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A decision-making model based on energy concepts in Fermatean Quadripartitioned Neutrosophic fuzzy graphs

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Abstract

Fermatean Quadripartitioned Neutrosophic fuzzy graphs *(FQNFG)* is the integrating form of Fermatean and Quadripartitioned Neutrosophic fuzzy graphs. Graph energy is recognized as a crucial concept in fuzzy graph theory for its ability to handle random events, thus capturing the attention of numerous researchers. Moreover, the study of graph energy has been a notable rise in recent years. Energy of Graphs have significant applications in various domains, including network analysis, decision making, Image processing, modelling uncertainty etc. This paper introduces energy and Laplacian energy for *FQNFG*. Adjacency matrix, eigen values, energy and Laplacian energy of *FQNFG* are defined with suitable illustrations. Furthermore, we obtain lower and upper bounds of energy and Laplacian energy for *FQNFG*. Additionally, this study presents a decision-making method that uses a scoring approach to assess and compare Laptops based on critical attributes such as processing power, memory & storage, Battery life and Display quality.

Keywords: Neutrosophic fuzzy graph, Quadripartitioned, Fermatean fuzzy graph, Eigen Values, Energy of *FQNFG*, Laplacian energy of *FQNFG*, Multi-criteria decision making.

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1. Introduction

Fuzzy Set Theory concept was initiated by Zadeh (1965)[41]. Fuzzy graph theory merges the principles of fuzzy sets with the framework of graphs and has widespread applications in various fields. Fuzzy graphs help in modelling and analyzing the complex systems with uncertain or imprecise relationships [23]. In 1975, Rosenfeld created the theory of Fuzzy Graphs [31]. An idea about intutionistic fuzzy set relationships was initially introduced by Attanssov. They have putforth a number of applications, theorems and properties [5, 6]. The classical, fuzzy and intuitionistic fuzzy sets were basic extensions of neutrosophic set. Smarandache's generalization of fuzzy set, led to the development of neutrosophic set [36]. It can handle any real world issue containing ambiguous, inconsistent, uncertain and indeterminate data. Every item in a neutrosophic set has three membership grades: truth, indeterminacy and false. These three membership tiers fall between [0, 1] and are always independent [39]. The notion of Pythagorean

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fuzzy graphs was incorporated into the fuzzy graph framework in [28].More recently, the combination of fuzzy graph with Pythagorean Neutrosopic set has resulted in the formulation of Pythagorean neutrosophic fuzzy graphs [1, 2, 3, 11, 12, 13]. Senapati et al. (2020) [33] presented a novel idea called Fermatean Fuzzy Set.Building on the Fermatean Neutrosophic Set [4], Said Broumi et al. (2022) proposed a new model for Fermatean Neutrosophic Graphs [7].

Belnap [8] introduced a four-valued logical system to address incomplete and inconsistent information, which inspired the development of Quadripartitioned Neutrosophic Sets, where the indeterminacy component is divided into contradiction and ignorance. Building on this concept, Satham Hussain et al. [20, 32] further extended the framework, enhancing its ability to represent complex forms of uncertainty. Bimal Shil et al. [34] further strengthened this theory by examining various structural properties. More relevant works on quadripartitioned neutrosophic graphs are provided in [21, 25]. Following this, V. Divya and J. Jesintha Rosline [14, 15] introduced *FQNFG*, providing enhanced flexibility for modeling complex uncertain information. I.Gutman [17] define Graph Energy as sum of the magnitudes of eigen values of the adjacency matrix in graph. The energy bounds are covered in [10, 18, 24]. Energy of different graphs like regular, non-regular and circulant are studied in [16, 19, 22, 35]. The graph's energy is expanded to energy of fuzzy graph [26]. Later, energy of fuzzy graph broadened to intuitionistic and Neutrosophic graph [27, 29]. The energy is applied to picture fuzzy graph, Pythagorean Fuzzy Graph, single valued neutrosophic fuzzy graph etc. Fuzzy graph Laplacian energy was introduced by Sadegh Rahimi Sharbaf et al. (2014) [30]. Later this was extended to Intuitionistic and Neutrosophic fuzzy graph [9, 27]. The energy of fuzzy graph has found applications in various fields like Chemical Graph Theory, Social Network analysis, Decision Making under uncertainty. The ability to model and analyse these uncertainties using fuzzy graph energy offers valuable insights and tools for decision making analysis and understand complex systems. The proposed FQNFG energy measure can be effectively applied in domains involving complex uncertainty. In multi-criteria decision-making, it supports accurate ranking of alternatives by capturing quadripartitioned uncertainty information. It is also useful in social networks, medical diagnosis, and risk assessment, where data may be incomplete or inconsistent. Additionally, engineering and management systems can benefit from its improved capability to model and evaluate uncertain relationships. Related works in the literature have also been reviewed to support the development of the proposed approach [37, 38, 40].

This study examines the energy and Laplacian energy (*LE*) of *FQNFG*. The paper is structured as follows: Section 2 presents the preliminaries . Section 3 defines energy of FQNFG. The lower and upper bounds of energy for *FQNFG* are also derived. In Section 4, we define and characterize *LE* of *FQNFG*. Section 5 derives an application of determining the best Laptop involving four characteristics employing multi-criteria decision-making technique. Section 6 ends with a discussion of further research.

2. Preliminaries

Definition 2.1. [26] Consider a fuzzy graph, $G = (V, \sigma, \mu)$ with adjacency matrix A. The energy G is termed as sum of the magnitudes of its eigenvalues.

Definition 2.2. [30] Consider a fuzzy graph, $G = (\sigma, \mu)$ with $|V| = n$ vertices and $\mu_1 \geqslant \mu_2 \geqslant, \cdots, \geqslant \mu_n$ are Laplacian eigen values of $G = (\sigma, \mu)$. Laplacian energy of G is denoted by

$$LE(G) = \left| \mu_i - \frac{2 \sum\limits_{1 \leqslant i \leqslant j \leqslant n} \mu(u_i u_j)}{n} \right|$$

Definition 2.3. [14] A FQNFG is defined on a Universal Set X with a pair $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$, $\sigma_{FQ} : X \to [0, 1]$ representing Fermatean Quadripartitioned Neutrosophic Set on X and $\mu_{FQ} : X \times X \to [0, 1]$ denoting

