T^k} \right) \mathcal{E}\{\eta(t)\}(\sigma) = \sum_{k=1}^{n} a_k \sum_{j=0}^{k-1} \frac{\sigma^{k-j+1}}{T^{k-j}} \eta^{\Delta^j}(0). \tag{3.5}$$

Example 3.1. Consider the second-order dynamic equation,

$$\eta^{\Delta\Delta}(\phi) - \eta^{\Delta}(\phi) - 6\eta(\phi) = 0, \quad \eta(0) = 1, \quad \eta^{\Delta}(0) = -7. \tag{3.6}$$

Here, $a_2 = 1$, $a_1 = -1$, and $a_0 = -6$. Applying the Elazi transform,

$$\left(\frac{\sigma^2}{T^2} - \frac{\sigma^2}{T} - 6\sigma^2\right) \mathcal{E}\{\eta(\phi)\}(\sigma) = -\frac{\sigma^2}{T} \cdot \eta(0) + \frac{\sigma^3}{T^2} \cdot \eta(0) + \frac{\sigma^2}{T} \cdot \eta^{\Delta}(0). \tag{3.7}$$

using, $\eta(0) = 1$, $\eta^{\Delta}(0) = -7$, we get:

$$\left(\frac{\sigma^2}{T^2} - \frac{\sigma^2}{T} - 6\sigma^2\right) \mathcal{E}\{\eta(t)\}(\sigma) = -\frac{\sigma^2}{T} + \frac{\sigma^3}{T^2} - \frac{7\sigma^2}{T}. \tag{3.8}$$

Simplify the right-hand side,

$$\mathcal{E}\{\eta(\phi)\}(\sigma) = \frac{\sigma^3 - 7T\sigma^2 - T^2\sigma^2}{\sigma^2\left(\frac{1}{T^2} - \frac{1}{T} - 6\right)}. \tag{3.9}$$

Factoring the numerator and simplifying,

$$\mathcal{E}\{\eta(\phi)\}(\sigma) = \frac{\sigma(\sigma - 7T - T^2)}{1 - T - 6T^2}. \tag{3.10}$$

We aim to invert the ET. Let us express the result as a sum of simpler terms:

$$\mathcal{E}\{\eta(\phi)\}(\sigma) = \frac{-1}{1 - 3T\sigma} + \frac{2}{1 + 2T\sigma}. \tag{3.11}$$

Applying the inverse ET,

$$\eta(\phi) = -e_3(\phi, 0) + 2e_{-2}(\phi, 0). \tag{3.12}$$

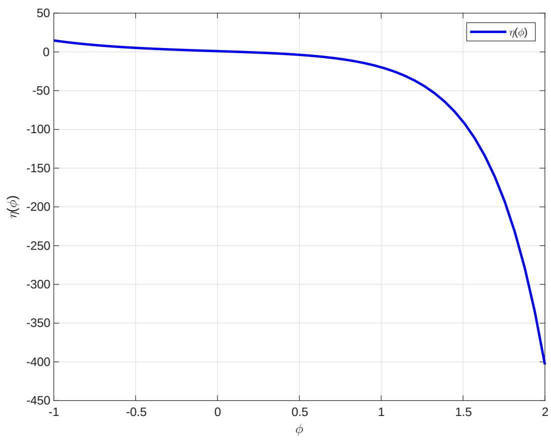

Figure 1: 2-dimensional plot of solution $\eta(\phi)$ of Example 3.1

Remark 3.2. 1. For the timescale $\mathbb{T} = \mathbb{R}$, the dynamic equation reduces to the classical differential equation:

$$\eta''(\phi) - \eta'(\phi) - 6\eta(\phi) = 0, \quad \eta(0) = 1, \quad \eta'(0) = -7. \tag{3.13}$$

The explicit solution is given by

$$\eta(\phi) = -e^{3\phi} + 2e^{-2\phi}. \tag{3.14}$$

2. When the timescale is $\mathbb{T} =$, the equation takes the form of a second-order difference equation:

$$\Delta^2\eta(\phi) - \Delta\eta(\phi) - 6\eta(\phi) = 0, \quad \eta(0) = 1, \quad \Delta\eta(0) = -7, \tag{3.15}$$

where Δ denotes the forward difference operator.

