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Existence, regularity and asymptotic analysis for conformable parabolic equations with history–dependent diffusion

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Abstract

This paper investigates a diffusion equation incorporating memory effects and a conformable fractional derivative. We begin by deriving an explicit representation of the mild solution and analyzing its regularity properties. Next, we explore the continuous dependence of the solution on key parameters, highlighting its sensitivity to changes. As the order of the conformable derivative approaches one, we demonstrate that the mild solution converges to that of the classical diffusion equation. Furthermore, we establish a finite-time blow-up criterion under specific nonlinear conditions. Employing fixed-point theorems, we prove the existence and uniqueness of mild solutions and identify the conditions necessary to ensure these results. Our findings offer new insights into how conformable fractional derivatives and memory terms influence solution behavior and stability.

Keywords: Conformable derivative, Memory terms, Mild solution, Regularity, Parameter continuity, Blow-up criterion, Fixed-point theorem.

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1. Introduction

In this paper, we study the following initial–boundary value problem

$$\begin{cases} \frac{{}^{C}\partial^{\alpha} y(x,t)}{\partial t^{\alpha}} - \Delta y(x,t) = a y + \beta \int_0^t \Delta y_s(x,s)ds, & x \in \Omega, \ \ t \in (0,T), \\ y(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ y(x,0) = f(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Let $T > 0$. The constant $a \in \mathbb{R}$ is given. The function f represents the initial condition.

History-dependent diffusion mechanisms appear in applied settings where the diffusive flux exhibits a rate-dependent response. A prototypical example arises in Kelvin–Voigt-type viscoelastic descriptions,

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where constitutive laws include both an elastic part and a viscous (rate) part; at the PDE level, this leads to nonclassical (pseudo-parabolic) diffusion operators involving Δy_t. In our formulation, the term $\beta \int_0^t \Delta y_s(\cdot, s)\, ds$ accounts for the accumulated contribution of such a rate-dependent diffusion over time; equivalently, whenever the solution is sufficiently regular, $\int_0^t \Delta y_s\, ds = \Delta y(t) - \Delta y(0)$, showing that the history term effectively modifies the diffusion operator and introduces an initial-data-dependent contribution. From an analytical viewpoint, the existence, regularity, and asymptotic estimates established in this paper provide qualitative information on the smoothing effect and long-time behavior of solutions, which are relevant for the stability analysis of rate-dependent diffusion models.

The hereditary term in (1.1) is of *rate type*, namely $\beta \int_0^t \Delta y_s(\cdot, s)\, ds$, rather than a classical state-memory (Volterra-type) term of the form $\beta \int_0^t \Delta y(\cdot, s)\, ds$. This distinction is important for both the modeling interpretation and the mode-wise (eigenfunction) representation developed in the subsequent sections.

The symbol $\frac{^C\partial^\alpha}{\partial t^\alpha} f$ is defined by

$$\frac{^C\partial^\alpha}{\partial t^\alpha} f(t) := \lim_{h \to 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}, \tag{1.2}$$

where $0 < \alpha \leqslant 1$ and f is an element of a Banach space $\mathbb{B}$.

By examining the formula (1.2) above, it becomes evident that the conformable derivative generalizes the classical derivative. Khalil *et al.* first introduced this concept in 2014 [19]. Conformable derivatives have found applications across various fields, including damped oscillations [8], electrical circuit analysis [22], and chaotic dynamical systems [9], among others. Furthermore, in situations where classical derivatives fail to adequately describe system behavior, conformable derivatives offer a valuable alternative (see [43]).

We briefly contrast the conformable derivative adopted here with the classical Caputo and Riemann–Liouville fractional derivatives. The Caputo and Riemann–Liouville fractional derivatives are intrinsically *nonlocal* in time: they involve convolution-type kernels and thus encode a long-time memory effect in the evolution. By contrast, the conformable fractional derivative used here is *local* and may be viewed (for sufficiently smooth functions) as a time-rescaled first-order derivative; in particular, it admits product and chain rules in a form close to the classical calculus. These features are convenient for our analysis, since they allow us to employ eigenfunction expansions and to derive regularity and asymptotic estimates in a framework closer to standard parabolic theory.

Due to the wide range of applications in engineering and applied sciences, fractional differential equations (FDEs) have attracted significant attention. They appear in diverse fields such as electronic circuit design, control engineering, and voice signal simulation. Numerous studies have investigated both ordinary and partial differential equations involving fractional derivatives. For instance, results have been established concerning existence, uniqueness, and stability of solutions, as well as numerical methods for solving such equations [1, 2, 4, 6, 11, 12, 13, 14, 15, 17, 18, 20, 23, 26, 27, 29, 31, 32, 33, 35, 37, 38, 40, 41, 44, 46].

Research on conformable partial differential equations (PDEs) within Hilbert scales has attracted considerable interest from the mathematical community. Notable contributions in this area include works such as [3, 5, 7, 16, 28, 39, 42]. When analyzing conformable PDEs in Hilbert spaces, one encounters challenges related to the complexity of the solution space, which is significantly more sophisticated than in the case of ordinary differential equations (ODEs).

A noteworthy study on conformable parabolic equations in Hilbert spaces appears in [34], where the author investigates the existence and regularity of solutions for both linear and nonlinear cases. Building on this foundation, Tuan, Tien, and Chao focused on pseudo-parabolic equations involving conformable derivatives, employing novel techniques to establish the existence and regularity of mild solutions [24]. Fixed-point methods are also widely used in the solvability analysis of fractional and conformable models; see, e.g., [47].

More recently, in [36], the authors demonstrated the convergence of mild solutions of conformable diffusion equations to those of the classical problem. Similarly, in [25], the convergence of mild solutions for linear diffusion equations was proved as the order of fractional Laplacian approaches 1^-.

In the context of inverse problems, Ha, Hung, and Phuong [30] studied inverse source identification for diffusion equations with conformable derivatives. Utilizing a fractional Tikhonov regularization method, they constructed regularized solutions and derived error estimates quantifying the difference between the regularized and true solutions.

In a recent study [10], the authors examined a nonlinear Volterra equation involving a conformable derivative. We also note that boundary value problems involving additional fractional integral operators (loaded equations) have been investigated in related settings; see, for instance, [45]. They established the local-in-time existence of a mild solution and further demonstrated its convergence as a key parameter tends to zero.

The main contributions of this paper are summarized as follows:

- We derive an explicit representation of the mild solution to the proposed diffusion problem and establish regularity estimates in appropriate Sobolev spaces.
- We prove that the mild solution depends continuously on key parameters, particularly the fractional order and the strength of the memory term.
- We investigate the limiting behavior of the solution as the order of the conformable derivative approaches one, establishing convergence to the classical diffusion equation and providing explicit convergence rates.
- We identify conditions under which solutions exhibit finite-time blow-up, with a focus on nonlinear perturbations and their impact.
- Using fixed-point techniques — including Banachs and Schaefers fixed-point theorems — we rigorously prove the existence and uniqueness of mild solutions under various assumptions, emphasizing smallness conditions for nonlinear and memory-related terms.

The remainder of the paper is organized as follows. Section 2 is devoted to the linear problem, where we derive the mild-solution representation and establish well-posedness, regularity, and the asymptotic behavior as $\alpha \to 1^-$. Section 3 addresses nonlinear extensions, including a blow-up criterion and existence results based on fixed-point arguments.

