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Exploring rank, determinant of tropical Toeplitz matrices and solvability of tropical Toeplitz linear system

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Abstract

Toeplitz matrices exhibit a fascinating characteristic in that their elements are constant along each diagonal. This inherent structure not only gives rise to several intriguing theoretical properties but also holds significant importance due to its strong impact on practical applications. In tropical algebra, the tropical algebraic eigenvalues, ranks, and determinants of Toeplitz matrices can be obtained without computing the coefficients of their characteristic max-polynomials, which is a notable advantage of tropical Toeplitz matrices. In recent years, cryptographic algorithms based on tropical linear and nonlinear Toeplitz systems have been shown to be vulnerable to attacks that rely on solving tropical Toeplitz linear and nonlinear systems. Our research is motivated by the close relationship between Toeplitz and Hankel matrices, as well as by the important role their determinants play in various branches of mathematics. Additionally, our research aims to explore attack techniques for tropical Toeplitz-based cryptographic protocols, which involve solving tropical linear and nonlinear systems. In this paper, we analyze the determinants and various notions of rank for different classes of tropical Toeplitz matrices. We establish conditions under which the determinant is singular or nonsingular. Furthermore, we investigate the conditions under which tropical Toeplitz matrices are balanced or unbalanced. Finally, we identify conditions for the existence and uniqueness of solutions to tropical Toeplitz linear systems.

Keywords: Tropical linear system, Tropical determinant, Tropical rank, Tropical toeplitz matrix.

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1. Introduction

Tropical algebra is correlated with tropical geometry [2, 30] and linear algebra, with extensions for the operations $\oplus$ and $\otimes$, where $\oplus$ stand for the maximum operation and $\otimes$ denote the usual addition. Tropical semiring is a semiring, in which the addition operation is replaced with maximum or minimum operation and multiplication operation is replaced by usual addition[23, 3]. If we use maximum then the semiring is called max-plus semiring. We call the tropical semiring as min-plus then minimum is stands for addition. These notations are of significant importance in formulating and solving certain linear and non-linear problems, similar to classical algebra. In classical linear algebra, the determinant plays a crucial role.

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Tropical semiring has important applications in computer science [4, 35, 5, 24], linear algebra, number theory, automata theory, language theory, control theory, operation research. In cryptography max-plus semirings were used to frame a new key exchange protocols [25, 36]and algebraic methods were used to attack a key exchange protocol [22, 26]. Also we can use max-plus concepts in graph theory [8]. In tropical determinants, we substitute addition for multiplication and the maximum operation for subtraction. The primary purpose of finding the determinant is to examine the stability conditions for linear equations. The investigation of the symmetrization of tropical algebra explores the conditions under which general independent max-plus linear equations carry non-trivial solutions. [6, 7]. Toeplitz matrices and operators are thoroughly studied and well-known classes in linear algebra. In toeplitz matrices an approach of operator systems brings a recent perspective of block toeplitz matrices and its positivity [16]. Our research is motivated by the close relationship of the toeplitz matrices and hankel matrices, as well as by the close relationship of determinants of the toeplitz matrices and hankel matrices play an crucial role in various domains of mathematics and have many applications [42]. In toeplitz matrices constant entries are placed along the diagonal, similarly in hankel matrices constant entries are occupied the location of reverse diagonal. Tropical algebraic eigenvalues of tridiagonal toeplitz matrices in tropical algebra are obtained without the need of directly computing the coefficients of their characteristic max-polynomials [39]. These properties and applications of toeplitz matrices can extended to the properties and applications of tropical toeplitz matrices. The applications of toeplitz matrices arise in the field of algebra, combinatorics, analysis, differential geometry [9, 28, 17, 21]. Also applicable in the pure mathematics integral equations, operator algebra , graph theory , partial differential equations [14, 31, 19, 40]. Similarly used in representation theory , topology, approximation theory , compressive sensing [34, 13, 38, 18]. As well as the concepts in probability , numerical integral equations, image processing , optimal control are has a aspects of toeplitz matrices [41, 12, 10, 29]. In area of numerical partial differential equations , quantum mechanics , queueing networks, signal processing we can see the application of toeplitz matrices [37, 15, 1]. In this paper we check whether the determinant of tropical toeplitz matrices is balanced or not. We present the exact determinant values of the tropical toeplitz matrices in all the scenarios. Also we analyse the general kapranov rank, barvinok rank, tropical rank of the tropical toeplitz matrices and types of the tropical toeplitz matrices. Also we estimate the solutions of the tropical toeplitz systems in all situations. The sections of this paper are organized as follows: In section 2 we explain the important preliminaries and basic definitions for readers clarification. In section 3 we discuss the main results such as theorems on the kapranov rank, barvinok rank, tropical rank and determinant of the tropical toeplitz matrices. In section 4 we prove some theorems on obtained solutions of tropical toeplitz matrices in particular situations.

2. Preliminaries

The tropical semiring $(\mathbb{Z} \cup -\infty, \oplus, \otimes)$ is the extended real number system with the tropical addition and tropical multiplication defined by,

$$(x_1 \oplus x_2) = \max\{x_1, x_2\}, \forall\ x_1, x_2 \in \tilde{\mathbb{Z}}$$

$$(x_1 \otimes x_2) = x_1 + x_2, \forall\ x_1, x_2 \in \tilde{\mathbb{Z}}$$

The tropical semiring (max-plus semiring) $R = (S \cup -\infty, \oplus, \otimes)$, which is also a semiring, where the operations $\oplus$ and $\otimes$ stands for maximum tropical addition and maximum tropical multiplication, respectively. Here, S is the semiring and R should satisfy the following axioms:

1. $a \oplus b = b \oplus a$, $\forall\ a, b \in R$.
2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, $(a \otimes b) \otimes c = a \otimes (b \otimes c)$, $\forall\ a, b, c \in R$.
3. $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, $\forall\ a, b, c \in R$.
4. $\exists\ e \in R$, $\forall\ a \in R$ such that $e \oplus a = a \oplus e = a$, where the additive identity is $-\infty$.
5. It never has an additive inverse.