3. For the quantum timescale $\mathbb{T} = q\mathbb{N}_0 \cup \{0\}$, the equation becomes a second-order q-difference equation:

$$\Delta_q^2\eta(\phi) - \Delta_q\eta(\phi) - 6\eta(\phi) = 0, \quad \eta(0) = 1, \quad \Delta_q\eta(0) = -7, \tag{3.16}$$

with the q-difference operator defined by

$$\Delta_q\eta(\phi) = \frac{\eta(q\phi) - \eta(\phi)}{(q-1)\phi}, \quad \phi \geqslant 1. \tag{3.17}$$

Utilizing the solution form from equation (3.3), the solution values satisfy

$$\eta(1) = -6, \tag{3.18}$$

and for $\phi > 1$,

$$\eta(\phi) = -\prod_{s\in[1,\phi)} [1 + 3(q-1)s] + 2\prod_{s\in[1,\phi)} [1 - 2(q-1)s]. \tag{3.19}$$

3.2. System of Dynamic Equations with Constant Coefficients:

Consider the system of equations of the form:

$$P(D)\eta = \omega, \tag{3.20}$$

where

$$P(D) = \begin{bmatrix} P_{11}(D) & \cdots & P_{1n}(D) \\ \vdots & \ddots & \vdots \\ P_{n1}(D) & \cdots & P_{nn}(D) \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}, \quad \omega = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \tag{3.21}$$

and

$$P_{ij}(D) = \sum_{k=0}^{n_i} a_{ik}D^k, \quad \text{with } D^k\eta = \eta^{\Delta^k}. \tag{3.22}$$

Analogous to equation (3.1), we apply the ET to both sides:

$$\mathcal{E}\{P(D)\eta\}(p) = \left(\mathcal{E}\{P_{ij}(D)\eta_j\}(p)\right) = \mathcal{E}\{\omega\}(p). \tag{3.23}$$

We apply the linearity of the ET:

$$\mathcal{E}\left\{\sum_{k=0}^{n_i} a_{ik}\eta_j^{\Delta^k}(\phi)\right\}(p) = \sum_{k=0}^{n_i} a_{ik}\mathcal{E}\{\eta_j^{\Delta^k}(\phi)\}(p). \tag{3.24}$$

Using the ET of delta derivatives, we have,

$$\mathcal{E}\{\eta_j^{\Delta^k}(\phi)\}(p) = \frac{N_k(p)}{p^k}\eta_j(p) - \text{initial condition terms}, \tag{3.25}$$

where $\eta_j(p) = \mathcal{E}\{\eta_j(\phi)\}(p)$ and $N_k(p)$ denotes the transform coefficients derived from the kernel $K(\phi, p)$ of the ET on the given timescale.

Substituting, the transformed system becomes:

$$\mathcal{P}(p)\eta(p) - \text{initial condition vector} = \omega, \tag{3.26}$$

where:

- $\eta(p) = \begin{bmatrix} \eta_1(p) \\ \vdots \\ \eta_n(p) \end{bmatrix}$,
- $\omega = \begin{bmatrix} \mathcal{E}\{\xi_1\}(p) \\ \vdots \\ \mathcal{E}\{\xi_n\}(p) \end{bmatrix}$,
- $\mathcal{P}(p)$ is a matrix with entries $\mathcal{P}_{ij}(p) = \sum_{k=0}^{n_i} a_{ik} \frac{N_k(p)}{p^k}$.

We solve for $\eta(p)$ by,

$$\eta(p) = \mathcal{P}(p)^{-1} [\omega + \text{initial condition vector}]. \tag{3.27}$$

Finally, apply the inverse ET to retrieve the time-domain solution,

$$\eta_j(\phi) = \mathcal{E}^{-1}\{\eta_j(p)\}(\phi), \quad j = 1, 2, \dots, n. \tag{3.28}$$

Example 3.3. Consider the system of dynamic equations

$$\begin{aligned} -m^{\Delta\Delta}(\phi) + 3n^{\Delta}(\phi) + 2m(\phi) &= 2e_3(\phi, 0), \quad m(0) = 1, \quad m^{\Delta}(0) = 5, \\ n^{\Delta\Delta}(\phi) - 4m^{\Delta}(\phi) + 3n(\phi) &= -9\cos_2(\phi, 0), \quad n(0) = 2, \quad n^{\Delta}(0) = 3. \end{aligned}$$