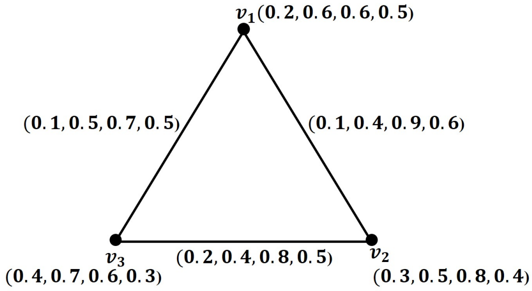

Figure 1: Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

Fermatean Quadripartitioned Neutrosophic mapping on $X \times X$ so that

$$\begin{aligned} T_{\mu_{FQ}}(uv) &\leqslant \min(T_{\sigma_{FQ}}(u), T_{\sigma_{FQ}}(v)) \\ C_{\mu_{FQ}}(uv) &\leqslant \min(C_{\sigma_{FQ}}(u), C_{\sigma_{FQ}}(v)) \\ I_{\mu_{FQ}}(uv) &\geqslant \max(I_{\sigma_{FQ}}(u), I_{\sigma_{FQ}}(v)) \\ F_{\mu_{FQ}}(uv) &\geqslant \max(F_{\sigma_{FQ}}(u), F_{\sigma_{FQ}}(v)) \end{aligned}$$

with $0 \leqslant T^3_{\mu_{FQ}}(uv) + C^3_{\mu_{FQ}}(uv) + I^3_{\mu_{FQ}}(uv) + F^3_{\mu_{FQ}}(uv) \leqslant 3 \ \forall u, v \in X$, where $T_{\mu_{FQ}} : X \times X \to [0,1]$, $C_{\mu_{FQ}} : X \times X \to [0,1]$, $U_{\mu_{FQ}} : X \times X \to [0,1]$, $F_{\mu_{FQ}} : X \times X \to [0,1]$, indicating the degree of truth, contradiction, ignorance and false membership of μ_{FQ}. σ_{FQ} and μ_{FQ} indicates Fermatean Quadripartitioned Neutrosophic vertex and edge set of G_{FQ}.

3. Energy in Fermatean Quadripartitioned Neutrosophic Fuzzy Graphs

Definition 3.1. The Adjacency matrix $A(G_{FQ})$ of FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is termed as square matrix $A(G_{FQ}) = [a_{ij}]$, a_{ij}=$\big(T_{\mu_{FQ}}(\omega_i\omega_j), C_{\mu_{FQ}}(\omega_i\omega_j), I_{\mu_{FQ}}(\omega_i\omega_j), F_{\mu_{FQ}}(\omega_i\omega_j)\big)$ where $T_{\mu_{FQ}}(\omega_i\omega_j)$, $C_{\mu_{FQ}}(\omega_i\omega_j)$, $I_{\mu_{FQ}}(\omega_i\omega_j)$ and $F_{\mu_{FQ}}(\omega_i\omega_j)$ indicates the strength of relationship, undecided relationship, unknown relationship and non-relationship of ω_i and ω_j accordingly.

The adjacency matrix $A(G_{FQ})$ of FQNFG involves 4 matrices, namely truth, contradiction, ignorance and falsity membership values. ie)

$$A(G_{FQ}) = \big(A(T_{\mu_{FQ}}(\omega_i\omega_j)), A(C_{\mu_{FQ}}(\omega_i\omega_j)), A(I_{\mu_{FQ}}(\omega_i\omega_j)), A(F_{\mu_{FQ}}(\omega_i\omega_j))\big)$$

Definition 3.2. The "spectrum of adjacency matrix" of a FQNFG $A(G_{FQ})$ is termed as (Q, R, S, T) where Q, R, S and T represent the eigen values of $A(T_{\mu_{FQ}}(\omega_i\omega_j))$, $A(C_{\mu_{FQ}}(\omega_i\omega_j))$, $A(I_{\mu_{FQ}}(\omega_i\omega_j))$ and $A(F_{\mu_{FQ}}(\omega_i\omega_j))$ respectively.

Example 1: Let $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ be a graph, where $\sigma_{FQ} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mu_{FQ} = \{\omega_1\omega_2, \omega_2\omega_3, \omega_3\omega_4, \omega_4\omega_1, \omega_1\omega_3\}$.

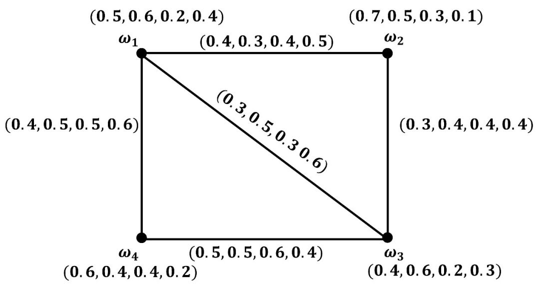

Figure 2: Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

σ_{FQ}	ω_1	ω_2	ω_3	ω_4
$T_{\sigma_{FQ}}$	0.5	0.7	0.4	0.6
$C_{\sigma_{FQ}}$	0.6	0.5	0.6	0.4
$I_{\sigma_{FQ}}$	0.2	0.3	0.2	0.4
$F_{\sigma_{FQ}}$	0.4	0.1	0.3	0.2

μ_{FQ}	$\omega_1\omega_2$	$\omega_2\omega_3$	$\omega_3\omega_4$	$\omega_4\omega_1$	$\omega_1\omega_3$
$T_{\mu_{FQ}}$	0.4	0.3	0.5	0.4	0.3
$C_{\mu_{FQ}}$	0.3	0.4	0.5	0.5	0.5
$I_{\mu_{FQ}}$	0.4	0.4	0.6	0.5	0.3
$F_{\mu_{FQ}}$	0.5	0.4	0.4	0.6	0.6

The adjacency matrix of a FQNFG is given by

$$\begin{pmatrix} (0.0,0.0,0.0,0.0) & (0.4,0.3,0.4,0.5) & (0.3,0.5,0.3,0.6) & (0.4,0.5,0.5,0.6) \\ (0.4,0.3,0.4,0.5) & (0.0,0.0,0.0,0.0) & (0.3,0.4,0.4,0.4) & (0.0,0.0,0.0,0.0) \\ (0.3,0.5,0.3,0.6) & (0.3,0.4,0.4,0.4) & (0.0,0.0,0.0,0.0) & (0.5,0.5,0.6,0.4) \\ (0.4,0.5,0.5,0.6) & (0.0,0.0,0.0,0.0) & (0.5,0.5,0.6,0.4) & (0.0,0.0,0.0,0.0) \end{pmatrix}$$

From Figure 2, Spectrum of FQNFG is

$spec(T_{\mu_{FQ}}(\omega_i\omega_j)) = \{0.9701, -0.6703, -0.3296, 0.0299\}$
$spec(C_{\mu_{FQ}}(\omega_i\omega_j)) = \{1.1490, -0.6556, -0.5, 0.0066\}$
$spec(I_{\mu_{FQ}}(\omega_i\omega_j)) = \{1.1244, -0.8270, -0.3031, 0.0057\}$
$spec(F_{\mu_{FQ}}(\omega_i\omega_j)) = \{1.3018, -0.7830, -0.5218, 0.003\}$
$spec(G_{FQ}(\omega_i\omega_j)) = \{(0.9701, 1.1490, 1.1244, 1.3018),$
$(-0.6703, -0.6556, -0.8270, -0.7830), (-0.3296, -0.5, -0.3031, -0.5218),$
$(0.0299, 0.0066, 0.0057, 0.003)\}$

Definition 3.3. The energy of a FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is defined as

$$\begin{aligned} E(G_{FQ}) &= \big(E(T_{\mu_{FQ}}(\omega_i\omega_j)), E(C_{\mu_{FQ}}(\omega_i\omega_j)), E(I_{\mu_{FQ}}(\omega_i\omega_j)), E(F_{\mu_{FQ}}(\omega_i\omega_j))\big) \\ &= \left(\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} |\alpha_i|, \sum_{\substack{i=1 \\ \beta_i \in R}}^{n} |\beta_i|, \sum_{\substack{i=1 \\ \lambda_i \in S}}^{n} |\lambda_i|, \sum_{\substack{i=1 \\ \eta_i \in T}}^{n} |\eta_i| \right) \end{aligned}$$

Definition 3.4. Two FQNFGs with the same number of vertices and the same energy are called equienergetic.