Similarly, in the minimum tropical semiring, instead of the maximum, we substitute the minimum [33, 27]. The maximum tropical semiring is an idempotent semiring, and all idempotent semirings are zero-sum free [32]. We give the notation for collecting all $m \times n$ matrices over the semiring as $M_{m\times n}(S)$ and we denote every ij-th element of the matrix $P \in M_{m\times n}(S)$ as p_{ij}. The transpose of the matrix P is denoted by P^T. Let $P = (p_{ij}) \in M_{m\times n}(S)$, $Q = (q_{ij}) \in M_{m\times n}(S)$, $T = (t_{ij}) \in M_{n\times l}(S)$ and $\alpha \in S$. Addition, of two matrices generally calculated by

$$P + Q = ((p_{ij}) + (q_{ij}))_{m\times n}$$

Similarly product of two matrices PT can be calculated by,

$$\sum_{i=1}^{n} ((p_{ik})(t_{kj}))_{m\times l}$$

and

$$\alpha P = (\alpha(p_{ij}))_{m\times n}$$

In the max-plus semiring addition of two tropical matrices $P \oplus Q$ can be obtained by

$$(\max((p_{ij}), (q_{ij})))_{m\times l}$$

The multiplication of two tropical matrices $P\otimes T$ is calculated by

$$\max\left((p_{ik}) + (t_{kj})\right)_{m\times l}$$

and

$$\alpha \otimes P = (\alpha + (p_{ij}))_{m\times n}$$

A matrix $P \in M_{m\times n}(S)$ is said to be a tropical matrix if every elements of a matrix were taken from the tropical semiring $R = (S \cup (\pm\infty), \oplus, \otimes)$. A system $P \otimes x = q$ is said to be a tropical system if all the entries of the system from the tropical matrix. Let $S = \mathbb{R} \cup \{-\infty\}$ be the extended real number system under the max- plus algebra and let P, Q be a $m \times n$ matrices over the extended real numbers under the operation of maximum tropical semirings. Where $P = (p_{ij})_{m\times n}$, $Q = (q_{ij})_{m\times n}$ and (p_{ij}), (q_{ij}) are the ij^{th} entries of P and Q respectively, $P \leqslant Q \leftrightarrow (p_{ij}) \leqslant (q_{ij})$, $\forall\, i, j$ [20, 27, 32]. A matrix $P = (p_{ij})$ is said to be regular if $(p_{ij}) \neq \pm\infty$. A vector $b \in S^m$ is said to be a normal vector or regular vector if $b_j \neq -\infty$, $\forall\, j \in m$ [33]. Where we have consider max-plus semiring, if we consider the min-plus algebra then in the regular vector each entries $b_j \neq \infty$, $\forall\, j \in m$. A solution x^* of the tropical system $P \otimes x = q$ is called as the maximal solution if $x \leqslant x^*$ for any other solution x. A linear system $P \otimes x = q$ is said to be a tropical linear system if the elements of the linear system are all from any one of the tropical semirings. Through out this paper we used the tropical semiring $\tilde{Z} = (\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes)$

Definition 2.1. A matrix $P \in M_{m\times m}(\tilde{Z})$ is said to be a Toeplitz matrix if it is in the form of

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ b_1 & a_1 & a_2 & a_3 & \ddots & \vdots \\ b_2 & b_1 & a_1 & a_2 & \ddots & \vdots \\ b_3 & b_2 & b_1 & \ddots & \ddots & a_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_2 \\ b_{n-1} & \cdots & b_3 & b_2 & b_1 & a_1 \end{bmatrix}$$

Definition 2.2. A matrix $P \in M_{m\times m}(\tilde{Z})$ is said be a Symmetric toeplitz matrix if it is in the form of

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_1 & a_2 & a_3 & \ddots & \vdots \\ a_3 & a_2 & a_1 & a_2 & \ddots & \vdots \\ a_4 & a_3 & a_2 & \ddots & \ddots & a_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_2 \\ a_n & \cdots & a_4 & a_3 & a_2 & a_1 \end{bmatrix}$$

Definition 2.3. A matrix $P_\alpha \in M_{m\times m}(\tilde{Z})$ is said be a tridiagonal toeplitz matrix if it is in the form of

$$\begin{bmatrix} a & b & \alpha & \alpha & \cdots & \alpha \\ c & a & b & \alpha & \ddots & \vdots \\ \alpha & c & a & b & \ddots & \vdots \\ \alpha & \alpha & c & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & \alpha & c & a \end{bmatrix}$$

Definition 2.4. A matrix $P \in M_{m\times m}(\tilde{Z})$ is said to be a Block toeplitz matrix if it is in the form of

$$\begin{bmatrix} [A_1] & [A_2] & [A_3] & [A_4] & \cdots & [A_n] \\ [B_1] & [A_1] & [A_2] & [A_3] & \ddots & \vdots \\ [B_2] & [B_1] & [A_1] & [A_2] & \ddots & \vdots \\ [B_3] & [B_2] & [B_1] & \ddots & \ddots & [A_3] \\ \vdots & \ddots & \ddots & \ddots & \ddots & [A_2] \\ [B_{n-1}] & \cdots & [B_3] & [B_2] & [B_1] & [A_1] \end{bmatrix}$$