Using the ET $\mathcal{E}\{\cdot\}$ and the properties,

$$\mathcal{E}\{\xi^{\Delta}(\phi)\} = \frac{\omega}{p} - \xi(0), \quad \mathcal{E}\{\xi^{\Delta\Delta}(\phi)\} = \frac{\omega}{p^2} - \frac{\xi(0)}{p} - \xi^{\Delta}(0), \tag{3.29}$$

along with,

$$\mathcal{E}\{e_a(\phi, 0)\} = \frac{p}{p-a}, \quad \mathcal{E}\{\cos_a(\phi, 0)\} = \frac{p}{p^2 + a^2}, \tag{3.30}$$

We transform each equation,

$$\begin{aligned} -\left(\frac{M(p)}{p^2} - \frac{m(0)}{p} - m^{\Delta}(0)\right) + 3\left(\frac{N(p)}{p} - n(0)\right) + 2M(p) &= 2 \cdot \frac{p}{p-3}, \\ \left(\frac{N(p)}{p^2} - \frac{n(0)}{p} - n^{\Delta}(0)\right) - 4\left(\frac{M(p)}{p} - m(0)\right) + 3N(p) &= -9 \cdot \frac{p}{p^2+4}. \end{aligned}$$

Substitute initial values $m(0) = 1, \quad m^{\Delta}(0) = 5, \quad n(0) = 2, \quad n^{\Delta}(0) = 3$:

$$\begin{aligned} -\left(\frac{M(p)}{p^2} - \frac{1}{p} - 5\right) + 3\left(\frac{N(p)}{p} - 2\right) + 2M(p) &= \frac{2p}{p-3}, \\ \left(\frac{N(p)}{p^2} - \frac{2}{p} - 3\right) - 4\left(\frac{X(p)}{p} - 1\right) + 3N(p) &= -\frac{9p}{p^2+4}. \end{aligned}$$

First equation:

$$\begin{aligned} -\frac{M(p)}{p^2} + \frac{1}{p} + 5 + \frac{3N(p)}{p} - 6 + 2M(p) &= \frac{2p}{p-3}, \\ -\frac{M(p)}{p^2} + \frac{3N(p)}{p} + 2M(p) &= \frac{2p}{p-3} + 1. \end{aligned}$$

Second equation:

$$\frac{N(p)}{p^2} - \frac{2}{p} - 3 - \frac{4M(p)}{p} + 4 + 3N(p) = -\frac{9p}{p^2+4},$$
$$\frac{N(p)}{p^2} - \frac{4M(p)}{p} + 3N(p) = -\frac{9p}{p^2+4} - 1.$$

$$\begin{bmatrix} 2p^2-1 & 3p \\ -4p & 3p^2+1 \end{bmatrix} \begin{bmatrix} M(p) \\ N(p) \end{bmatrix} = \begin{bmatrix} \frac{2p^3}{p-3} + p^2 \\ -\frac{9p^3}{p^2+4} - p^2 \end{bmatrix}. \tag{3.31}$$

$$\det = (2p^2-1)(3p^2+1) - (3p)(-4p) = 6p^4 + 11p^2 - 1. \tag{3.32}$$

$$A^{-1} = \frac{1}{6p^4+11p^2-1} \begin{bmatrix} 3p^2+1 & -3p \\ 4p & 2p^2-1 \end{bmatrix}. \tag{3.33}$$

$$\begin{bmatrix} M(p) \\ N(p) \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} \frac{2p^3}{p-3} + p^2 \\ -\frac{9p^3}{p^2+4} - p^2 \end{bmatrix}. \tag{3.34}$$

$$\boxed{\begin{cases} m(\phi) = e_3(\phi, 0) + \sin_2(\phi, 0), \\ n(\phi) = e_3(\phi, 0) + \cos_2(\phi, 0). \end{cases}} \tag{3.35}$$