Theorem 3.5. *Let* $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ *be a FQNFG and* $A(G_{FQ})$ *be its adjacency matrix. If* $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$, $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_n$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, *and* $\eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_n$ *are the eigen values of* $A(T_{\mu_{FQ}}(\omega_i\omega_j))$, $A(C_{\mu_{FQ}}(\omega_i\omega_j))$, $A(I_{\mu_{FQ}}(\omega_i\omega_j))$ *and* $A(F_{\mu_{FQ}}(\omega_i\omega_j))$ *respectively, then,*

1. $\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} \alpha_i = 0$, $\sum_{\substack{i=1 \\ \beta_i \in R}}^{n} \beta_i = 0$, $\sum_{\substack{i=1 \\ \lambda_i \in S}}^{n} \lambda_i = 0$, $\sum_{\substack{i=1 \\ \eta_i \in T}}^{n} \eta_i = 0$
2. $\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} \alpha_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \big(T_{\mu_{FQ}}(\omega_i\omega_j)\big)^2$, $\sum_{\substack{i=1 \\ \beta_i \in R}}^{n} \beta_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \big(C_{\mu_{FQ}}(\omega_i\omega_j)\big)^2$,
   $\sum_{\substack{i=1 \\ \lambda_i \in S}}^{n} \lambda_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \big(I_{\mu_{FQ}}(\omega_i\omega_j)\big)^2$, $\sum_{\substack{i=1 \\ \eta_i \in T}}^{n} \eta_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \big(F_{\mu_{FQ}}(\omega_i\omega_j)\big)^2$.

Proof. **(1)** W.k.t. $A(G_{FQ})$ denotes a symmetric matrix with a trace value of zero (its eigen values sum to zero).

$$\text{i.e.)} \sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} \alpha_i = 0$$

Similarly, contradiction, ignorance and false membership are also obtained

(2) To prove: $\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} \alpha_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \left(T_{\mu_{FQ}}(\omega_i \omega_j)\right)^2$

Based on a Matrix Property,

$$tr((A(T_{\mu_{FQ}}(\omega_i \omega_j)))^2) = \sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} \alpha_i^2$$

where

$$\begin{aligned} tr((A(T_{\mu_{FQ}}(\omega_i \omega_j)))^2) &= (0 + T^2_{\mu_{FQ}}(\omega_1 \omega_2)) + \cdots + T^2_{\mu_{FQ}}(\omega_1 \omega_n)) + \\ &\quad (T^2_{\mu_{FQ}}(\omega_2 \omega_1)) + 0 + \cdots + T^2_{\mu_{FQ}}(\omega_2 \omega_n)) + \cdots \\ &\quad + (T^2_{\mu_{FQ}}(\omega_n \omega_1)) + T^2_{\mu_{FQ}}(\omega_n \omega_2) + \cdots + 0) \\ &= 2 \sum_{1 \leqslant i < j \leqslant n} \left(T_{\mu_{FQ}}(\omega_i \omega_j)\right)^2 \end{aligned}$$

Hence,

$$\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} \alpha_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \left(T_{\mu_{FQ}}(\omega_i \omega_j)\right)^2$$

Likewise, it follows that

$$\sum_{\substack{i=1 \\ \beta_i \in R}}^{n} \beta_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \left(C_{\mu_{FQ}}(\omega_i \omega_j)\right)^2, \quad \sum_{\substack{i=1 \\ \lambda_i \in S}}^{n} \lambda_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \left(I_{\mu_{FQ}}(\omega_i \omega_j)\right)^2, \quad \sum_{\substack{i=1 \\ \eta_i \in T}}^{n} \eta_i^2 = 2 \sum_{1 \leqslant i < j \leqslant n} \left(F_{\mu_{FQ}}(\omega_i \omega_j)\right)^2.$$

□

Example 2:

From Figure 2, Calculating the graph's energy yields:

$E(T_{\mu_{FQ}}(\omega_i \omega_j) = 1.9999$, $E(C_{\mu_{FQ}}(\omega_i \omega_j) = 2.3112$, $E(I_{\mu_{FQ}}(\omega_i \omega_j) = 2.2602$, $E(F_{\mu_{FQ}}(\omega_i \omega_j) = 2.6096$

So, $E(G_{FQ}) = (1.9999, 2.3112, 2.2602, 2.6096)$

The following values are also found:

$$\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{4} \alpha_i = 0.9701 - 0.6703 - 0.3296 + 0.0299 = 0$$

$$\sum_{\substack{i=1 \\ \beta_i \in R}}^{4} \beta_i = 1.1490 - 0.6556 - 0.5 + 0.0066 = 0$$

$$\sum_{\substack{i=1 \\ \lambda_i \in S}}^{4} \lambda_i = 1.1244 - 0.8270 - 0.3031 + 0.0057 = 0$$

$$\sum_{\substack{i=1 \\ \eta_i \in T}}^{4} \eta_i = 1.3018 - 0.7830 - 0.5218 + 0.003 = 0$$

Then

$$\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{4} \alpha_i^2 = 1.5 = 2(0.75) = 2 \sum_{1 \leqslant i < j \leqslant 4} \left(T_{\mu_{FQ}}(\omega_i \omega_j)\right)^2$$

$$\sum_{\substack{i=1\\ \beta_i\in R}}^{4} \beta_i^2 = 2 = 2(1) = 2\sum_{1\leqslant i<j\leqslant 4}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2$$
$$\sum_{\substack{i=1\\ \lambda_i\in S}}^{4} \lambda_i^2 = 2.04 = 2(1.02) = 2\sum_{1\leqslant i<j\leqslant 4}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2$$
$$\sum_{\substack{i=1\\ \eta_i\in T}}^{4} \eta_i^2 = 2.58 = 2(1.29) = 2\sum_{1\leqslant i<j\leqslant 4}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2$$

Theorem 3.6. *Consider a FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ with n vertices and adjacency matrix*

$$A(G_{FQ}) = (A(T_{\mu_{FQ}}(\omega_i\omega_j)), A(C_{\mu_{FQ}}(\omega_i\omega_j)), A(I_{\mu_{FQ}}(\omega_i\omega_j)), A(F_{\mu_{FQ}}(\omega_i\omega_j))).$$

Then,

1. $\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n(n-1)|T|^{\frac{2}{n}}} \leqslant E(T_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}$
2. $\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n(n-1)|C|^{\frac{2}{n}}} \leqslant E(C_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}$
3. $\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n(n-1)|I|^{\frac{2}{n}}} \leqslant E(I_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}$
4. $\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n(n-1)|F|^{\frac{2}{n}}} \leqslant E(F_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}$

where $|T|$, $|C|$, $|I|$ and $|F|$ represent determinant of $A(T_{\mu_{FQ}}(\omega_i\omega_j))$, $A(C_{\mu_{FQ}}(\omega_i\omega_j))$, $A(I_{\mu_{FQ}}(\omega_i\omega_j))$, and $A(F_{\mu_{FQ}}(\omega_i\omega_j))$ respectively.