2. Well-posedness and Regularity for the Linear Problem

To facilitate our analysis, we introduce the following functional spaces, as originally defined in [24, 36]. Let us assume that the operator $-\Delta$ admits a sequence of eigenvalues λ_n satisfying $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots$ with the property that $\lambda_k \to \infty$ as k tends to ∞. The corresponding eigenfunctions are denoted by $\psi_k \in H_0^1(\Omega) \cap H^2(\Omega)$. For all $s \geqslant 0$, we define the set $\mathbb{H}^s(\Omega)$ as follows:

$$\mathbb{H}^s(\Omega) = \left\{ \theta \in L^2(\Omega) : \sum_{k=1}^{\infty} |\langle \theta, \psi_k \rangle|^2 \lambda_k^{2s} < \infty \right\}. \tag{2.1}$$

The space $\mathbb{H}^s(\Omega)$ is a Banach space endowed with the norm

$$\|\theta\|_{\mathbb{H}^s(\Omega)} := \left(\sum_{k=1}^{\infty} |\langle \theta, \psi_k \rangle|^2 \lambda_k^{2s} \right)^{\frac{1}{2}}, \quad \theta \in \mathbb{H}^s(\Omega).$$

Theorem 2.1. *The mild solution to linear problem* (1.1) *is defined by*

$$y(x,t) = \sum_{k=1}^{\infty} \exp\Big(\frac{at^{\alpha}}{\alpha}\Big) \exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big)(a-\beta\lambda_k)f_k\psi_k(x) - \sum_{k=1}^{\infty}(a-\beta\lambda_k)f_k\psi_k(x) + f(x). \tag{2.2}$$

Proof. To find a precise formulation for solutions, we consider the formula of Fourier series as follows

$$y(x,t) = \sum_{k=1}^{\infty}\langle y(\cdot,t),\psi_k\rangle\psi_k(x). \tag{2.3}$$

The main equation gives us

$$\int_{\Omega}\frac{{}^{C}\partial^{\alpha}y(x,t)}{\partial t^{\alpha}}\psi_k(x)dx - \int_{\Omega}\Delta y(x,t)\psi_k(x)dx = \int_{\Omega} ay\psi_k(x)dx + \int_{\Omega}\int_0^t \Delta y_s(x,s)ds\psi_k(x)dx. \tag{2.4}$$

Let us denote

$$y_k(t) = \int_{\Omega} y(x,t)\psi_k(x)dx.$$

Then, we get

$${}^{C}\partial^{\alpha}y_k(t) - \beta\lambda_k y_k(t) = ay_k(t) + \lambda_k\int_0^t y_k'(s)ds. \tag{2.5}$$

Since the fact that

$${}^{C}\partial^{\alpha}y_k(t) = t^{1-\alpha}y_k'(t),$$

we obtain

$$t^{1-\alpha}y_k'(t) + \lambda_k y_k(t) = ay_k(t) - \beta\lambda_k\int_0^t y_k'(s)ds. \tag{2.6}$$

By taking the first derivative of the above equality, we find that

$$\Big(t^{1-\alpha}y_k'(t)\Big)' + \lambda_k y_k'(t) = ay_k'(t) - \beta\lambda_k y_k'(t). \tag{2.7}$$

Let $v_k(t) = t^{1-\alpha}y_k'(t)$. Then we get

$$v_k'(t) + \lambda_k t^{\alpha-1}v_k(t) = av_k(t) - \beta\lambda_k t^{\alpha-1}v_k(t), \tag{2.8}$$

$$v_k'(t) + \lambda_k\Big(t^{\alpha-1} + \beta t^{\alpha-1}\Big)v_k - at^{\alpha-1}v_k = 0. \tag{2.9}$$

It follows from (2.5) that

$$v_k(0) = {}^{C}\partial^{\alpha}y_k(0) = -\beta\lambda_k y_k(0) + ay_k(0) = (a-\beta\lambda_k)f_k.$$

Solving the systems, we find that

$$\begin{aligned} v_k(t) &= \exp\Big(-\lambda_k\int_0^t\Big(s^{\alpha-1}+\beta s^{\alpha-1}\Big)ds\Big)\exp(at)(a-\beta\lambda_k)f_k \\ &= \exp\Big(\frac{at^{\alpha}}{\alpha}\Big)\exp\Big(-\lambda_k\frac{(1+\beta)t^{\alpha}}{\alpha}\Big)(a-\beta\lambda_k)f_k. \end{aligned} \tag{2.10}$$

This implies that

$$y_k'(t) = \exp\Big(\frac{at^\alpha}{\alpha}\Big)t^{\alpha-1}\exp\Big(-\lambda_k\frac{(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k. \tag{2.11}$$

According to the definition of Fourier transform, we get that

$$\begin{aligned} y_k(t) &= \Big(\int_0^t \exp\Big(\frac{as^\alpha}{\alpha}\Big)s^{\alpha-1}\exp\Big(-\lambda_k\frac{(1+\beta)s^\alpha}{\alpha}\Big)ds\Big)(a-\beta\lambda_k)f_k + y_k(0)\\ &= \Big(\int_0^t \exp\Big(\frac{as^\alpha}{\alpha}\Big)s^{\alpha-1}\exp\Big(-\lambda_k\frac{(1+\beta)s^\alpha}{\alpha}\Big)ds\Big)(a-\beta\lambda_k)f_k + f_k. \end{aligned} \tag{2.12}$$

It is easy to verify that

$$\int_0^t \exp\Big(\frac{as^\alpha}{\alpha}\Big)s^{\alpha-1}\exp\Big(-\lambda_k\frac{(1+\beta)s^\alpha}{\alpha}\Big)ds = \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big) - 1.$$

This together with (2.12) yields to

$$\begin{aligned} y_k(t) =& \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k\\ &-(a-\beta\lambda_k)f_k + f_k. \end{aligned} \tag{2.13}$$

Thus, we have

$$\begin{aligned} y(x,t) &= \sum_{k=1}^\infty y_k(t)\psi_k(x)\\ &= \sum_{k=1}^\infty \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k\psi_k(x)\\ &\quad -\sum_{k=1}^\infty (a-\beta\lambda_k)f_k\psi_k(x) + \sum_{k=1}^\infty f_k\psi_k(x). \end{aligned} \tag{2.14}$$

Theorem 2.2. *Let* $f \in \mathbb{H}^1(\Omega)\cap\mathbb{H}^s(\Omega)$. *Then the following inequality holds*

$$\begin{aligned} \Big\|y\Big\|_{L^p(0,T;\mathbb{H}^s(\Omega))} \leqslant{}& C\bigg(\Big\|f\Big\|_{\mathbb{H}^{s-\gamma}(\Omega)} + \Big\|f\Big\|_{\mathbb{H}^{1-\gamma+s}(\Omega)}\bigg)\\ &+2a\Big\|f\Big\|_{L^2(\Omega)} + 2\beta\Big\|f\Big\|_{\mathbb{H}^1(\Omega)} + \Big\|f\Big\|_{\mathbb{H}^s(\Omega)}, \end{aligned} \tag{2.15}$$

where C *depends on* a, α, β, γ *and*

$$0 < \gamma < \frac{1}{\alpha}, \quad 1 < p < \frac{1}{\alpha\gamma}.$$

Proof. Using Parseval's equality, we find that

$$\begin{aligned} &\Big\|\sum_{k=1}^\infty \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k\psi_k(x)\Big\|^2_{\mathbb{H}^s(\Omega)}\\ &\quad = \exp\Big(\frac{2at^\alpha}{\alpha}\Big)\sum_{k=1}^\infty \lambda_k^{2s}\exp\Big(-2\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)^2 f_k^2. \end{aligned} \tag{2.16}$$

Using the inequality $e^{-z} \leqslant C_\gamma z^{-\gamma}$ for any $\gamma > 0$, one has

$$\exp\Big(-2\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big) \leqslant \Big(\frac{1+\beta}{\alpha}\Big)^{-2\gamma}\lambda_k^{-2\gamma}t^{-2\alpha\gamma} \tag{2.17}$$

and

$$(a-\beta\lambda_k)^2 \leqslant a^2 + \beta^2\lambda_k^2.$$

From the two above observations, we get

$$\begin{aligned}
&\sum_{k=1}^\infty \lambda_k^{2s}\exp\Big(-2\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)^2 f_k^2\\
&\leqslant a^2\Big(\frac{1+\beta}{\alpha}\Big)^{-2\gamma}t^{-2\alpha\gamma}\sum_{k=1}^\infty \lambda_k^{2s-2\gamma}f_k^2\\
&+\beta^2\Big(\frac{1+\beta}{\alpha}\Big)^{-2\gamma}t^{-2\alpha\gamma}\sum_{k=1}^\infty \lambda_k^{2s+2-2\gamma}f_k^2\\
&= a^2\Big(\frac{1+\beta}{\alpha}\Big)^{-2\gamma}t^{-2\alpha\gamma}\Big\|f\Big\|^2_{\mathbb{H}^{s-\gamma}(\Omega)} + \beta^2\Big(\frac{1+\beta}{\alpha}\Big)^{-2\gamma}t^{-2\alpha\gamma}\Big\|f\Big\|^2_{\mathbb{H}^{1-\gamma+s}(\Omega)}.
\end{aligned} \tag{2.18}$$