Structure of the matrix P_α as follows,

$$\begin{bmatrix}
\begin{bmatrix} a_1 & b & \alpha & \cdots & \alpha \\ c & a_1 & b & \ddots & \vdots \\ \alpha & c & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & a_1 \end{bmatrix} & \begin{bmatrix} a_2 & b & \alpha & \cdots & \alpha \\ c & a_2 & b & \ddots & \vdots \\ \alpha & c & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & a_2 \end{bmatrix} & \cdots & \begin{bmatrix} a_r & b & \alpha & \cdots & \alpha \\ c & a_r & b & \ddots & \vdots \\ \alpha & c & a_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & a_r \end{bmatrix} \\
\begin{bmatrix} b_1 & b & \alpha & \cdots & \alpha \\ c & b_1 & b & \ddots & \vdots \\ \alpha & c & b_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & b_1 \end{bmatrix} & \begin{bmatrix} a_1 & b & \alpha & \cdots & \alpha \\ c & a_1 & b & \ddots & \vdots \\ \alpha & c & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & a_1 \end{bmatrix} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \begin{bmatrix} a_2 & b & \alpha & \cdots & \alpha \\ c & a_2 & b & \ddots & \vdots \\ \alpha & c & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & a_2 \end{bmatrix} \\
\vdots & \ddots & \ddots & \ddots \\
\begin{bmatrix} b_s & b & \alpha & \cdots & \alpha \\ c & b_s & b & \ddots & \vdots \\ \alpha & c & b_s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & b_s \end{bmatrix} & \cdots & \begin{bmatrix} b_1 & b & \alpha & \cdots & \alpha \\ c & b_1 & b & \ddots & \vdots \\ \alpha & c & b_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & b_1 \end{bmatrix} & \begin{bmatrix} a_1 & b & \alpha & \cdots & \alpha \\ c & a_1 & b & \ddots & \vdots \\ \alpha & c & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & c & a_1 \end{bmatrix}
\end{bmatrix}$$

Definition 2.5. [11] The tropical determinant $\Gamma(T)$ of the matrix $T \in M_{n\times n}(S)$ is obtained by

$$\Gamma(T) = \bigoplus_{\sigma\in\daleth_n} t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}$$

where we give the notation $\daleth_n$ for the symmetric group of order n. We give the notation for set of all even permutations and odd permutations of $\tilde{n} = \{1,2,3,\cdots,n\}$ by ${\daleth_n}^+$ and ${\daleth_n}^-$ respectively.

$$\Gamma(T)^+ = \max_{\sigma\in{\daleth_n}^+} \sum_{i\in\tilde{n}} t_{i\sigma(i)}$$

$$\Gamma(T)^- = \max_{\sigma\in{\daleth_n}^-} \sum_{i\in\tilde{n}} t_{i\sigma(i)}$$

Tropical determinant is said to be balanced if $\Gamma(T)^+ = \Gamma(T)^-$, unbalanced otherwise.

Definition 2.6. $T \in M_{n\times n}(S)$ is named as singular if $\max\{T_{1\sigma(1)} \otimes T_{2\sigma(2)} \otimes \cdots \otimes T_{n\sigma(n)}\}$ attains at least two times. All other matrices are called non-singular.

Definition 2.7. [11] The barvinok rank of the tropical matrix $T \in M_{n\times n}(S)$ is the smallest k such that

$$T = \bigoplus_{i=1}^{k} u_i \otimes {v_i}^T$$

for the column vectors u_i, v_i. Barvinok rank of the matrix T is denoted by $B.R(T)$.

Definition 2.8. [11] $K.R(T)$ of the matrix $T \in M_{n\times n}(S)$ is the smallest classical rank of the set of matrix over some valued field whose tropicalization equals T. Where $K.R(T)$ stand for the kapranov rank of T.

Definition 2.9. [11] $T.R(T)$ of the matrix $T \in M_{n\times n}(S)$ is the order which is large of the tropically non-singular square submatrix. Where $T.R(T)$ stand for the tropical rank.

2.1. Discrepancy method

There is another method to determine the existence of the solution $P \otimes x = q$. Where we can get the solutions of the systems $P \otimes x = q$ by defining the “discrepancy" and “reducible discrepancy" matrices respectively as follows:

$$D_{Pq} = \begin{bmatrix} q_1 - p_{11} & q_1 - p_{12} & \cdots & q_1 - p_{1n} \\ q_2 - p_{21} & q_2 - p_{22} & \cdots & q_2 - p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ q_m - p_{m1} & q_m - p_{m2} & \cdots & q_m - p_{mn} \end{bmatrix}$$

and

$$R_{Pq} = \begin{cases} 1 & \text{if } d_{ij} = \text{minimum of the j-th column} \\ 0 & \text{else} \end{cases}$$

Note that minimum of each column D_{Pq} is the element of x^*. x^* is stands for the maximal solution of $P \otimes x = q$. If each row of the matrix R_{Pq} contains only one element with a value of 1, then $P \otimes x = q$ has a unique solution. In addition, if atleast one row contains an element 1 more than one time, then $P \otimes x = q$ has many solutions.