Proof. **(1) Upper Bound (UB)**
Applying Cauchy-Schwarz inequality to the n numbers $1, 1, \cdots 1$ and $|\lambda_1|, |\lambda_2|, \cdots |\lambda_n|$, we get

$$\sum_{i=1}^{n}|\lambda_i| \leqslant \sqrt{n}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} \tag{3.1}$$
$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\leqslant i<j\leqslant n}\lambda_i\lambda_j \tag{3.2}$$

Equating coefficients of λ^{n-2} in the characteristic polynomial

$$\prod_{i=1}^{n}(\lambda-\lambda_i) = |A(G_{FQ}) - \lambda I|$$

then

$$\sum_{1\leqslant i<j\leqslant n}\lambda_i\lambda_j = -\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 \tag{3.3}$$

Substituting (3.3) into (3.2) gives

$$\sum_{i=1}^{n}|\lambda_i|^2 = 2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 \tag{3.4}$$

Substituting (3.4) into (3.1) gives

$$\begin{aligned}\sum_{i=1}^{n}|\lambda_i| &\leqslant \sqrt{n}\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2} = \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}\\ E(T_{\mu_{FQ}}(\omega_i\omega_j)) &\leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}\end{aligned}$$

Lower bound (LB)

$$\begin{aligned}\left(E(T_{\mu_{FQ}}(\omega_i\omega_j))\right)^2 &= \left(\sum_{i=1}^{n}|\lambda_i|^2\right)^2\\ &= \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\leqslant i<j\leqslant n}|\lambda_i\lambda_j|\\ &= 2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{2n(n-1)}{2}AM\{|\lambda_i\lambda_j|\}\end{aligned}$$

W.k.t. $AM\{|\lambda_i\lambda_j|\} \geqslant GM\{|\lambda_i\lambda_j|\}, 1 \leqslant i < j \leqslant n$

$$\left(E(T_{\mu_{FQ}}(\omega_i\omega_j))\right) \geqslant \sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n(n-1)GM\{|\lambda_i\lambda_j|\}}$$

and then

$$\begin{aligned}GM\{|\lambda_i\lambda_j|\} &= \left(\prod_{1\leqslant i<j\leqslant n}|\lambda_i\lambda_j|\right)^{\frac{2}{n(n-1)}}\\ &= \left(\prod_{i=1}^{n}|\lambda_i|^{n-1}\right)^{\frac{2}{n(n-1)}}\\ &= \left(\prod_{i=1}^{n}|\lambda_i|\right)^{\frac{2}{n}}\\ &= |T|^{\frac{2}{n}}\end{aligned}$$

$$\left(E(T_{\mu_{FQ}}(\omega_i\omega_j))\right) \geqslant \sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n(n-1)|T|^{\frac{2}{n}}}$$

$$\begin{aligned}\therefore \sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n(n-1)|T|^{\frac{2}{n}}} &\leqslant \left(E(T_{\mu_{FQ}}(\omega_i\omega_j))\right)\\ &\leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}\end{aligned}$$

Accordingly, it is found that

$$\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2+n(n-1)|C|^{\frac{2}{n}}} \leqslant \left(E(C_{\mu_{FQ}}(\omega_i\omega_j))\right) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}$$

$$\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2+n(n-1)|I|^{\frac{2}{n}}} \leqslant \left(E(I_{\mu_{FQ}}(\omega_i\omega_j))\right) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}$$

$$\sqrt{2\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2+n(n-1)|F|^{\frac{2}{n}}} \leqslant \left(E(F_{\mu_{FQ}}(\omega_i\omega_j))\right) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2}$$

□

Example 3: From Figure 2, the LB and UB for the FQNFG can be calculated as follows:
$\left(E(T_{\mu_{FQ}}(\omega_i\omega_j))\right) = 1.9999$, LB = 1.5684 and UB = 2.4495.

$$\therefore 1.5684 \leqslant 1.9999 \leqslant 2.4495$$

$\left(E(C_{\mu_{FQ}}(\omega_i\omega_j))\right) = 2.3112$, LB = 1.6125 and UB = 2.8284.

$$\therefore 1.6125 \leqslant 2.3112 \leqslant 2.8284$$

$\left(E(I_{\mu_{FQ}}(\omega_i\omega_j))\right) = 2.2602$, LB = 1.5875 and UB = 2.8566.

$$\therefore 1.5875 \leqslant 2.2602 \leqslant 2.8566$$

$\left(E(F_{\mu_{FQ}}(\omega_i\omega_j))\right) = 2.6096$, LB = 1.7493 and UB = 3.2125.

$$\therefore 1.7493 \leqslant 2.6096 \leqslant 3.2125$$

4. Laplacian Energy of Fermatean Quadripartitioned Neutrosophic Fuzzy Graphs

Definition 4.1. Consider $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is a FQNFG with n vertices. Then degree matrix $D(G_{FQ}) = (D(T_{\mu_{FQ}}(\omega_i\omega_j)), D(C_{\mu_{FQ}}(\omega_i\omega_j)), D(I_{\mu_{FQ}}(\omega_i\omega_j)), D(F_{\mu_{FQ}}(\omega_i\omega_j))) = [d_{ij}]$, of G_{FQ} is an $n \times n$ diagonal matrix termed as

$$d_{ij} = \begin{cases} d_{G_{FQ}}(\omega_i), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Definition 4.2. The Laplacian matrix (LM) of a FQNFG $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is termed as $L(G_{FQ}) = (L(T_{\mu_{FQ}}(\omega_i\omega_j)), L(C_{\mu_{FQ}}(\omega_i\omega_j)), L(I_{\mu_{FQ}}(\omega_i\omega_j)), L(F_{\mu_{FQ}}(\omega_i\omega_j))) = D(G_{FQ}) - A(G_{FQ})$, where $A(G_{FQ})$ and $D(G_{FQ})$ represent an adjacency matrix and degree matrix of FQNFG.

Definition 4.3. The spectrum of LM of FQNFG, $L(G_{FQ})$ is denoted by (Q_L, R_L, S_L, T_L), where Q_L, R_L, S_L and T_L are the sets of Laplacian eigen values of $L(T_{\mu_{FQ}}(\omega_i\omega_j))$, $L(C_{\mu_{FQ}}(\omega_i\omega_j))$, $L(I_{\mu_{FQ}}(\omega_i\omega_j))$ and $L(F_{\mu_{FQ}}(\omega_i\omega_j))$ respectively.