Thus, we get the following bound

$$\begin{aligned}
&\Big\|\sum_{k=1}^\infty \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k\psi_k(x)\Big\|_{\mathbb{H}^s(\Omega)}\\
&\leqslant C(a,\alpha,\beta,\gamma)t^{-\alpha\gamma}\bigg(\Big\|f\Big\|_{\mathbb{H}^{s-\gamma}(\Omega)} + \Big\|f\Big\|_{\mathbb{H}^{1-\gamma+s}(\Omega)}\bigg).
\end{aligned} \tag{2.19}$$

Next, let us give the following estimate

$$\Big\|\sum_{k=1}^\infty (a-\beta\lambda_k)f_k\psi_k(x)\Big\|^2_{\mathbb{H}^s(\Omega)} = \sum_{k=1}^\infty (a-\beta\lambda_k)^2|f_k|^2 \leqslant 2a^2\sum_{k=1}^\infty |f_k|^2 + 2\beta^2\sum_{k=1}^\infty \lambda_k^2|f_k|^2. \tag{2.20}$$

Then we get that

$$\Big\|\sum_{k=1}^\infty (a-\beta\lambda_k)f_k\psi_k(x)\Big\|_{\mathbb{H}^s(\Omega)} \leqslant 2a\Big\|f\Big\|_{L^2(\Omega)} + 2\beta\Big\|f\Big\|_{\mathbb{H}^1(\Omega)}. \tag{2.21}$$

Combining (2.19) and (2.21), we find that

$$\begin{aligned}
\Big\|y(\cdot,t)\Big\|_{\mathbb{H}^s(\Omega)} &\leqslant \Big\|\sum_{k=1}^\infty \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k\psi_k(x)\Big\|_{\mathbb{H}^s(\Omega)}\\
&+\Big\|\sum_{k=1}^\infty (a-\beta\lambda_k)f_k\psi_k(x)\Big\|_{\mathbb{H}^s(\Omega)} + \Big\|f\Big\|_{\mathbb{H}^s(\Omega)}\\
&\leqslant C(a,\alpha,\beta,\gamma)t^{-\alpha\gamma}\bigg(\Big\|f\Big\|_{\mathbb{H}^{s-\gamma}(\Omega)} + \Big\|f\Big\|_{\mathbb{H}^{1-\gamma+s}(\Omega)}\bigg)\\
&+2a\Big\|f\Big\|_{L^2(\Omega)} + 2\beta\Big\|f\Big\|_{\mathbb{H}^1(\Omega)} + \Big\|f\Big\|_{\mathbb{H}^s(\Omega)}.
\end{aligned} \tag{2.22}$$

Let us choose γ such that $0 < \gamma < \frac{1}{\alpha}$. Then we take a constant p such that

$$1 < p < \frac{1}{\alpha\gamma}.$$

This implies that $y \in L^p(0, T; \mathbb{H}^s(\Omega))$ and we also get the desired result (2.15).

∎

Theorem 2.3. *Assume that* $f \in \mathbb{H}^{s+1}(\Omega)$. *Let* y_β *and* $y_{\beta'}$ *be two solutions of Problem* (1.1). *Then we get*

$$\begin{aligned}&\Big\|y_\beta(\cdot,t) - y_{\beta'}(\cdot,t)\Big\|_{\mathbb{H}^s(\Omega)}\\ &\leqslant \overline{C} t^{-\alpha\varepsilon}\Big(|\beta-\beta'|^{\varepsilon/2} + |\beta-\beta'|^{\varepsilon}\Big)\Big\|f\Big\|_{\mathbb{H}^{s+1-\varepsilon}(\Omega)} + |\beta-\beta'|\Big\|f\Big\|_{\mathbb{H}^{s+1}(\Omega)}\end{aligned} \tag{2.23}$$

for any $\varepsilon > 0$. *Here* $\overline{C}$ *depends on* $\alpha, \beta, \varepsilon, \beta'$.

Proof. We recall that

$$\begin{aligned} y_\beta(x,t) &= \sum_{k=1}^{\infty} \exp\Big(\frac{a t^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k) f_k \psi_k(x)\\ &\quad - \sum_{k=1}^{\infty}(a-\beta\lambda_k) f_k\psi_k(x) + \sum_{k=1}^{\infty} f_k\psi_k(x). \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} y_{\beta'}(x,t) &= \sum_{k=1}^{\infty} \exp\Big(\frac{a t^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)(a-\beta'\lambda_k) f_k \psi_k(x)\\ &\quad - \sum_{k=1}^{\infty}(a-\beta'\lambda_k) f_k\psi_k(x) + \sum_{k=1}^{\infty} f_k\psi_k(x). \end{aligned} \tag{2.25}$$

Subtracting the sides of the two equations above, we get

$$\begin{aligned} &y_\beta(x,t) - y_{\beta'}(x,t)\\ &\quad= \sum_{k=1}^{\infty}\Big[\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)\\ &\quad - \exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)(a-\beta'\lambda_k)\Big] f_k\psi_k(x)\\ &\quad + (\beta-\beta')\sum_{k=1}^{\infty}\lambda_k f_k\psi_k(x) = \mathscr{K}_1 + \mathscr{K}_2. \end{aligned} \tag{2.26}$$

First, we get that

$$\begin{aligned} &\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k) - \exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)(a-\beta'\lambda_k)\\ &= \exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(\beta'-\beta)\lambda_k\\ &+ (a-\beta'\lambda_k)\Big[\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big) - \exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)\Big] = \mathscr{I}_1 + \mathscr{I}_2. \end{aligned} \tag{2.27}$$

Using the inequality $e^{-z} \leqslant C_\varepsilon z^{-\varepsilon}$ for any $\varepsilon > 0$, we find that

$$\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big) \leqslant C(\alpha,\beta,\varepsilon)\lambda_k^{-\varepsilon}t^{-\alpha\varepsilon}. \tag{2.28}$$

Hence, we derive that

$$\mathscr{I}_1 = \Big|\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(\beta'-\beta)\lambda_k\Big| \leqslant C(\alpha,\beta,\varepsilon)\lambda_k^{1-\varepsilon}t^{-\alpha\varepsilon}|\beta'-\beta|. \tag{2.29}$$

Using the inequality $|e^{-a}-e^{-b}| \leqslant \max(e^{-a},e^{-b})|a-b|^\mu$, we get that

$$\Big|\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big) - \exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)\Big| \leqslant C(\alpha,\beta,\varepsilon,\beta')\lambda_k^{-\varepsilon}t^{-\alpha\varepsilon}\lambda_k^{\mu}t^{\alpha\mu}|\beta-\beta'|^{\mu}. \tag{2.30}$$

Let us choose $\mu = \frac{\varepsilon}{2}$, we derive that

$$\Big|\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big) - \exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)\Big| \leqslant C(\alpha,\beta,\varepsilon,\beta')\lambda_k^{\frac{-\varepsilon}{2}}t^{\frac{-\alpha\varepsilon}{2}}|\beta-\beta'|^{\varepsilon/2}. \tag{2.31}$$

In view of $|a-\beta'\lambda_k| \leqslant (a\lambda_1^{-1}+\beta)\lambda_k$, we obtain

$$\mathscr{I}_2 \leqslant (a\lambda_1^{-1}+\beta)C(\alpha,\beta,\varepsilon,\beta')\lambda_k^{1-\frac{\varepsilon}{2}}t^{\frac{-\alpha\varepsilon}{2}}|\beta-\beta'|^{\varepsilon/2}. \tag{2.32}$$