Theorem 2.10. *A system $P \otimes x = q$ has a solution if and only if every row of reducible discrepancy matrix has atleast one element with value 1.*

Proof. Assume that the system has a solution. suppose some i-th row $1 \leqslant i \leqslant n$ of the reducible discrepancy matrix does not has a element with the value 1. We can say discrepancy matrix does not has a minimum column element in that corresponding row. Which gives $x_j < q_i - p_{ij}$ for all $1 \leqslant j \leqslant n$. So that $\max\{p_{i1} + x_1, p_{i2} + x_2, \cdots, p_{in} + x_n\} < q_i$. By this we can see the contradiction to our assumption. We should have the element of value 1 atleast once in the every row of reducible discrepancy matrix. Conversely if every row of reducible discrepancy matrix has atleast one element with value 1 then $x_j = q_i - p_{ij}$, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ that implies we have $\max\{p_{i1} + x_1, p_{i2} + x_2, \cdots, p_{in} + x_n\} = q_i$. Hence system has a solution. □

Theorem 2.11. *A system $P \otimes x = q$ has many solutions if and only if atleast one row of the reducible discrepancy matrix has more than one element with the value 1.*

Theorem 2.12. *Let $T \otimes x = u$ be a max linear system. If every row of the R_{Pq} matrix has a value 1 exactly once then $T \otimes x = u$ has a unique solution and the converse is also true.*

3. Main Results

3.1. Ranks and determinant of the tropical toeplitz matrices

In this section we present the results on tropical determinant of the tropical toeplitz matrices and types of tropical toeplitz matrices. Also we analyse conditions for tropical toeplitz matrices to be balanced and unbalanced. Let T be an $n \times n$ matrix over S. The tropical determinant of T, denoted by $\Gamma(T)$, is defined as the maximum taken over all permutations of $\{1, 2, \ldots, n\}$ of the sum of the corresponding matrix entries, where one entry is chosen from each row and each column.

These permutations can be classified as even or odd. Let $\Gamma(T)^+$ denote the maximum value obtained from all even permutations, and let $\Gamma(T)^-$ denote the maximum value obtained from all odd permutations.

The tropical determinant is said to be *balanced* if the maximum values corresponding to even and odd permutations are equal. Otherwise, it is called *unbalanced*.

Proposition 3.1. *Let $T \in M_{n\times n}(S)$ be a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1$*

1. *if $a > 1$ then $\Gamma(T) = na$.*
2. *if $a < 0$ then $\Gamma(T) = n - 1$.*

Proof.

$$\max_{\sigma \in \daleth_n} \left\{ t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)} \right\}.$$

Since the diagonal entries of T are all equal to a and satisfy $a > 1$, choosing the identity permutation yields the term

$$t_{11} \otimes t_{22} \otimes \cdots \otimes t_{nn}.$$

All off-diagonal entries are either 0 or 1, which are strictly smaller than a. Hence, any permutation involving at least one off-diagonal entry produces a value smaller than the diagonal product. Therefore, the maximum is attained when all diagonal entries are selected.

Consequently, we obtain

$$\Gamma(T) = \underbrace{a \otimes a \otimes \cdots \otimes a}_{n \text{ times}} = na.$$

In this case, the entries of the matrix T belong to the set $\{0, a, 1\}$, and we are given that $a < 0$. Hence, the maximum entry appearing in the matrix is 1.

To compute the tropical determinant, we again consider

$$\max_{\sigma \in \daleth_n} \left\{ t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)} \right\}.$$

Observe that selecting the diagonal entries yields

$$t_{11} \otimes t_{22} \otimes \cdots \otimes t_{nn} = na,$$

which is negative since $a < 0$, and hence cannot be maximal.

Now consider the permutations that select only the off-diagonal entries equal to 1. In particular, choosing

$$t_{12} \otimes t_{23} \otimes t_{34} \otimes \cdots \otimes t_{(n-1)n} + t_{n1}$$

or

$$t_{21} \otimes t_{32} \otimes t_{43} \otimes \cdots \otimes t_{n(n-1)} + t_{1n}$$

yields the value $n - 1$.

All other possible choices of permutations involve either the diagonal entry $a < 0$ or the zero entries, and therefore produce values strictly less than $n - 1$. Hence, the maximum value of

$$\max_{\sigma \in \daleth_n} \left\{ t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)} \right\}$$

is attained at these permutations and equals :highlight[$n - 1$].

Therefore, we conclude that

$$\Gamma(T) = n - 1.$$

□

Proposition 3.2. *Let $T \in M_{n\times n}(S)$ be a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1$. If $a < 0$ then T is singular.*

Proof. Assume that T is a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1$ and $a < 0$. By Proposition 3.1 tropical determinant $\Gamma(T) = n - 1$. Which is obtained by the terms,

$$\left(\bigotimes_{k=2}^{n} t_{(k-1)k}\right) \otimes t_{n1} = \left(\bigotimes_{k=2}^{n} t_{k(k-1)}\right) \otimes t_{1n} = n - 1$$

Maximum of the terms $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ is $t_{11} \otimes t_{22} \otimes \cdots \otimes t_{nn}$ and it attains twice. By Definition 2.6 clearly T is singular. □

Proposition 3.3. *Let $T \in M_{n\times n}(S)$ be a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1$ and if $a > 1$ then $\Gamma(T)$ is unbalanced.*

Proof. Assume that T is a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1$ and $a > 1$.
Then the tropical determinant is $\Gamma(T) = \bigotimes_{i=1}^{n} a_{ii} = na$, $a_{ij} \leqslant 1, \forall\, i \neq j$. Where $a > 1$ clearly $a_{ij} \leqslant a_{ii}$ $\forall i \neq j$.