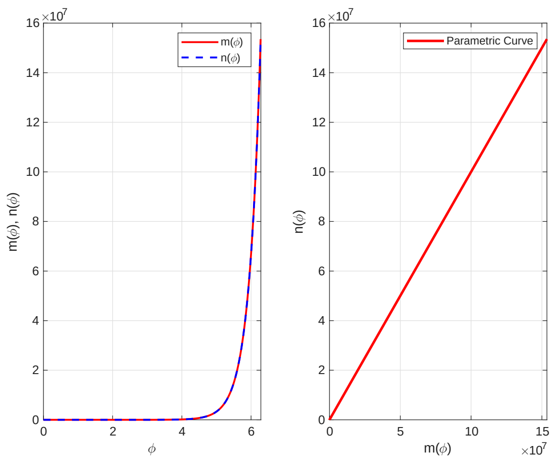

Figure 2: 2-dimensional plot of solution $m(\phi), n(\phi)$ with respect to time and Phase Portrait $n(\phi)$ verses $m(\phi)$ of Example 3.3.

Example 3.4. Consider the integral equation:

$$\zeta(\phi) = e_2(\phi, 0) + 4\int_0^{\phi} \zeta(\tau)\Delta\tau. \tag{3.36}$$

Taking the ET on both sides, we get,

$$\mathcal{E}\{\zeta(\phi)\} = \mathcal{E}\{e_2(\phi, 0)\} + 4\mathcal{E}\left\{\int_0^{\phi} \zeta(\tau)\Delta\tau\right\}. \tag{3.37}$$

$$\mathcal{E}\{e_a(\phi,0)\} = \frac{1}{1-ap}, \quad \mathcal{E}\left\{\int_0^{\phi} \zeta(\tau)\Delta\tau\right\} = p\mathcal{E}\{\zeta(\phi)\}, \tag{3.38}$$

We obtain,

$$\mathcal{E}\{\zeta(\phi)\} = \frac{1}{1-2p} + 4p\mathcal{E}\{\zeta(\phi)\}. \tag{3.39}$$

We write,

$$\frac{1}{(1-2p)(1-4p)} = \frac{\kappa}{1-2p} + \frac{\rho}{1-4p}. \tag{3.40}$$

Thus,

$$\mathcal{E}\{\zeta(t)\} = \frac{-1}{1-2p} + \frac{2}{1-4p}. \tag{3.41}$$

Using the inverse ET,

$$\mathcal{E}^{-1}\left\{\frac{1}{1-ap}\right\} = e_a(\phi,0), \tag{3.42}$$

We obtain,

$$\boxed{\zeta(\phi) = -e_2(\phi,0) + 2e_4(\phi,0)} \tag{3.43}$$

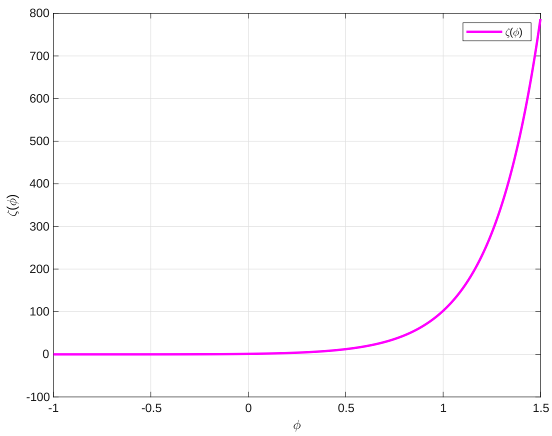

Figure 3: 2-dimensional plot of solution $\zeta(\phi)$ of Example 3.4 .

3.3. Transforms of Generalized Exponential and Trigonometric Functions:

Assume the timescale $\mathbb{T}$ has a constant graininess function, i.e., $\mu(\phi) \equiv \mu$. The transforms of generalized exponential and trigonometric functions derived here are consistent with and complement recent results on fractional formulas involving extended hypergeometric-type functions on timescales [1, 3, 23]. From equations (2.46), we have

$$\mathcal{E}\left(e^{\delta}_{\ominus\frac{1}{p}}(\phi,0)\right)(p) = \frac{p^2}{p + \frac{\mu}{1+\mu\cdot(-\frac{1}{p})} - 1}, \quad p \in \mathbb{R},\ p > 0,$$

$$\mathcal{E}\left(\phi\, e^{\delta}_{\ominus\frac{1}{p}}(\phi,0)\right)(p) = p\int_0^{\infty} \phi\, e^{\delta}_{\ominus\frac{1}{p}}(\phi,0)\,\Delta\phi = \frac{p^2}{\left(\frac{1}{p}\right)^2} = p^4,$$

$$\mathcal{E}\left(e_{\beta}(\phi,0)\, e^{\delta}_{\ominus\frac{1}{p}}(\phi,0)\right)(p) = \frac{p^2}{p-\beta}, \quad p > \beta.$$