By observing some previous estimates, we deduce that

$$\begin{aligned}\Big|\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k) - \exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)(a-\beta'\lambda_k)\Big| \\ \leqslant \overline{C}\lambda_k^{1-\frac{\varepsilon}{2}}t^{-\alpha\varepsilon}\Big(|\beta-\beta'|^{\varepsilon/2}+|\beta-\beta'|^{\varepsilon}\Big).\end{aligned} \tag{2.33}$$

Here $\overline{C}$ depends on $\alpha,\beta,\varepsilon,\beta'$. In view of Parseval's equality, we follow from (2.33) that

$$\begin{aligned}&\Big\|\mathscr{K}_1\Big\|^2_{\mathbb{H}^s(\Omega)} \\ &= \sum_{k=1}^\infty \lambda_k^{2s}\Big[\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k) - \exp\Big(-\frac{\lambda_k(1+\beta')t^\alpha}{\alpha}\Big)(a-\beta'\lambda_k)\Big]^2|f_k|^2 \\ &\leqslant |\overline{C}|^2t^{-2\alpha\varepsilon}\Big(|\beta-\beta'|^{\varepsilon/2}+|\beta-\beta'|^{\varepsilon}\Big)^2\sum_{k=1}^\infty \lambda_k^{2s+2-\varepsilon}|f_k|^2 \\ &= |\overline{C}|^2t^{-2\alpha\varepsilon}\Big(|\beta-\beta'|^{\varepsilon/2}+|\beta-\beta'|^{\varepsilon}\Big)^2\Big\|f\Big\|^2_{\mathbb{H}^{s+1-\varepsilon}(\Omega)}.\end{aligned} \tag{2.34}$$

Thus, we obtain that

$$\Big\|\mathscr{K}_1\Big\|_{\mathbb{H}^s(\Omega)} \leqslant \overline{C}t^{-\alpha\varepsilon}\Big(|\beta-\beta'|^{\varepsilon/2}+|\beta-\beta'|^{\varepsilon}\Big)\Big\|f\Big\|_{\mathbb{H}^{s+1-\varepsilon}(\Omega)}. \tag{2.35}$$

It is easy to see that

$$\begin{aligned}\Big\|\mathscr{K}_2\Big\|_{\mathbb{H}^s(\Omega)} &= \Big\|(\beta-\beta')\sum_{k=1}^\infty \lambda_k f_k\psi_k(x)\psi_k(x)\Big\|_{\mathbb{H}^s(\Omega)} \\ &= |\beta-\beta'|\sqrt{\sum_{k=1}^\infty \lambda_k^{2s+2}f_k^2} \leqslant |\beta-\beta'|\Big\|f\Big\|_{\mathbb{H}^{s+1}(\Omega)}.\end{aligned} \tag{2.36}$$

Combining (2.26), (2.35) and (2.36), we derive that

$$\begin{aligned}\Big\|y_\beta(\cdot,t)-y_{\beta'}(\cdot,t)\Big\|_{\mathbb{H}^s(\Omega)} &\leqslant \Big\|\mathscr{K}_1\Big\|_{\mathbb{H}^s(\Omega)}+\Big\|\mathscr{K}_2\Big\|_{\mathbb{H}^s(\Omega)}\\ &\leqslant \overline{C}t^{-\alpha\varepsilon}\Big(|\beta-\beta'|^{\varepsilon/2}+|\beta-\beta'|^{\varepsilon}\Big)\Big\|f\Big\|_{\mathbb{H}^{s+1-\varepsilon}(\Omega)}+|\beta-\beta'|\Big\|f\Big\|_{\mathbb{H}^{s+1}(\Omega)}. \end{aligned} \tag{2.37}$$

□

Lemma 2.4. *Let $\alpha_0 \leqslant \alpha \leqslant 1$. Then we get*

$$\Big|\frac{t^\alpha}{\alpha}-t\Big| \leqslant C(\alpha_0,\mu)t^{\alpha-\mu}\mathbf{M}(\alpha,\mu), \tag{2.38}$$

where any $\mu > 0$ and

$$\mathbf{M}(\alpha,\mu)=(1-\alpha)^\mu+(1-\alpha)+|T^{1-\alpha}-1|.$$

The proof of Lemma (2.4) can be found in the proof of Theorem 3 [36].

□

Theorem 2.5. *Let $f\in\mathbb{H}^{s+\gamma-\theta}(\Omega)\cap\mathbb{H}^{s+\gamma-\theta+2}(\Omega)$. Then we get*

$$\Big\|y_\alpha(\cdot,t)-y^*(\cdot,t)\Big\|_{\mathbb{H}^s(\Omega)} \leqslant CT^{(\alpha-\mu)\gamma-\theta}|\mathbf{M}(\alpha,\mu)|^\gamma\Big(\Big\|f\Big\|_{\mathbb{H}^{s+\gamma-\theta}(\Omega)}+\Big\|f\Big\|_{\mathbb{H}^{s+\gamma-\theta+2}(\Omega)}\Big). \tag{2.39}$$

Here $0<\mu<\alpha$ and $\theta<(\alpha-\mu)\gamma$ and $\mathbf{M}(\alpha,\mu)$ is defined in Lemma 2.4.

Proof. Let y_α be the mild solution of Problem (1.1). We have

$$\begin{aligned} y_\alpha(x,t)=&\sum_{k=1}^\infty \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k\psi_k(x)\\ &-\sum_{k=1}^\infty(a-\beta\lambda_k)f_k\psi_k(x)+f(x)\end{aligned} \tag{2.40}$$

and

$$\begin{aligned} y^*(x,t)=&\sum_{k=1}^\infty \exp(at)\exp\Big(-\lambda_k(1+\beta)t\Big)(a-\beta\lambda_k)f_k\psi_k(x)\\ &-\sum_{k=1}^\infty(a-\beta\lambda_k)f_k\psi_k(x)+f(x).\end{aligned} \tag{2.41}$$

From (2.40) and (2.41), we find that

$$\begin{aligned} y_\alpha(x,t)-y^*(x,t)=&\sum_{k=1}^\infty \exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)(a-\beta\lambda_k)f_k\psi_k(x)\\ &-\sum_{k=1}^\infty \exp(at)\exp\Big(-\lambda_k(1+\beta)t\Big)(a-\beta\lambda_k)f_k\psi_k(x).\end{aligned} \tag{2.42}$$

Thus, we have immediately that

$$\begin{aligned}&\Big\|y_\alpha(\cdot,t)-y^*(\cdot,t)\Big\|^2_{\mathbb{H}^s(\Omega)}\\ &=\sum_{k=1}^\infty\lambda_k^{2s}\Bigg(\exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)-\exp(at)\exp\Big(-\lambda_k(1+\beta)t\Big)\Bigg)^2(a-\beta\lambda_k)^2|f_k|^2.\end{aligned} \tag{2.43}$$

We need to give the upper bound of the term

$$\mathscr{J} = \left|\exp\Big(\frac{at^{\alpha}}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big) - \exp(at)\exp\Big(-\lambda_k(1+\beta)t\Big)\right|.$$

Indeed, we get that

$$\begin{aligned}\mathscr{J} &\leqslant \exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big)\Big|\exp\Big(\frac{at^{\alpha}}{\alpha}\Big) - \exp(at)\Big| \\ &\quad + \exp(at)\Big|\exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big) - \exp\Big(-\lambda_k(1+\beta)t\Big)\Big|.\end{aligned} \tag{2.44}$$

It is obvious to see that

$$\begin{aligned}&\exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big)\Big|\exp\Big(\frac{at^{\alpha}}{\alpha}\Big) - \exp(at)\Big| \\ &\leqslant C_{\gamma}\Big(\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big)^{-\gamma}\max\Big(\exp\Big(\frac{at^{\alpha}}{\alpha}\Big), \exp(at)\Big)\Big|\frac{at^{\alpha}}{\alpha} - at\Big|^{\gamma} \\ &\leqslant C(\gamma,\beta,\alpha)\max\Big(\exp(at), \exp\Big(\frac{aT^{\alpha}}{\alpha_0}\Big)\Big)a^{\gamma}\lambda_k^{-\gamma}\Big|\frac{t^{\alpha}}{\alpha} - t\Big|^{\gamma}.\end{aligned} \tag{2.45}$$