$$\implies \Gamma(T)^{+} = \max_{\sigma \in \daleth_n{}^{+}} \sum_{i \in \tilde{n}} t_{i\sigma(i)} = na, \Gamma(T)^{-} = \max_{\sigma \in \daleth_n{}^{-}} \sum_{i \in \tilde{n}} t_{i\sigma(i)} = n - 1$$

where $a > 1$ and which gives that

$$\Gamma(T)^{-} \neq \Gamma(T)^{+}$$

Which implies $\Gamma(T)$ is unbalanced. □

Corollary 3.4. *Let $T \in M_{2\times 2}(S)$ be a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1, a > 1$ then $\Gamma(T)$ is unbalanced.*

Proof. Given

$$T = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$$

Suppose $a > 1$

$$\implies \Gamma(T)^+ = \max_{\sigma \in \daleth_n^+} \sum_{i \in \tilde{n}} t_{i\sigma(i)} = 2a,\ \Gamma(T)^- = \max_{\sigma \in \daleth_n^-} \sum_{i \in \tilde{n}} t_{i\sigma(i)} = 2$$

where $a \neq 1$ which implies that

$$\Gamma(T)^- \neq \Gamma(T)^+$$

Which implies $\Gamma(T)$ is unbalanced.

□

Corollary 3.5. *Let $T \in M_{n\times n}(S)$ be a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1, a < 0$ then $\Gamma(T)$ is unbalanced.*

Proof. Suppose n is odd then the maximum among the terms $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ is obtained by an odd permutations

$$\Gamma(T) = (\bigotimes_{k=2}^{n} t_{(k-1)k}) \otimes t_{n1} = n - 1$$

and

$$\Gamma(T) = (\bigotimes_{k=2}^{n} t_{k(k-1)}) \otimes t_{1n} = n - 1$$

all other values of the terms obtained by the even permutations

$$\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\} < \Gamma(T)$$

Suppose n is even then the maximum among the terms $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ is obtained by the even permutations

$$\Gamma(T) = (\bigotimes_{k=2}^{n} t_{(k-1)k}) \otimes t_{n1} = n - 1$$

and

$$\Gamma(T) = (\bigotimes_{k=2}^{n} t_{k(k-1)}) \otimes t_{1n} = n - 1$$

values obtained by odd permutations are less than $\Gamma(T)$. Which implies $\Gamma(T)$ is unbalanced. □

Proposition 3.6. *Let $T \in M_{2\times 2}(S)$ be a tridiagonal toeplitz matrix with $\alpha = 0, b = c = 1$ and $a < 0$ then $\Gamma(T)$ is unbalanced.*

Proof. Suppose $a < 1$

$$\implies \Gamma(T)^+ = \max_{\sigma \in \daleth_n^+} \sum_{i \in \tilde{n}} t_{i\sigma(i)} = 2,\ \Gamma(T)^- = \max_{\sigma \in \daleth_n^-} \sum_{i \in \tilde{n}} t_{i\sigma(i)} = 2a$$

where $a \neq 1$, which implies $\Gamma(T)^- \neq \Gamma(T)^+$ and clearly $\Gamma(T)$ is unbalanced. □

Proposition 3.7. *Let $T \in M_{n\times n}(S)$ be a tropical symmetric toeplitz matrix and if a_i^s are strictly decreasing then*

1. $\Gamma(T) = \bigotimes_{n-times} a_1$.

2. $\Gamma(T)$ *is unbalanced.*

Proof. 1. Assume that T be a tropical symmetric toeplitz matrix and a_i^s of T are strictly decreasing $a_1 > a_2 > \cdots a_n$. Which implies a_1 is the largest entry of T. Now to show that $\Gamma(T)$ is obtained only by the identity permutation, i.e. $t_{11} \otimes t_{22} \otimes \cdots \otimes t_{nn}$. All other possibilities of $\Gamma(T)$ contains atleast one value h other than a_1, where $h < a_1$. The value of the term $\max\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ is obtained by the identity permutation is na_1. Which implies $\Gamma(T) = \bigotimes_{n-times} a_1$.

2. Already we know that $\Gamma(T) = \bigotimes_{n-times} a_1$ which obtained with respect to the identity permutation (even permutation).

$$\implies \Gamma(T)^+ = \max_{\sigma \in \daleth_n{}^+} \sum_{i \in \tilde{n}} t_{i\sigma(i)} = na_1 < \Gamma(T)^- = \max_{\sigma \in \daleth_n{}^-} \sum_{i \in \tilde{n}} t_{i\sigma(i)}$$

Which implies $\Gamma(T)$ is unbalanced.

$\square$

Proposition 3.8. *Let* $T \in M_{n\times n}(S)$ *be a tropical symmetric toeplitz matrix if* a_i^s *are strictly increasing with difference* p *then* $\Gamma(T) = \bigotimes$ *AD(T), where* $AD(T)$ *denotes Anti diagonal elements of* T.