Example 4:
Let $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ be a FQNFG, where $\sigma_{FQ} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and

$$\mu_{FQ} = (\omega_1\omega_2, \omega_2\omega_3, \omega_3\omega_1, \omega_3\omega_4, \omega_2\omega_4)$$

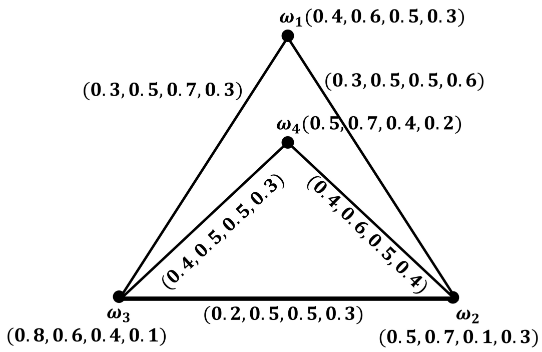

Figure 3: Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

σ_{FQ}	ω_1	ω_2	ω_3	ω_4
$T_{\sigma_{FQ}}$	0.4	0.5	0.8	0.5
$C_{\sigma_{FQ}}$	0.6	0.7	0.6	0.7
$I_{\sigma_{FQ}}$	0.5	0.1	0.4	0.4
$F_{\sigma_{FQ}}$	0.3	0.3	0.1	0.2

μ_{FQ}	$\omega_1\omega_2$	$\omega_2\omega_3$	$\omega_3\omega_1$	$\omega_3\omega_4$	$\omega_2\omega_4$
$T_{\mu_{FQ}}$	0.3	0.2	0.3	0.4	0.4
$C_{\mu_{FQ}}$	0.5	0.5	0.5	0.5	0.6
$I_{\mu_{FQ}}$	0.5	0.5	0.7	0.5	0.5
$F_{\mu_{FQ}}$	0.6	0.4	0.4	0.3	0.4

The adjacency matrix of a FQNFG is given by

$$A(G_{FQ}) = \begin{pmatrix} (0.0,0.0,0.0,0.0) & (0.3,0.5,0.5,0.6) & (0.3,0.5,0.7,0.3) & (0.0,0.0,0.0,0.0) \\ (0.3,0.5,0.5,0.6) & (0.0,0.0,0.0,0.0) & (0.2,0.5,0.5,0.3) & (0.4,0.6,0.5,0.4) \\ (0.3,0.5,0.7,0.3) & (0.2,0.5,0.5,0.3) & (0.0,0.0,0.0,0.0) & (0.4,0.5,0.5,0.3) \\ (0.0,0.0,0.0,0.0) & (0.4,0.6,0.5,0.4) & (0.4,0.5,0.5,0.3) & (0.0,0.0,0.0,0.0) \end{pmatrix}$$

The Laplacian matrix of a FQNFG is given by

$$L(G_{FQ}) = \begin{pmatrix} (0.6,1.0,1.2,0.9) & (-0.3,-0.5,-0.5,-0.6) & (-0.3,-0.5,-0.7,-0.4) & (0.0,0.0,0.0,0.0) \\ (-0.3,-0.5,-0.5,-0.6) & (1.1,1.6,1.5,1.3) & (-0.2,-0.5,-0.5,-0.4) & (-0.4,-0.6,-0.5,-0.4) \\ (-0.3,-0.5,-0.7,-0.4) & (-0.2,-0.5,-0.5,-0.4) & (0.9,1.5,1.7,0.9) & (-0.4,-0.5,-0.5,-0.3) \\ (0.0,0.0,0.0,0.0) & (-0.4,-0.6,-0.5,-0.4) & (-0.4,-0.5,-0.5,-0.3) & (0.8,1.1,1.0,0.7) \end{pmatrix}$$

The Laplacian spectrum of a FQNFG is
Laplacian $spec(T_{\mu_{FQ}}(\omega_i\omega_j))$ = {0.0442, 0.6740, 1.4952, 1.1866}

Laplacian $spec(C_{\mu_{FQ}}(\omega_i\omega_j))$ = {0, 1.0432, 2.1568, 2.0}
Laplacian $spec(I_{\mu_{FQ}}(\omega_i\omega_j))$ = {0, 1.0755, 2.3245, 2.0}
Laplacian $spec(F_{\mu_{FQ}}(\omega_i\omega_j))$ = {−0.1044, 0.7650, 1.8304, 1.3090}
Laplacian $spec(G_{FQ}(\omega_i\omega_j))$ = {(0.0442, 0, 0, −0.1044), (0.6740, 1.0432, 1.0755, 0.7650), (1.4952, 2.1568, 2.3245, 1.8304), (1.1866, 2.0, 2.0, 1.3090)}

Theorem 4.4. *Let* $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ *be a FQNFG and*

$$L(G_{FQ}) = (L(T_{\mu_{FQ}}(\omega_i\omega_j)), L(C_{\mu_{FQ}}(\omega_i\omega_j)), L(I_{\mu_{FQ}}(\omega_i\omega_j)), L(F_{\mu_{FQ}}(\omega_i\omega_j)))$$

be the LM of G_{FQ}. *If* $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n$, $\phi_1 \geqslant \phi_2 \geqslant \cdots \geqslant \phi_n$, $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_n$, *and* $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_n$ *are the eigen values of* $L(T_{\mu_{FQ}}(\omega_i\omega_j))$, $L(C_{\mu_{FQ}}(\omega_i\omega_j))$, $L(I_{\mu_{FQ}}(\omega_i\omega_j))$ *and* $L(F_{\mu_{FQ}}(\omega_i\omega_j))$ *respectively. Then,*

1. $$\sum_{\substack{i=1\\ \gamma_i\in Q_L}}^{n} \gamma_i = 2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right), \quad \sum_{\substack{i=1\\ \phi_i\in R_L}}^{n} \phi_i = 2\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right),$$
$$\sum_{\substack{i=1\\ \tau_i\in S_L}}^{n} \tau_i = 2\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right), \quad \sum_{\substack{i=1\\ \rho_i\in T_L}}^{n} \rho_i = 2\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)$$
2. $$\sum_{\substack{i=1\\ \gamma_i\in Q_L}}^{n} \gamma_i^2 = 2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n} d^2_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i),$$
$$\sum_{\substack{i=1\\ \phi_i\in R_L}}^{n} \phi_i^2 = 2\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n} d^2_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i),$$
$$\sum_{\substack{i=1\\ \tau_i\in S_L}}^{n} \tau_i^2 = 2\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n} d^2_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i),$$
$$\sum_{\substack{i=1\\ \rho_i\in T_L}}^{n} \rho_i^2 = 2\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n} d^2_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i).$$

Proof. **(1)** We know that $L(G_{FQ})$ denotes a symmetric matrix of positive values. Then,

$$\begin{aligned}\sum_{\substack{i=1\\ \gamma_i\in Q_L}}^{n} \gamma_i &= tr(L(G_{FQ}))\\ &= \sum_{i=1}^{n} d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i)\\ &= 2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)\end{aligned}$$