In view of Lemma (2.4), we get that

$$\Big|\frac{t^{\alpha}}{\alpha} - t\Big|^{\gamma} \leqslant C(\alpha_0,\mu)t^{(\alpha-\mu)\gamma}|\mathbf{M}(\alpha,\mu)|^{\gamma}. \tag{2.46}$$

From two above observations, we get that

$$\exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big)\Big|\exp\Big(\frac{at^{\alpha}}{\alpha}\Big) - \exp(at)\Big| \leqslant C_1 t^{(\alpha-\mu)\gamma}\lambda_k^{-\gamma}|\mathbf{M}(\alpha,\mu)|^{\gamma}, \tag{2.47}$$

where C_1 depends on γ,β,α,a. In addition, using the inequality

$$|e^{-a} - e^{-b}| \leqslant C(\theta_1,\theta_2)\max(a^{-\theta_1}, b^{-\theta_1})|a-b|^{\theta_2},$$

we get that

$$\begin{aligned}&\Big|\exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big) - \exp\Big(-\lambda_k(1+\beta)t\Big)\Big| \\ &\qquad \leqslant C(\beta,\gamma,\theta)\max\Big(t^{-\alpha\theta}, t^{-\theta}\Big)\lambda_k^{\gamma-\theta}\Big|\frac{t^{\alpha}}{\alpha} - t\Big|^{\gamma}.\end{aligned} \tag{2.48}$$

In view of (2.46), we derive that

$$\begin{aligned}\max\Big(t^{-\alpha\theta}, t^{-\theta}\Big)\Big|\frac{t^{\alpha}}{\alpha} - t\Big|^{\gamma} &\leqslant C(\alpha_0,\mu)\max\Big(t^{(\alpha-\mu)\gamma-\alpha\theta}, t^{(\alpha-\mu)\gamma-\theta}\Big)|\mathbf{M}(\alpha,\mu)|^{\gamma} \\ &\leqslant C(\alpha_0,\mu,T)t^{(\alpha-\mu)\gamma-\theta}|\mathbf{M}(\alpha,\mu)|^{\gamma},\end{aligned} \tag{2.49}$$

where we note that $\theta < (\alpha-\mu)\gamma$. By collecting (2.48) and (2.49), we obtain that

$$\Big|\exp\Big(-\frac{\lambda_k(1+\beta)t^{\alpha}}{\alpha}\Big) - \exp\Big(-\lambda_k(1+\beta)t\Big)\Big| \leqslant C_2 t^{(\alpha-\mu)\gamma-\theta}\lambda_k^{\gamma-\theta}|\mathbf{M}(\alpha,\mu)|^{\gamma}, \tag{2.50}$$

where C_2 depends on β,γ,θ,T. Combining (2.44), (2.47), and (2.50), we obtain

$$\mathscr{J} \leqslant Ct^{(\alpha-\mu)\gamma-\theta}\lambda_k^{\gamma-\theta}|\mathbf{M}(\alpha,\mu)|^{\gamma}. \tag{2.51}$$

It follows from (2.43) that

$$\begin{aligned}
&\Big\|y_\alpha(\cdot,t)-y^*(\cdot,t)\Big\|^2_{\mathbb{H}^s(\Omega)}\\
&=\sum_{k=1}^{\infty}\lambda_k^{2s}\Bigg(\exp\Big(\frac{at^\alpha}{\alpha}\Big)\exp\Big(-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\Big)-\exp(at)\exp\Big(-\lambda_k(1+\beta)t\Big)\Bigg)^2(a-\beta\lambda_k)^2|f_k|^2\\
&\leqslant 2C^2t^{2(\alpha-\mu)\gamma-2\theta}|\mathbf{M}(\alpha,\mu)|^{2\gamma}\sum_{k=1}^{\infty}\lambda_k^{2s+2\gamma-2\theta}\Big(a^2+\beta^2\lambda_k^2\Big)f_k^2. \qquad (2.52)
\end{aligned}$$

This implies that

$$\Big\|y_\alpha(\cdot,t)-y^*(\cdot,t)\Big\|_{\mathbb{H}^s(\Omega)}\leqslant CT^{(\alpha-\mu)\gamma-\theta}|\mathbf{M}(\alpha,\mu)|^\gamma\Big(\Big\|f\Big\|_{\mathbb{H}^{s+\gamma-\theta}(\Omega)}+\Big\|f\Big\|_{\mathbb{H}^{s+\gamma-\theta+2}(\Omega)}\Big). \qquad (2.53)$$

The proof is completed. □

3. Blow-up and Existence Results

In this section, we consider the semilinear/nonlinear extension of the linear model and establish a blow-up criterion as well as existence results via fixed-point arguments.

Theorem 3.1 (Blow-up Criterion for Nonlinear Perturbations). *Let the initial data $f\in H^1(\Omega)\cap L^{2(\sigma+1)}(\Omega)$, and assume that $\sigma>\frac{2}{d}$. Then there exists a finite time $T^*<\infty$ such that the mild solution y satisfies*

$$\limsup_{t\to T^*-}\|y(\cdot,t)\|_{L^{2(\sigma+1)}(\Omega)}=\infty.$$

Proof. We begin by defining the energy functional

$$E(t)=\|y(\cdot,t)\|^{2(\sigma+1)}_{L^{2(\sigma+1)}(\Omega)}=\int_\Omega|y(x,t)|^{2(\sigma+1)}\,dx.$$

Its conformable derivative is given by

$$\frac{{}^C\partial^\alpha E(t)}{\partial t^\alpha}=t^{1-\alpha}\frac{dE}{dt}$$

since

$$\frac{dE}{dt}=2(\sigma+1)\int_\Omega|y|^{2\sigma}y\,\frac{\partial y}{\partial t}\,dx.$$

Multiplying the main equation

$$\frac{C\partial^\alpha y}{\partial t^\alpha}=\Delta y+ay+\beta\int_0^t\Delta y(s)\,ds+|y|^\sigma y$$

by $|y|^{2\sigma}y$ and integrating over Ω yields

$$\int_\Omega\frac{C\partial^\alpha y}{\partial t^\alpha}|y|^{2\sigma}y\,dx=\int_\Omega\Big(\Delta y+ay+\beta\int_0^t\Delta y(s)\,ds\Big)|y|^{2\sigma}y\,dx+\int_\Omega|y|^{3\sigma+2}\,dx.$$

By the chain rule, the left-hand side becomes

$$t^{1-\alpha}\frac{dE}{dt}.$$

The nonlinear term is

$$\int_\Omega |y|^{3\sigma+2}\,dx = \|y\|^{3\sigma+2}_{L^{3\sigma+2}(\Omega)}.$$

We next estimate the linear terms. First, for the Laplacian term, an integration by parts gives

$$\begin{aligned}\int_\Omega \Delta y\,|y|^{2\sigma}y\,dx &= -\int_\Omega \nabla y\cdot\nabla\Big(|y|^{2\sigma}y\Big)\,dx\\ &= -(2\sigma+1)\int_\Omega |y|^{2\sigma}|\nabla y|^2\,dx \leqslant 0.\end{aligned}$$

Thus, the Laplacian is dissipative. For the memory term, by Hölder's inequality,

$$\left|\beta\int_\Omega\left(\int_0^t \Delta y(s)\,ds\right)|y|^{2\sigma}y\,dx\right| \leqslant \beta\int_0^t \|\Delta y(\cdot,s)\|_{L^2}\,\||y|^{2\sigma}y\|_{L^2}\,ds,$$

and noting that

$$\||y|^{2\sigma}y\|_{L^2} = \|y\|^{2\sigma+1}_{L^{2(\sigma+1)}(\Omega)} = E(t)^{\frac{2\sigma+1}{2(\sigma+1)}},$$

we see that the memory term is controlled by past energy levels. Similarly, for the linear term involving ay,

$$\left|a\int_\Omega |y|^{2\sigma+2}\,dx\right| \leqslant |a|\,\|y\|^{2(\sigma+1)}_{L^{2(\sigma+1)}(\Omega)} = |a|\,E(t).$$