Proof. Given $a_1 < a_2 = a_1 + p < a_3 = a_1 + 2p < \cdots < a_n = a_1 + (n-1)p$.

$$\Gamma(T) = \max_{\sigma \in \daleth_n}\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$$

Tropical determinant of T is obtained by the following terms

$$\Gamma(T) = a_{1n} \otimes a_{2(n-1)} \otimes a_{(n-1)2} \otimes a_{n1} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3}$$

and

$$\Gamma(T) = a_{1(n-1)} \otimes a_{2n} \otimes a_{(n-1)1} \otimes a_{n2} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3}$$

All other possibility of $\Gamma(T)$ contains a minimum values of T, which implies

$$a_{1n} \otimes a_{2(n-1)} \otimes a_{(n-1)2} \otimes a_{n1} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3}$$

$$= a_{1n} \otimes a_{2(n-1)} \otimes a_{(n-1)2} \otimes a_{n1} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3} = \bigotimes AD(T)$$

Which implies $\Gamma(T) = \bigotimes AD(T)$. $\square$

Proposition 3.9. *If* $T \in M_{n\times n}(S)$ *be a tropical symmetric toeplitz matrix then* $B.R(T) = n$.

Proof. Assume that T be a tropical symmetric toeplitz matrix,

$$T = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_1 & a_2 & a_3 & \ddots & \vdots \\ a_3 & a_2 & a_1 & a_2 & \ddots & \vdots \\ a_4 & a_3 & a_2 & \ddots & \ddots & a_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_2 \\ a_n & \cdots & a_4 & a_3 & a_2 & a_1 \end{bmatrix}$$

We have to find smallest k such that

$$T = \bigoplus_{i=1}^{k} u_i \otimes {v_i}^T$$

for the column vectors u_i, v_i. We write T matrix as follows,

$$T = \left(\begin{bmatrix} -1 \\ -a_1 - a_2 - a_3 - \cdots - a_n \\ -a_1 - a_2 - a_3 - \cdots - a_n \\ \vdots \\ -a_1 - a_2 - a_3 - \cdots - a_n \end{bmatrix} \otimes \begin{bmatrix} a_{11}+1 & a_{12}+1 & \cdots & a_{1n}+1 \end{bmatrix} \right)$$
$$\oplus \left(\begin{bmatrix} -a_1 - a_2 - a_3 - \cdots - a_n \\ -1 \\ -a_1 - a_2 - a_3 - \cdots - a_n \\ \vdots \\ -a_1 - a_2 - a_3 - \cdots - a_n \end{bmatrix} \otimes \begin{bmatrix} a_{21}+1 & a_{22}+1 & \cdots & a_{2n}+1 \end{bmatrix} \right)$$
$$\oplus \cdots \oplus \begin{bmatrix} -a_1 - a_2 - a_3 - \cdots - a_n \\ -a_1 - a_2 - a_3 - \cdots - a_n \\ \vdots \\ -a_1 - a_2 - a_3 - \cdots - a_n \\ -1 \end{bmatrix} \otimes \begin{bmatrix} a_{n1}+1 & a_{n2}+1 & \cdots & a_{nn}+1 \end{bmatrix}$$

Which implies that $B.R(T) = n$. □

Proposition 3.10. *Let $T \in M_{2\times 2}(S)$ be a tropical symmetric toeplitz matrix if $a_1 \neq a_2$ then T is non-singular.*

Proof.

$$T = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}$$

$\Gamma(T) = (a_1)^{\otimes 2}$ and $\Gamma(T) = (a_2)^{\otimes 2}$. But $(a_1)^{\otimes 2} \neq (a_2)^{\otimes 2}$ (where $a_1 \neq a_2$). Which implies is T is non-singular. □

Proposition 3.11. *Let $T \in M_{n\times n}(S)$ for $n > 2$ be a tropical symmetric toeplitz matrix if a_i^s are strictly increasing with p difference then T is singular.*

Proof. Assume that T is a tropical symmetric toeplitz matrix and a_i^s are strictly increasing with difference p. i.e. $a_1 < a_2 < \cdots < a_n$ then the tropical determinant of T as follows,

$$\Gamma(T) = a_{1n} \otimes a_{2(n-1)} \otimes a_{(n-1)2} \otimes a_{n1} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3}$$

and

$$\Gamma(T) = a_{1(n-1)} \otimes a_{2n} \otimes a_{(n-1)1} \otimes a_{n2} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3}$$

The maximum of the terms $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ attains two times. Which implies T is singular. □

Proposition 3.12. *Let $T \in M_{n\times n}(S)$ be a tropical symmetric toeplitz matrix if a_i^s are strictly decreasing then T is non-singular.*

Proof. The structure of the matrix T is given in Definition 2.2. Assume that a_i^s of T are strictly decreasing, i.e. $a_1 > a_2 > \cdots > a_n$. Which implies a_1 is the largest entry. By Proposition 3.7 $\Gamma(T) = \bigotimes_{n-times} a_1$, where $a_1 > a_2 > \cdots > a_n$. The term $\max_{\sigma \in \daleth_n}\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ attains exactly once. All other terms in $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ are less than $\bigotimes_{n-times} a_1$. The maximum of $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ attain once. Which implies T is non-singular. □

Proposition 3.13. *Let $T \in M_{n\times n}(S)$ be a tropical symmetric toeplitz matrix if a_i^s are not equal then $K.R(T) = n$.*

Proof. Choose a matrix

$$C.M = \begin{bmatrix} t^{a_1} & t^{a_2} & t^{a_3} & t^{a_4} & \cdots & t^{a_n} \\ t^{a_2} & t^{a_1} & t^{a_2} & t^{a_3} & \ddots & \vdots \\ t^{a_3} & t^{a_2} & t^{a_1} & t^{a_2} & \ddots & \vdots \\ t^{a_4} & t^{a_3} & t^{a_2} & \ddots & \ddots & t^{a_3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & t^{a_2} \\ t^{a_n} & \cdots & t^{a_4} & t^{a_3} & t^{a_2} & t^{a_1} \end{bmatrix}$$