Likewise, it follows that,

$$\sum_{\substack{i=1\\ \phi_i\in R_L}}^{n} \phi_i = 2\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right), \quad \sum_{\substack{i=1\\ \tau_i\in S_L}}^{n} \tau_i = 2\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right), \quad \sum_{\substack{i=1\\ \rho_i\in T_L}}^{n} \rho_i = 2\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)$$

(2) By the definition of LM,

$$L(T_{\mu_{FQ}}(\omega_i\omega_j)) = \begin{pmatrix} d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_1) & -T_{\mu_{FQ}}(\omega_1\omega_2) & \cdots & -T_{\mu_{FQ}}(\omega_1\omega_n)\\ -T_{\mu_{FQ}}(\omega_2\omega_1) & d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_2) & \cdots & -T_{\mu_{FQ}}(\omega_2\omega_n)\\ \vdots & \vdots & \ddots & \vdots\\ -T_{\mu_{FQ}}(\omega_n\omega_1) & -T_{\mu_{FQ}}(\omega_n\omega_2) & \cdots & d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_n)\end{pmatrix}$$ By matrix trace theory,

$$tr((L(T_{\mu_{FQ}}(\omega_i\omega_j)))^2) = \sum_{\substack{i=1\\ \gamma_i\in Q_L}}^{n} \gamma_i^2$$

where

$$\begin{aligned} tr((L(T_{\mu_{FQ}}(\omega_i\omega_j)))^2) &= \Big(d^2_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_1) + T^2_{\mu_{FQ}}(\omega_1\omega_2) + \cdots + T^2_{\mu_{FQ}}(\omega_1\omega_n)\Big) + \\ &\quad \Big(T^2_{\mu_{FQ}}(\omega_2\omega_1) + d^2_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_2) + \cdots + T^2_{\mu_{FQ}}(\omega_2\omega_n)\Big) + \cdots \\ &\quad + \Big(T^2_{\mu_{FQ}}(\omega_n\omega_1) + T^2_{\mu_{FQ}}(\omega_n\omega_2) + \cdots + d^2_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_n)\Big) \\ &= 2\sum_{1\leqslant i<j\leqslant n}\big(T_{\mu_{FQ}}(\omega_i\omega_j)\big)^2 + \sum_{i=1}^{n} d^2_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) \\ \therefore \sum_{\substack{i=1\\ \gamma_i\in Q_L}}^{n}\gamma_i^2 &= 2\sum_{1\leqslant i<j\leqslant n}\big(T_{\mu_{FQ}}(\omega_i\omega_j)\big)^2 + \sum_{i=1}^{n} d^2_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) \end{aligned}$$

Likewise,

$$\sum_{\substack{i=1\\ \phi_i\in R_L}}^{n}\phi_i^2 = 2\sum_{1\leqslant i<j\leqslant n}\big(C_{\mu_{FQ}}(\omega_i\omega_j)\big)^2 + \sum_{i=1}^{n} d^2_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i),$$

$$\sum_{\substack{i=1\\ \tau_i\in S_L}}^{n}\tau_i^2 = 2\sum_{1\leqslant i<j\leqslant n}\big(I_{\mu_{FQ}}(\omega_i\omega_j)\big)^2 + \sum_{i=1}^{n} d^2_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i),$$

$$\sum_{\substack{i=1\\ \rho_i\in T_L}}^{n}\rho_i^2 = 2\sum_{1\leqslant i<j\leqslant n}\big(F_{\mu_{FQ}}(\omega_i\omega_j)\big)^2 + \sum_{i=1}^{n} d^2_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i). \qquad \square$$

Definition 4.5. The LE of FQNFG $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is denoted by

$$\begin{aligned} LE(G_{FQ}) &= (LE(T_{\mu_{FQ}}(\omega_i\omega_j)), LE(C_{\mu_{FQ}}(\omega_i\omega_j)), LE(I_{\mu_{FQ}}(\omega_i\omega_j)), LE(F_{\mu_{FQ}}(\omega_i\omega_j))) \\ &= \left(\sum_{i=1}^{n}|\delta_i|, \sum_{i=1}^{n}|\Delta_i|, \sum_{i=1}^{n}|\theta_i|, \sum_{i=1}^{n}|\zeta_i|\right) \end{aligned}$$

where,

$$\delta_i = \gamma_i - \frac{2\sum\limits_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}; \quad \Delta_i = \phi_i - \frac{2\sum\limits_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n};$$

$$\theta_i = \tau_i - \frac{2\sum\limits_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}; \quad \zeta_i = \rho_i - \frac{2\sum\limits_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}.$$

Theorem 4.6. *Let $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ be a FQNFG and $L(G_{FQ})$ be the LM of G_{FQ}. If $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n$, $\phi_1 \geqslant \phi_2 \geqslant \cdots \geqslant \phi_n$, $\zeta_1 \geqslant \zeta_2 \geqslant \cdots \geqslant \zeta_n$, and $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_n$ were eigen values of $L(T_{\mu_{FQ}}(\omega_i\omega_j))$, $L(C_{\mu_{FQ}}(\omega_i\omega_j))$, $L(I_{\mu_{FQ}}(\omega_i\omega_j))$ and $L(F_{\mu_{FQ}}(\omega_i\omega_j))$ respectively and $\delta_i = \gamma_i - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}$; $\Delta_i = \phi_i - \frac{2\sum_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n}$; $\theta_i = \tau_i - \frac{2\sum_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}$; $\zeta_i = \rho_i - \frac{2\sum_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}$. Then $\sum_{i=1}^{n}\delta_i = 0$, $\sum_{i=1}^{n}\Delta_i = 0$, $\sum_{i=1}^{n}\theta_i = 0$, $\sum_{i=1}^{n}\zeta_i = 0$, $\sum_{i=1}^{n}\delta_i^2 = 2\mathbb{N}_T$, $\sum_{i=1}^{n}\Delta_i^2 = 2\mathbb{N}_C$, $\sum_{i=1}^{n}\theta_i^2 = 2\mathbb{N}_I$, $\sum_{i=1}^{n}\zeta_i^2 = 2\mathbb{N}_F$, where*

$$\mathbb{N}_T = \sum_{1\leqslant i<j\leqslant n}\big(T_{\mu_{FQ}}(\omega_i\omega_j)\big)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum\limits_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2$$

$$\mathbb{N}_C = \sum_{1\leqslant i<j\leqslant n}\big(C_{\mu_{FQ}}(\omega_i\omega_j)\big)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum\limits_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2$$

$$\mathbb{N}_I = \sum_{1\leqslant i<j\leqslant n} \left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2$$

$$\mathbb{N}_F = \sum_{1\leqslant i<j\leqslant n} \left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2$$

Theorem 4.7. *Consider FQNFG,* $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ *with* n *vertices and*

$$L(G_{FQ}) = (L(T_{\mu_{FQ}}(\omega_i\omega_j)), L(C_{\mu_{FQ}}(\omega_i\omega_j)), L(I_{\mu_{FQ}}(\omega_i\omega_j)), L(F_{\mu_{FQ}}(\omega_i\omega_j)))$$

be the LM of G_{FQ}*. Then,*

1. $LE(T_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n} \left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$
2. $LE(C_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n} \left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$
3. $LE(I_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n} \left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$
4. $LE(F_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n} \left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$