Similarly, the transforms involving generalised trigonometric functions yield

$$\mathcal{E}\left(e^{\delta}_{\ominus\frac{1}{p}}(\phi,0)\cos_{\beta}(\phi,0)\right)(p) = \frac{p^2\cdot p}{p^2+\beta^2} = \frac{p^3}{p^2+\beta^2},$$
$$\mathcal{E}\left(e^{\delta}_{\ominus\frac{1}{p}}(\phi,0)\sin_{\beta}(\phi,0)\right)(p) = \frac{p^3\beta}{p^2+\beta^2}.$$

Example 3.5. Consider the dynamic equation with a non-standard forcing term on the timescale $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$:

$$\xi^{\Delta}(\phi) + 2\xi(\phi) = \phi\, e_1(\phi,0), \quad \xi(0) = 1. \tag{3.44}$$

The Laplace transform on this timescale requires evaluating the transform of $\phi\, e_1(\phi,0)$ using convolution, which involves a complicated graininess-dependent kernel that does not simplify to a closed form on $\mathbb{T} = \mathbb{N}_0^2$. By contrast, applying the ET and using Theorem 2.11 and the result $\mathcal{E}\{\phi\}(p) = p$ (established in Problem 1 following Theorem 2.12), together with $\mathcal{E}\{\phi\, e_1(\phi,0)\}(p) = p^4$ from Section 3.3, the transformed equation immediately yields:

$$\frac{\mathcal{E}\{\xi\}(p)}{p} - p + 2\mathcal{E}\{\xi\}(p) = p^4,$$

giving

$$\mathcal{E}\{\xi\}(p) = \frac{p^5+p^2}{1+2p},$$

from which the solution is recovered by the inverse ET. The computational reduction compared to the Laplace-based convolution approach is substantial on this timescale.

3.4. *Special Case:* $\mathbb{T} = \mathbb{R}$

When the timescale reduces to the real numbers, the ETs reduce to classical forms:

$$\mathcal{E}\{e^{-\alpha\phi}\}(p) = \frac{p^2}{p+\alpha}, \quad p > -\alpha,$$

$$\mathcal{E}\{\phi e^{-\alpha\phi}\}(p) = \frac{p^3}{(p+\alpha)^2}, \quad p > -\alpha,$$

$$\mathcal{E}\{e^{\beta t}e^{-\alpha\phi}\}(p) = \frac{p^2}{p+\alpha-\beta}, \quad p > \beta-\alpha,$$

$$\mathcal{E}\{e^{-\alpha\phi}\cos(\beta\phi)\}(p) = \frac{p^2(p+\alpha)}{(p+\alpha)^2+\beta^2}, \quad p > -\alpha.$$

These results are consistent with earlier findings (see, for example, [6]), and the current formulation offers a more general framework.

4. Conclusion

The ET is introduced in this work as a novel integral transform defined on an arbitrary timescale $\mathbb{T}$, designed to facilitate the solution of systems of dynamic equations. This development represents an original contribution since the ET on general timescales has not been addressed before. The proposed framework generalizes and unifies classical approaches, making it applicable not only to ordinary differential equations when $\mathbb{T} = \mathbb{R}$, and difference equations when $\mathbb{T} = \mathbb{N}_0$, but also extending to more complex cases such as q-difference equations$\mathbb{T} = q\mathbb{N}_0 := \{q^{\phi} : \phi \in \mathbb{N}_0, q > 1\}$, as well as to timescales like $\mathbb{T} = q \cup \{0\}$, $\mathbb{T} = h\mathbb{N}_0$, $\mathbb{T} = \mathbb{N}_{20}$, and $\mathbb{T} = \mathbb{T}_n$, the set of harmonic numbers. This unified theoretical approach equips the ET with the capacity to address dynamic and integral equations on various timescales effectively. For an in-depth comparison of the Elzaki and Laplace transforms, the interested reader is directed to the works [13, 14, 15, 29].

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