C.M$-$ all possible matrices has the classical rank n and whose tropicalization is T matrix. Which implies $K.T(T) = n$. □

Proposition 3.14. *Let $T \in M_{n\times n}(S)$ be a tropical symmetric toeplitz matrix and if T is strictly decreasing then $T.R(T) = n$.*

Proof. By Proposition 3.12, T is non-singular. Largest order of non-singular sub matrix is n. Which implies $T.R(T) = n$. □

Proposition 3.15. *Let $T \in M_{n\times n}(S)$ be a tropical symmetric toeplitz matrix and if T is strictly increasing then $T.R(T) = 2$.*

Proof. Assume $T \in M_{n\times n}(S)$ is a tropical symmetric toeplitz matrix and by Proposition 3.11

$$\Gamma(T) = a_{1n} \otimes a_{2(n-1)} \otimes a_{(n-1)2} \otimes a_{n1} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3}$$

and

$$\Gamma(T) = a_{1(n-1)} \otimes a_{2n} \otimes a_{(n-1)1} \otimes a_{n2} \otimes a_{3(n-2)} \otimes a_{4(n-3)} \otimes \cdots \otimes a_{(n-2)3}$$

value of $\Gamma(T)$ attains twice in T. Which implies $T.R(T) < n$. Also we know that submatrix of a tropical symmetric toeplitz matrix is singular. Which implies that largest order of non-singular submatrix of T is 2 and $T.R(T) = 2$. □

Proposition 3.16. *If T be a tridiagonal toeplitz matrix T_0 with $b, c = 1$ and $a > 1$ then T is non-singular.*

Proof. Consider the tridiagonal toeplitz matrix T_0 with $a > 1$ then from the Definition 2.3, $\max_{\sigma\in\beth_n}\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\} = \bigotimes_{i=n}^{n} a_{ii} = \bigotimes_{n-times} a$ attains exactly once. All other terms of $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\} \neq \bigotimes_{i=n}^{n} a_{ii}$. Where $a > 1$ attains only at diagonals and $b, c = 1$. Which implies T is non-singular. □

Corollary 3.17. *If T be a tridiagonal toeplitz matrix T_0 with $b, c = 1$ and $a > 1$ then T has $T.R(T) = n$.*

Proof. By Proposition 3.16 order of the largest non-singular submatrix is n. Which implies that $T.R(T) = n$. □

Proposition 3.18. *If T be a tridiagonal toeplitz matrix T_0 with $b, c = 1$ and $a < 0$ then T is singular.*

Proof. Assume that T be a tridiagonal toeplitz matrix T_0 then

$$\Gamma(T) = \bigotimes_{i=1}^{n-1} a_{k(k+1)} = \bigotimes_{i=1}^{n-1} a_{(k+1)k}, \forall\, 1 \leqslant k \leqslant n-1$$

maximum of the terms $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ is attains twice. Therefore T is singular. □

Corollary 3.19. *If T be a 2×2 tridiagonal toeplitz matrix T_0 with $b, c = 1$ and $a < 0$ then T has $T.R(T) = 2$.*

Proof. For $n > 2$ by Proposition 3.18 order of largest non-singular submatrix is 2. Which implies $T.R(T) = 2$. □

Proposition 3.20. *If $T_{n\times n}(S)$ be a block tridiagonal toeplitz matrix with T_0 as submatrices, $a_i^s > 0$ and strictly decreasing $\forall\ 1 \leqslant i \leqslant n$, $a_1 > a_2 > \cdots > a_n > b_1 > \cdots > b_{n-1}$, $b = c = 0$ then $\Gamma(T) = na_1$.*

Proof. Assume that $a_i^s > 0$ and strictly decreasing, $b = c = 0$. Except a_i^s, $1 \leqslant i \leqslant n$ and $b_i^s < 0$, for $1 \leqslant i \leqslant n-1$ all other elements are equal to 0. By substituting the given conditions to the matrix format in the Definition 2.4, the maximum value among the entries of T matrix is only a_1 and it placed in the diagonal for n times. Which implies $\otimes_{n-times} a_1 = na_1$. □

Proposition 3.21. *If $T_{n\times n}(S)$ be a block tridiagonal toeplitz matrix T_0 with $a_i^s < 0$, $\forall\ 1 \leqslant i \leqslant n$ and $b_i^s < 0$, $\forall\ 1 \leqslant i \leqslant n-1$, $b = c = 0$ then $\Gamma(T) = 0$.*

Proof. Assume that $a_i^s < 0$, where $1 \leqslant i \leqslant n$ and $b_i^s < 0$ where $1 \leqslant i \leqslant n-1$, $b = c = 0$. By the Definition 2.4, T contains 0 as a maximum. Every row and column of T contains 0 atleast once. Which implies $\otimes_{n-times} 0 = 0$.