Proof. Applying Cauchy-Schwarz inequality to the n numbers $1, 1, \cdots, 1$ and $|\delta_1|, |\delta_2|, \cdots, |\delta_n|$ results in,

$$\sum_{i=1}^{n} |\delta_i| \leqslant \sqrt{n}\sqrt{\sum_{i=1}^{n} |\delta_i|^2}$$

$$LE(T_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{n}\sqrt{2\mathbb{N}_T} = \sqrt{2n\mathbb{N}_T}$$

Since $\mathbb{N}_T = \sum_{1\leqslant i\leqslant j\leqslant n} \left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2$

$$\therefore LE(T_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n} \left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$$

Likewise, we get

$$LE(C_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n} \left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$$

$$LE(I_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$$

$$LE(F_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant \sqrt{2n\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2} \qquad \square$$

Theorem 4.8. *Consider FQNFG,* $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ *with* n *vertices and*

$$L(G_{FQ}) = (L(T_{\mu_{FQ}}(\omega_i\omega_j)), L(C_{\mu_{FQ}}(\omega_i\omega_j)), L(I_{\mu_{FQ}}(\omega_i\omega_j)), L(F_{\mu_{FQ}}(\omega_i\omega_j)))$$

are LM of G_{FQ}. *Then,*

1. $LE(T_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$
2. $LE(C_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$
3. $LE(I_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$
4. $LE(F_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$

Proof.

$$\left(\sum_{i=1}^{n}|\delta_i|\right)^2 = \sqrt{\sum_{i=1}^{n}|\delta_i|^2} + 2\sum_{1\leqslant i<j\leqslant n}|\delta_i\delta_j| \geqslant 4\mathbb{N}_T$$

$$LE(T_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\mathbb{N}_T}$$

We know that

$$\mathbb{N}_T = \sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2$$

$$\therefore LE(T_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$$

Similary, it follows that

$$LE(C_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$$

$$LE(I_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2}$$

$$LE(F_{\mu_{FQ}}(\omega_i\omega_j)) \geqslant 2\sqrt{\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2} \qquad \square$$

Theorem 4.9. *Let* $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ *be a FQNFG on* n *vertices and let*

$$L(G_{FQ}) = (L(T_{\mu_{FQ}}(\omega_i\omega_j)), L(C_{\mu_{FQ}}(\omega_i\omega_j)), L(I_{\mu_{FQ}}(\omega_i\omega_j)), L(F_{\mu_{FQ}}(\omega_i\omega_j)))$$

be the LM of G_{FQ}. *Then,*

1. $LE(T_{\mu_{FQ}}(\omega_i\omega_j))$
$$\leqslant |\delta_1|\sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2 - \delta_1^2\right)}$$
2. $LE(C_{\mu_{FQ}}(\omega_i\omega_j))$
$$\leqslant |\Delta_1|\sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} C_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2 - \Delta_1^2\right)}$$
3. $LE(I_{\mu_{FQ}}(\omega_i\omega_j))$
$$\leqslant |\theta_1|\sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} I_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2 - \theta_1^2\right)}$$
4. $LE(F_{\mu_{FQ}}(\omega_i\omega_j))$
$$\leqslant |\zeta_1|\sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} F_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2 - \zeta_1^2\right)}$$

Proof. The Cauchy-Schwarz Inequality yields, $\sum_{i=1}^{n}|\delta_i| \leqslant \sqrt{n\sum_{i=1}^{n}|\delta_i|^2}$, $\sum_{i=2}^{n}|\delta_i| \leqslant \sqrt{(n-1)\sum_{i=2}^{n}|\delta_i|^2}$

$$\begin{aligned} LE(T_{\mu_{FQ}}(\omega_i\omega_j)) - |\delta_1| &\leqslant \sqrt{(n-1)(2\mathbb{N}_T - \delta_1^2)} \\ LE(T_{\mu_{FQ}}(\omega_i\omega_j)) &\leqslant |\delta_1| + \sqrt{(n-1)(2\mathbb{N}_T - \delta_1^2)} \end{aligned}$$

We know that,

$$\mathbb{N}_T = \sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2$$

Therefore,

$LE(T_{\mu_{FQ}}(\omega_i\omega_j))$
$$\leqslant |\delta_1|\sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(T_{\mu_{FQ}}(\omega_i\omega_j)\right)^2 + \sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i) - \frac{2\sum_{1\leqslant i<j\leqslant n} T_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2 - \delta_1^2\right)}$$

Similarly, we can prove

$$LE(C_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant |\Delta_1| \sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(C_{\mu_{FQ}}(\omega_i\omega_j)\right)^2+\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i)-\frac{2\sum_{1\leqslant i<j\leqslant n}C_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2-\Delta_1^2\right)}.$$

$$LE(I_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant |\theta_1| \sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(I_{\mu_{FQ}}(\omega_i\omega_j)\right)^2+\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i)-\frac{2\sum_{1\leqslant i<j\leqslant n}I_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2-\theta_1^2\right)}.$$

$$LE(F_{\mu_{FQ}}(\omega_i\omega_j)) \leqslant |\zeta_1| \sqrt{(n-1)\left(2\sum_{1\leqslant i<j\leqslant n}\left(F_{\mu_{FQ}}(\omega_i\omega_j)\right)^2+\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(\omega_i\omega_j)}(\omega_i)-\frac{2\sum_{1\leqslant i<j\leqslant n}F_{\mu_{FQ}}(\omega_i\omega_j)}{n}\right)^2-\zeta_1^2\right)}. \qquad \square$$

5. Application: Decision Making Model for Laptop Selection using FQNFG

In this section, the proposed FQNFG based multi-criteria decision-making (MCDM) approach is applied to a practical problem, namely the selection of the best laptop for purchase.

5.1. *Algorithm*

The following algorithm is our proposed technique for MCDM.
Step 1: Define the alternatives and criteria involved in the decision-making problem.
Step 2: Construct the Fermatean Quadripartitioned Neutrosophic (FQN) decision matrix by assigning FQN values (T, C, I, F) to each alternative under each criterion.
Step 3: Determine the weight vector corresponding to the importance of each criterion.
Step 4: Compute the aggregated FQN value for each alternative using the arithmetic averaging operator.
Step 5: Evaluate the score function for each alternative.
Step 6: If necessary, compute the accuracy function to compare close alternatives.
Step 7: Rank the alternatives based on the score values and select the best one.

5.2. *An Illustrative example:*

Selecting a laptop is a complex decision since multiple criteria such as processor performance, memory capacity, battery life, and display quality must be evaluated simultaneously. As real-world assessments often involve uncertainty and partial truth, the FQN model is employed to represent truth, contradiction, ignorance, and falsity values effectively. To demonstrate the proposed approach, consider four laptops namely L_1, L_2, L_3, L_4.