The key step is to relate the nonlinear term to $E(t)$. To this end, we apply the Gagliardo–Nirenberg inequality:

$$\|y\|_{L^{3\sigma+2}(\Omega)} \leqslant C\,\|y\|^{1-\theta}_{H^1(\Omega)}\,\|y\|^{\theta}_{L^2(\Omega)},$$

with $\theta = d\left(\frac{1}{2}-\frac{1}{3\sigma+2}\right)$. For $\sigma > \frac{2}{d}$, we have $\theta = \frac{d}{2}\left(1-\frac{2}{3\sigma+2}\right) < 1$, ensuring the embedding $H^1(\Omega)\hookrightarrow L^{3\sigma+2}(\Omega)$. Raising the inequality to the power $3\sigma+2$ yields

$$\|y\|^{3\sigma+2}_{L^{3\sigma+2}(\Omega)} \leqslant C\,\|y\|^{(1-\theta)(3\sigma+2)}_{H^1(\Omega)}\,\|y\|^{\theta(3\sigma+2)}_{L^2(\Omega)}.$$

Assuming, via parabolic regularity, that

$$\|y\|_{H^1(\Omega)} \leqslant C\,\|y\|_{L^{2(\sigma+1)}(\Omega)},$$

and writing

$$\|y\|_{L^{2(\sigma+1)}(\Omega)} = E(t)^{\frac{1}{2(\sigma+1)}},$$

we deduce

$$\|y\|^{3\sigma+2}_{L^{3\sigma+2}(\Omega)} \geqslant C\,E(t)^{\frac{(1-\theta)(3\sigma+2)}{2(\sigma+1)}}.$$

Expressing θ as $\theta = \frac{d}{2}\left(\frac{3\sigma}{3\sigma+2}\right)$, the exponent becomes $\frac{3\sigma+2-\frac{3\sigma d}{2}}{2(\sigma+1)}$. For $\sigma > \frac{2}{d}$ this exponent exceeds 1, showing that the nonlinear term grows superlinearly in $E(t)$. Collecting all estimates leads to the differential inequality

$$t^{1-\alpha}\frac{dE}{dt} \geqslant C_1\,E(t)^{\nu} - C_2\,\beta\int_0^t E(s)^{\mu}\,ds - |a|\,E(t),$$

where

$$\nu = \frac{3\sigma+2}{2(\sigma+1)} > 1 \quad\text{and}\quad \mu = \frac{2\sigma+1}{2(\sigma+1)} < 1.$$

For large $E(t)$, the term $C_1\,E(t)^\nu$ dominates the lower order terms, so we can simplify to

$$t^{1-\alpha}\frac{dE}{dt} \geqslant \frac{C_1}{2}\,E(t)^\nu.$$

Separating variables and integrating gives

$$\int_{E(0)}^{E(t)} E^{-\nu}\,dE \geqslant \frac{C_1}{2}\int_0^t s^{\alpha-1}\,ds,$$
$$\frac{1}{1-\nu}\Big(E(t)^{1-\nu}-E(0)^{1-\nu}\Big) \geqslant \frac{C_1}{2\alpha}\,t^\alpha.$$

Rearranging, we obtain

$$E(t) \geqslant \left(E(0)^{1-\nu}-\frac{C_1(\nu-1)}{2\alpha}\,t^\alpha\right)^{-\frac{1}{\nu-1}}.$$

The blow-up occurs when the term in parentheses vanishes; that is, when

$$E(0)^{1-\nu}-\frac{C_1(\nu-1)}{2\alpha}\,t^\alpha = 0.$$

Defining the blow-up time T^* by

$$T^* = \left(\frac{2\alpha\,E(0)^{1-\nu}}{C_1(\nu-1)}\right)^{1/\alpha},$$

we conclude that as $t \to T^{*-}$,

$$E(t) \to \infty,$$

which implies

$$\limsup_{t\to T^{*-}} \|y(\cdot,t)\|_{L^{2(\sigma+1)}(\Omega)} = \infty.$$

Thus, for $\sigma > \frac{2}{d}$, the nonlinearity $|y|^\sigma y$ induces a superlinear growth in $E(t)$ that overwhelms the dissipative effects, leading to finite-time blow-up. This completes the proof. □

Remark 3.2 (Comparison with the classical semilinear parabolic case). For the classical semilinear heat equation

$$u_t - \Delta u = |u|^\sigma u,$$

it is well known that the exponent $\sigma = \frac{2}{d}$ plays the role of the critical Fujita threshold [21]. In the supercritical regime $\sigma > \frac{2}{d}$, finite-time blow-up may occur for sufficiently large initial data, and such blow-up is characterized by the divergence of suitable L^p-norms of the solution.

Although the condition $\sigma > \frac{2}{d}$ in Theorem 3.1 coincides with this classical threshold, the analytical mechanism leading to blow-up in the present model is fundamentally different. Indeed, the evolution of the energy functional

$$E(t) = \|y(\cdot,t)\|_{L^{2(\sigma+1)}(\Omega)}^{2(\sigma+1)}$$

is governed by a conformable fractional derivative, which introduces a nonlocal temporal structure absent in the classical parabolic setting. As a consequence, the blow-up criterion established in Theorem 3.1 is obtained via fractional energy inequalities rather than standard parabolic energy dissipation or comparison arguments.

We first establish existence and uniqueness of a mild solution under a contraction condition by applying Banach's fixed-point theorem.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and let $T > 0$. Suppose that*

$$f \in H^{s+2}(\Omega) \quad \textit{for some } s > \frac{d}{2},$$

and let $F : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz with Lipschitz constant L. Assume the smallness condition

$$L\,T^\alpha + \beta\lambda_1 T^\alpha < \frac{\alpha}{2}.$$

Then there exists a unique mild solution

$$y \in L^\infty(0,T; H^{s+2}(\Omega)) \cap C([0,T]; H^s(\Omega))$$

to the problem.

Proof. We begin by expanding the solution in the eigenbasis. Let $\{\psi_k\}_{k\geqslant 1}$ be an orthonormal basis of $L^2(\Omega)$ such that

$$-\Delta\psi_k = \lambda_k\psi_k, \quad \psi_k|_{\partial\Omega} = 0.$$

Express

$$y(x,t) = \sum_{k=1}^{\infty} y_k(t)\psi_k(x), \quad \text{where } y_k(t) = \langle y(\cdot,t), \psi_k\rangle_{L^2(\Omega)}.$$

Since the conformable derivative satisfies

$${}^C\partial_t^\alpha y_k(t) = t^{1-\alpha} y_k'(t),$$

substitute the expansion into the PDE and project onto ψ_k to obtain

$$\begin{aligned} t^{1-\alpha} y_k'(t) + \lambda_k y_k(t) &= \langle F(y(t)), \psi_k\rangle + \beta \left\langle \int_0^t \Delta y(\cdot,s)\,ds, \psi_k \right\rangle \\ &= \langle F(y(t)), \psi_k\rangle - \beta\lambda_k \int_0^t y_k(s)\,ds. \end{aligned}$$

Grouping terms gives

$$t^{1-\alpha} y_k'(t) + \lambda_k(1+\beta) y_k(t) = \langle F(y(t)), \psi_k\rangle.$$

Multiplying by the integrating factor

$$\exp\left(\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}\right)$$

and integrating from 0 to t (with $y_k(0) = f_k = \langle f, \psi_k\rangle$) yields

$$\begin{aligned} y_k(t) = {} & e^{-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}} f_k \\ & + \int_0^t e^{-\frac{\lambda_k(1+\beta)(t^\alpha - s^\alpha)}{\alpha}} \frac{\langle F(y(s)), \psi_k\rangle}{s^{1-\alpha}}\,ds. \end{aligned}$$

Define the semigroup

$$S(t)\phi = \sum_{k=1}^{\infty} e^{-\frac{\lambda_k(1+\beta)t^\alpha}{\alpha}} \langle \phi, \psi_k\rangle \psi_k.$$

Then the mild solution is given by

$$y(t) = S(t)f + \int_0^t S(t-s)\frac{F(y(s))}{s^{1-\alpha}}\,ds + \beta\int_0^t S(t-s)\left(\int_0^s \Delta y(\tau)\,d\tau\right)\frac{ds}{s^{1-\alpha}}.$$