Each laptop is evaluated under the following four criteria:

1. Processor
2. Memory & Storage
3. Battery life
4. Display quality

The evaluations are expressed using FQN values of the form (T, C, I, F). Selecting a new laptop requires careful evaluation of several important criteria. Processing power is essential for ensuring fast performance and efficient multitasking. Memory and storage capacity are equally important to support smooth operation and accommodate necessary files and applications. Battery life is a crucial factor for maintaining uninterrupted productivity throughout the day. In addition, display quality significantly influences

user experience by providing clear and vibrant visuals. Using these four criteria, a MCDM approach is applied to evaluate four laptops in order to determine the most suitable option for purchase.
Step 1: The criteria involved in the problem are as follows:
Laptop 1:

1. Processor -(0.3, 0.4, 0.3, 0.1)
2. Memory & Storage -(0.6, 0.5, 0.3, 0.2)
3. Battery life - (0.3, 0.5, 0.2, 0.1)
4. Display quality -(0.2, 0.1, 0.6, 0.3)

Laptop 2:

1. Processor -(0.6, 0.6, 0.5, 0.3)
2. Memory & Storage - (0.5, 0.4, 0.6, 0.2)
3. Battery life - (0.8, 0.2, 0.6, 0.2)
4. Display quality - (0.2, 0.3, 0.5, 0.3)

Laptop 3:

1. Processor -(0.8, 0.2, 0.1, 0.1)
2. Memory & Storage -(0.7, 0.4, 0.3, 0.2)
3. Battery life -(0.7, 0.6, 0.3, 0.4)
4. Display quality - (0.5, 0.8, 0.2, 0.3)

Laptop 4:

1. Processor - (0.6, 0.5, 0.3, 0.1)
2. Memory & Storage -(0.4, 0.4, 0.5, 0.1)
3. Battery life - (0.6, 0.3, 0.1, 0.2)
4. Display quality - (0.7, 0.6, 0.4, 0.2)

Step 2: The decision matrix P constructed such that each row represents a laptop and each column represents one of the four criteria.

$$P = \begin{pmatrix} (0.3, 0.4, 0.3, 0.1) & (0.6, 0.5, 0.3, 0.2) & (0.3, 0.5, 0.2, 0.1) & (0.2, 0.1, 0.6, 0.3) \\ (0.6, 0.6, 0.5, 0.3) & (0.5, 0.4, 0.6, 0.2) & (0.8, 0.2, 0.6, 0.2) & (0.2, 0.3, 0.5, 0.3) \\ (0.8, 0.2, 0.1, 0.1) & (0.7, 0.4, 0.3, 0.2) & (0.7, 0.6, 0.3, 0.4) & (0.5, 0.8, 0.2, 0.3) \\ (0.6, 0.5, 0.3, 0.1) & (0.4, 0.4, 0.5, 0.1) & (0.6, 0.3, 0.1, 0.2) & (0.7, 0.6, 0.4, 0.2) \end{pmatrix}$$

Step 3: The criteria have associated weights given by the vector W = (0.6, 0.4, 0.3, 0.2).
Step 4: Using the decision matrix P, we will compute the FQN arithmetic average across all laptops for each criterion, resulting in a single FQN value per criterion. The aggregated FQN values are as follows.

$$L_1 = \left(\frac{0.3 + 0.6 + 0.3 + 0.2}{4}, \frac{0.4 + 0.5 + 0.5 + 0.1}{4}, \frac{0.3 + 0.3 + 0.2 + 0.6}{4}, \frac{0.1 + 0.2 + 0.1 + 0.3}{4} \right)$$
$$= (0.35, 0.375, 0.35, 0.175)$$

Similarly we Calculate for other Laptops as
$L_2 = (0.525, 0.375, 0.55, 0.25)$; $L_3 = (0.675, 0.5, 0.225, 0.25)$; $L_4 = (0.575, 0.45, 0.325, 0.15)$.
Step 5: Score equation formula is

$$S = \frac{m}{2} \times \sum_{i=0}^{m} \left(\frac{W_T T_i + W_C C_i + W_I I_i + W_F F_i}{W_T + W_C + W_I + W_F} \right)$$

Calculating the Score value for each alternative is,
$S(L_1) = 0.67$; $S(L_2) = 0.9$; $S(L_3) = 0.96$; $S(L_4) = 0.87$
Step 6: The accuracy function formula is

$$A = \sum_{i=0}^{m} \sqrt{\left(\frac{(W_T T_i)^2 + (W_C C_i)^2 + (W_I I_i)^2 + (W_F F_i)^2}{2} \right)}$$

Computing the accuracy value for each alternative is,
$A(L_1) = 0.1985; \quad A(L_2) = 0.2751; \quad A(L_3) = 0.325; \quad A(L_4) = 0.2844.$
Step 7: we now rank the laptops based on their calculated scores and accuracy values as follows:
$L_3 > L_2 > L_4 > L_1$.
Based on the calculated score values, L_3 achieves the highest rank among all alternatives. This shows that it performs better overall across the selected criteria. The results indicate that the proposed method effectively supports decision-making under uncertainty. Therefore, L_3 is selected as the best laptop for purchase.

5.3. *Comparative Analysis*

A comparative analysis is conducted between the mobile network selection application presented in Hiremath et al., [21] and our proposed application using FQNFG. Both applications employ quadripartitioned neutrosophic aggregation to evaluate alternatives based on multiple criteria. The decision matrices for mobiles and laptops are processed to compute the aggregated neutrosophic values, scores, and accuracy values for each alternative. Table 1 presents a side-by-side comparison of the ranking results. In both cases, the methods successfully identify the best and worst alternatives according to their respective score functions. Although the decision context differs (mobile network vs. laptop selection), the proposed Fermatean approach demonstrates similar effectiveness and reliability in ranking alternatives.

Table 1: **Comparison Analysis**

Method	Score Values of Alternatives	Accuracy Values	Ranking Order
Hiremath et al. [21]	$N_1 = 0.729$ $N_2 = 0.8458$ $N_3 = 0.8745$ $N_4 = 0.8313$	$N_1 = 0.166$ $N_2 = 0.219$ $N_3 = 0.257$ $N_4 = 0.221$	$N_3 > N_2 > N_4 > N_1$
Our Proposed Method	$L_1 = 0.67$ $L_2 = 0.9$ $L_3 = 0.96$ $L_4 = 0.87$	$L_1 = 0.1985$ $L_2 = 0.2751$ $L_3 = 0.325$ $L_4 = 0.2844$	$L_3 > L_2 > L_4 > L_1$

6. Conclusion

Fermatean Quadripartitioned Neutrosophic Fuzzy Graphs provides a suitable modelling structure. These models generally offer improved precision, greater flexibility and enhanced compatibility compared to other fuzzy approaches. This manuscript dealt with the new idea of Energy and Laplacian Energy of FQNFG . The LB and UB of Energy and Laplacian Energy for FQNFG were derived. The energy of FQNFG holds potential for applications in Computer Science, Chemistry, Risk Assessment, Medical diagnosis etc. The proposed methodology offers the most effective method for laptop acquisition. The future work can extend the study of FQNFG energy to Double Layered Fuzzy Graphs, Hesitant Fuzzy Graphs, Spherical Fuzzy Graphs and other fuzzy graph models.

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