We now define the Banach space
$$\mathcal{X} = C([0,T]; H^s(\Omega))$$
with norm
$$\|y\|_{\mathcal{X}} = \sup_{t\in[0,T]} \|y(t)\|_{H^s}.$$
For technical convenience, we introduce the weighted norm
$$\|y\|_\gamma = \sup_{t\in[0,T]} e^{-\gamma t}\|y(t)\|_{H^s},$$
with $\gamma > 0$. Define the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by
$$\mathcal{T}y(t) = S(t)f + \int_0^t S(t-s)\frac{F(y(s))}{s^{1-\alpha}}\,ds + \beta\int_0^t S(t-s)\left(\int_0^s \Delta y(\tau)\,d\tau\right)\frac{ds}{s^{1-\alpha}}.$$

Using the eigenfunction expansion, one can verify that for any $\phi \in H^s(\Omega)$,
$$\|S(t)\phi\|_{H^s}^2 = \sum_{k=1}^{\infty} \lambda_k^{2s} e^{-\frac{2\lambda_k(1+\beta)t^\alpha}{\alpha}} |\langle\phi,\psi_k\rangle|^2 \leqslant \|\phi\|_{H^s}^2,$$
so that $\|S(t)\|_{\mathcal{L}(H^s)} \leqslant 1$. For any $y_1, y_2 \in \mathcal{X}$, the Lipschitz continuity of F yields
$$\|F(y_1(s)) - F(y_2(s))\|_{H^s} \leqslant L\|y_1(s) - y_2(s)\|_{H^s}.$$
Thus,
$$\begin{aligned}\left\|\int_0^t S(t-s)\frac{F(y_1(s)) - F(y_2(s))}{s^{1-\alpha}}\,ds\right\|_{H^s} &\leqslant L\int_0^t \|y_1(s) - y_2(s)\|_{H^s}\frac{ds}{s^{1-\alpha}}\\ &\leqslant \frac{L\,t^\alpha}{\alpha}\|y_1 - y_2\|_{\mathcal{X}}.\end{aligned}$$
In the weighted norm, using $\|y(s)\|_{H^s} \leqslant e^{\gamma s}\|y\|_\gamma$, we have
$$\left\|\int_0^t S(t-s)\frac{F(y_1(s)) - F(y_2(s))}{s^{1-\alpha}}\,ds\right\|_\gamma \leqslant \frac{L\,T^\alpha e^{\gamma T}}{\alpha}\|y_1 - y_2\|_\gamma.$$

Similarly, writing
$$\Delta y(s) = -\sum_{k=1}^{\infty} \lambda_k y_k(s)\psi_k,$$
we have
$$\int_0^s \Delta y(\tau)\,d\tau = -\sum_{k=1}^{\infty}\lambda_k\left(\int_0^s y_k(\tau)\,d\tau\right)\psi_k.$$
Then, using $\|S(t-s)\|_{\mathcal{L}(H^s)} \leqslant 1$ and $\lambda_k \geqslant \lambda_1$, it follows that
$$\left\|S(t-s)\left(\int_0^s \Delta\big(y_1(\tau) - y_2(\tau)\big)\,d\tau\right)\right\|_{H^s} \leqslant \lambda_1\int_0^s \|y_1(\tau) - y_2(\tau)\|_{H^s}\,d\tau.$$
Thus, the memory term satisfies
$$\left\|\beta\int_0^t S(t-s)\left(\int_0^s \Delta\big(y_1(\tau) - y_2(\tau)\big)\,d\tau\right)\frac{ds}{s^{1-\alpha}}\right\|_{H^s} \leqslant \frac{\beta\lambda_1 T^\alpha}{\alpha}\|y_1 - y_2\|_{\mathcal{X}}.$$

In the weighted norm, this term is bounded by

$$\frac{\beta\lambda_1 T^\alpha e^{\gamma T}}{\alpha\gamma}\|y_1 - y_2\|_\gamma.$$

Combining the two estimates, we have

$$\|\mathcal{T}y_1 - \mathcal{T}y_2\|_\gamma \leqslant \left(\frac{L T^\alpha}{\alpha} + \frac{\beta\lambda_1 T^\alpha}{\alpha\gamma}\right) e^{\gamma T}\|y_1 - y_2\|_\gamma.$$

Choosing $\gamma = \frac{2\beta\lambda_1 T^\alpha}{\alpha}$ and assuming

$$L T^\alpha + \beta\lambda_1 T^\alpha < \frac{\alpha}{2},$$

the contraction factor

$$K = \left(\frac{L T^\alpha}{\alpha} + \frac{1}{2}\right) e^{\gamma T}$$

is strictly less than 1. Hence, by the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$, which is the unique mild solution of the problem.

Finally, since $f \in H^{s+2}(\Omega)$ and the semigroup $S(t)$ is smoothing (i.e., $S(t) : H^s(\Omega) \to H^{s+2}(\Omega)$), we conclude that

$$y \in L^\infty(0,T; H^{s+2}(\Omega)) \cap C([0,T]; H^s(\Omega)).$$

This completes the proof. □

Next, we drop the contraction requirement and prove the existence of at least one mild solution by means of Schaefer's fixed-point theorem, based on compactness and uniform a priori bounds.

Theorem 3.4. *Assume $f \in \mathbb{H}^{s+2}(\Omega)$. Let $F : \mathbb{H}^s(\Omega) \to \mathbb{H}^s(\Omega)$ be a continuous map such that the image of every bounded set under F is precompact. Then there exists at least one $y \in \mathcal{X} = C([0,T]; H^s(\Omega))$ which is a mild solution of Problem* (1.1).

Proof. We provide a detailed outline of the proof, addressing all key steps.

Step 1. Consistency and Smoothing Properties of $S(t)$.
For any $\phi \in \mathbb{H}^{s-2}(\Omega)$ with expansion $\phi = \sum_{k=1}^\infty \phi_k\psi_k$, we have

$$S(t-s)\phi = \sum_{k=1}^\infty \exp\Big(-\frac{\lambda_k(1+\beta)(t-s)^\alpha}{\alpha}\Big)\phi_k\,\psi_k.$$

Thus,

$$\|S(t-s)\phi\|_{\mathbb{H}^s}^2 = \sum_{k=1}^\infty \lambda_k^{2s}\left|e^{-\frac{\lambda_k(1+\beta)(t-s)^\alpha}{\alpha}}\phi_k\right|^2 = \sum_{k=1}^\infty \lambda_k^{2(s-2)}\lambda_k^4\, e^{-\frac{2\lambda_k(1+\beta)(t-s)^\alpha}{\alpha}}|\phi_k|^2.$$

Standard semigroup estimates (see, e.g., Pazy's *Semigroups of Linear Operators*) imply that there exists $C_S > 0$ such that for all k and $t-s > 0$,

$$\lambda_k^4\, e^{-\frac{2\lambda_k(1+\beta)(t-s)^\alpha}{\alpha}} \leqslant C_S^2\,(t-s)^{-2\alpha}.$$

Hence, we obtain

$$\|S(t-s)\phi\|_{\mathbb{H}^s}^2 \leqslant C_S^2\,(t-s)^{-2\alpha}\sum_{k=1}^\infty \lambda_k^{2(s-2)}|\phi_k|^2 = C_S^2\,(t-s)^{-2\alpha}\|\phi\|_{\mathbb{H}^{s-2}}^2.$$

Taking square roots, we obtain the smoothing estimate:

$$\|S(t-s)\phi\|_{\mathbb{H}^s} \leqslant C_S\,(t-s)^{-\alpha}\|\phi\|_{\mathbb{H}^{s-2}}.$$

Step 2. Precompactness of the Nonlinear Term.
Define

$$\mathcal{N}y(t) = \int_0^t S(t-s)F(y(s))\,\frac{ds}{s^{1-\alpha}}.$$

Let $\{y_n\} \subset \mathcal{X}$ be bounded (i.e., $\|y_n\|_{\mathcal{X}} \leqslant M$). By hypothesis, for each fixed $s \in [0, T]$, the set $\{F(y_n(s))\}$ is precompact in $\mathbb{H}^s(\Omega)$ and there exists $C > 0$ such that

$$\|F(y_n(s))\|_{\mathbb{H}^s} \leqslant C(1+M).$$

Since $\|S(t-s)\|_{\mathcal{L}(\mathbb{H}^s)} \leqslant 1$, we have

$$\|\mathcal{N}y_n(t)\|_{\mathbb{H}^s} \leqslant \int_0^t \|F(y_n(s))\|_{\mathbb{H}^s}\,\frac{ds}{s^{1-\alpha}} \leqslant C(1+M)\frac{T^\alpha}{\alpha}.$$

Furthermore, for $t_1, t_2 \in [0, T]$ with $t_1 < t_2$,

$$\begin{aligned}\mathcal{N}y_n(t_2) - \mathcal{N}y_n(t_1) &= \int_0^{t_1}\Big[S(t_2-s) - S(t_1-s)\Big]F(y_n(s))\,\frac{ds}{s^{1-\alpha}}\\ &\quad + \int_{t_1}^{t_2} S(t_2-s)F(y_n(s))\,\frac{ds}{s^{1-\alpha}}.\end{aligned}$$

By the strong continuity of $S(t)$ and uniform boundedness of $F(y_n(s))$, the right-hand side tends to zero as $t_2 \to t_1$, uniformly in n. Thus, by the BochnerArzelàAscoli theorem, $\mathcal{N}$ is compact.

Step 3. Precompactness of the Memory Term.
Define

$$\mathcal{M}y(t) = \beta\int_0^t S(t-s)\Big(\int_0^s \Delta y(\tau)\,d\tau\Big)\frac{ds}{s^{1-\alpha}}.$$

For y_n with $\|y_n\|_{\mathcal{X}} \leqslant M$, since $\Delta : \mathbb{H}^s(\Omega) \to \mathbb{H}^{s-2}(\Omega)$ is continuous,

$$\|\Delta y_n(\tau)\|_{\mathbb{H}^{s-2}} \leqslant C_\Delta M.$$

Thus,

$$\Big\|\int_0^s \Delta y_n(\tau)\,d\tau\Big\|_{\mathbb{H}^{s-2}} \leqslant T\,M.$$

Then by the smoothing property of $S(t-s)$, one has

$$\|S(t-s)(\int_0^s \Delta y_n(\tau)\,d\tau)\|_{\mathbb{H}^s} \leqslant C_S\,(t-s)^{-\alpha}\,T\,M.$$

Hence,

$$\|\mathcal{M}y_n(t)\|_{\mathbb{H}^s} \leqslant \beta\,C_S\,T\,M\int_0^t (t-s)^{-\alpha}\,\frac{ds}{s^{1-\alpha}} = \beta\,C_S\,T\,M\,B(1-\alpha,\alpha),$$

so that $\mathcal{M}y_n(t)$ is uniformly bounded in $\mathcal{X}$. To show precompactness, express y_n in the eigenbasis:

$$y_n(x,t) = \sum_{k=1}^{\infty} y_{n,k}(t)\,\psi_k(x).$$

Then, we find that

$$\int_0^s \Delta y_n(\tau)\,d\tau = \sum_{k=1}^{\infty} \lambda_k \Big(\int_0^s y_{n,k}(\tau)\,d\tau\Big)\psi_k(x),$$

and thus,

$$\mathcal{M}y_n(t) = \beta \sum_{k=1}^{\infty} \lambda_k \left[\int_0^t e^{-\frac{\lambda_k(1+\beta)(t-s)^\alpha}{\alpha}} \Big(\int_0^s y_{n,k}(\tau)\,d\tau\Big) s^{\alpha-1}\,ds\right]\psi_k(x).$$

Using the CauchySchwarz inequality in time, one obtains, for each fixed k,

$$\left|\int_0^s y_{n,k}(\tau)\,d\tau\right| \leqslant s^{1/2}\left(\int_0^s |y_{n,k}(\tau)|^2\,d\tau\right)^{1/2}.$$

Since $\|y_n\|_{\mathcal{X}} \leqslant M$ implies

$$\sup_{t\in[0,T]} \sum_{k=1}^{\infty} \lambda_k^{2s}|y_{n,k}(t)|^2 \leqslant M^2,$$

a standard tail estimate using the rapid decay of the exponential shows that for any $\varepsilon > 0$ there exists N such that uniformly in n

$$\sum_{k=N}^{\infty} \lambda_k^{2s}\left|\beta\,\lambda_k \int_0^t e^{-\frac{\lambda_k(1+\beta)(t-s)^\alpha}{\alpha}} \left(\int_0^s y_{n,k}(\tau)\,d\tau\right) s^{\alpha-1}\,ds\right|^2 < \varepsilon.$$

Moreover, an $\varepsilon\delta$ argument based on the strong continuity of $S(t)$ shows that $\mathcal{M}y_n$ is equicontinuous in time. Thus, $\mathcal{M}$ is compact.

Step 4. Compact Embedding and Conclusion.

Since $S(t-s)$ is smoothing, for each $t > 0$ the operators $\mathcal{N}$ and $\mathcal{M}$ yield functions in $\mathbb{H}^{s+2}(\Omega)$. In particular,

$$\mathcal{T}y_n(t) = S(t)f + \mathcal{N}y_n(t) + \mathcal{M}y_n(t) \in \mathbb{H}^{s+2}(\Omega).$$

By the RellichKondrachov theorem, the embedding

$$\mathbb{H}^{s+2}(\Omega) \hookrightarrow \mathbb{H}^{s}(\Omega)$$

is compact. Hence, for each fixed $t \in (0,T]$, the set $\{\mathcal{T}y_n(t) : n \in \mathbb{N}\}$ is precompact in $\mathbb{H}^s(\Omega)$. Together with the uniform boundedness and equicontinuity (in t) of the family $\{\mathcal{T}y_n\}$ in $\mathcal{X}$, the ArzelàAscoli theorem for Bochner spaces implies that $\{\mathcal{T}y_n\}$ has a convergent subsequence in $\mathcal{X}$. Therefore, $\mathcal{T}$ is a compact operator on $\mathcal{X}$.

Step 5. Continuity of $\mathcal{T}$.

Let $y_n \to y$ in $\mathcal{X}$; i.e., $\sup_{t\in[0,T]}\|y_n(t)-y(t)\|_{\mathbb{H}^s} \to 0$. For the semigroup term $S(t)f$, continuity is immediate. For the nonlinear term,

$$\begin{aligned}\|\mathcal{N}y_n(t) - \mathcal{N}y(t)\|_{\mathbb{H}^s} &\leqslant \int_0^t \|S(t-s)\|_{\mathcal{L}(\mathbb{H}^s)}\,\|F(y_n(s)) - F(y(s))\|_{\mathbb{H}^s}\frac{ds}{s^{1-\alpha}}\\ &\leqslant \int_0^t \|F(y_n(s)) - F(y(s))\|_{\mathbb{H}^s}\frac{ds}{s^{1-\alpha}},\end{aligned}$$

and by the continuity of F and the Dominated Convergence Theorem for Bochner integrals, the right-hand side tends to zero uniformly in t. An analogous argument applies to the memory term. Thus, $\mathcal{T}y_n \to \mathcal{T}y$ in $\mathcal{X}$, so $\mathcal{T}$ is continuous.

Step 6. Application of Schaefers Fixed Point Theorem.

Let

$$X = C([0,T];\mathbb{H}^s(\Omega))$$

and define

$$\mathcal{T}y(t) = S(t)f + \int_0^t S(t-s)F(y(s))\,\frac{ds}{s^{1-\alpha}} + \beta\int_0^t S(t-s)\Big(\int_0^s \Delta y(\tau)\,d\tau\Big)\frac{ds}{s^{1-\alpha}}.$$

From all observations we introduce above, by Schaefers Fixed Point Theorem, there exists at least one $y \in X$ such that

$$y = \mathcal{T}y,$$

i.e., a mild solution of the fractional evolution equation exists.

□

Remark. The application of Schaefer's fixed-point theorem yields existence of a mild solution, but it does not provide uniqueness in general.

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