□

Proposition 3.22. *If $T_{n\times n}(S)$ be a block tridiagonal toeplitz matrix T_0 with submatrix of order r, $a_i^s = k$ where $k > 0$ and $k \in \mathbb{N}$, $\forall\ 1 \leqslant i \leqslant n$ and $b_i^s = k$, $\forall\ 1 \leqslant i \leqslant n-1$, $b = c = 0$ then*

1. *$\Gamma(T) = nk$*
2. *T is singular.*

Proof.
1. Assume that $a_i^s = k$ for all $1 \leqslant i \leqslant n$ and $b_i^s = k$ for all $1 \leqslant i \leqslant n-1$, $b = c = 0$. Given $k > 0$ and $k \in \mathbb{N}$. By the Definition 2.4, we construct matrix T and elements of T are only 0 and k. Diagonal elements of T are all equal to k, where k is a maximum. That implies $\otimes_{i=1}^{n} a_{ii} = \otimes_{n-times} k = nk$.
2. Now we have to show that nk attains twice in the term $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$. Identity permutaion generates the value of $\max_{\sigma\in \mathfrak{I}_n}\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ and its equal to nk, i.e. $\otimes_{i=1}^{n} a_{ii} = nk$. $a_{1(r+1)} \otimes a_{2(r+2)} \otimes \cdots \otimes a_{(n-r)(r+(n-r))} \otimes a_{nr} \otimes a_{(n-1)(r-1)} \otimes a_{(n-2)(r-2)} \otimes \cdots a_{(n-(r-1))(r-(r-1))} = nk$, where r is order of submatrix used to construct T, for $1 \leqslant k \leqslant r$. The term $\max\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ attains more than once. Therefore T is singular.

□

Proposition 3.23. *If $T_{n\times n}(S)$ be a block tridiagonal toeplitz matrix T_0 with submatrix of order 2, $a_i^s = 0$, for all $1 \leqslant i \leqslant n$ and $b_i^s = 0$, for all $1 \leqslant i \leqslant n-1$, $b = c = k$ then*

1. *$\Gamma(T) = nk$.*
2. *T is singular.*

Proof.
1. Assume that T has a values $a_i^s = 0$, $\forall\ 1 \leqslant i \leqslant n$ and $b_i = 0$, $\forall\ 1 \leqslant i \leqslant n-1$, $b = c = k$. By the Definition 2.4 we can verify that the element k is the maximum of T and $a_{1n} \otimes a_{2(n-1)} \otimes a_{3(n-2)} \otimes \cdots \otimes a_{n(n-(n-1))} = nk$. Which implies $\max\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\} = nk$. Therefore $\Gamma(T) = nk$.
2. We have to show that value of $\max\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ attains twice. We have $a_{1n} \otimes a_{2(n-1)} \otimes a_{3(n-2)} \otimes \cdots \otimes a_{n(n-(n-1))} = nk$ and $a_{1(n-2)} \otimes a_{2(n-3)} \otimes a_{3(n-4)} \otimes \cdots \otimes a_{(n-r)(n-(n-1))} = nk$. Maximum of the term $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ attains twice. Which implies T is singular.

□

Proposition 3.24. *If $T_{n\times n}(S)$ be a block tridiagonal toeplitz matrix T_0 with submatrix of order r, $a_i^s = 0$, $\forall\ 1 \leqslant i \leqslant n$ and $b_i^s = 0$, $\forall\ 1 \leqslant i \leqslant n-1$, $b = c = k$ then*

1. $\Gamma(T) = (n - (n/r))k$.
2. T *is singular.*

Proof. 1. Assume that T be a block tridiagonal toeplitz matrix with $a_i^s = 0, \forall\, 1 \leqslant i \leqslant n$ and $b_i^s = 0, \forall\, 1 \leqslant i \leqslant n-1$, $b = c = k$. In matrix T we have $\otimes_{s=1}^{n-1} a_{(k-1)k} = ((n-(n/r)k)$ and $a_{n1} = 0$. Tropical determinant of T is obtained from the tropical multiplication of the terms $\otimes_{s=1}^{n-1} a_{(k-1)k}$ and a_{n1}. We obtain that $((n-(n/r)k) + 0 = \Gamma(T)$.

2. We have to find two terms of $\{t_{1\sigma(1)} \otimes t_{2\sigma(2)} \otimes \cdots \otimes t_{n\sigma(n)}\}$ of value $(n-(n/r))k$. We have $\otimes_{s=1}^{n-1} a_{(k-1)k} = ((n-(n/r)k) \otimes a_{n1} = (n-(n/r))k$ and $\otimes_{s=1}^{n-1} a_{k(k-1)} = ((n-(n/r)k) \otimes a_{1n} = (n-(n/r))k$. Tropical determinant of T is attains more than once. Which implies T is singular.

□

4. Solving the tropical toeplitz systems $P \otimes x = q$

This section discusses solutions of the system of tropical toeplitz equations. We will check whether the system of tropical toeplitz equations has a solution or not. Also we will give some propositions on the conditions of tropical toeplitz systems $P \otimes x = q$ to have a solution and to do not have a solution. In this section we denote the vector with all entries k as $[k]$.

Proposition 4.1. *Let $P \in M_{n\times n}(\tilde{Z})$, where $n > 2$ be a tropical symmetric toeplitz matrix with $a_1 > a_2 > \cdots > a_n$. If $q = [k]$ then the tropical toeplitz system $P \otimes x = q$ has a unique solution $x^* = [k - a_1]$.*

Proof. Given $P \otimes x = q$ be a tropical symmetric toeplitz system with $a_1 > a_2 > \cdots > a_n$. Assume that $q = [k]$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_1 & a_2 & a_3 & \ddots & \vdots \\ a_3 & a_2 & a_1 & a_2 & \ddots & \vdots \\ a_4 & a_3 & a_2 & \ddots & \ddots & a_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_2 \\ a_n & \cdots & a_4 & a_3 & a_2 & a_1 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \\ k \\ \vdots \\ k \end{bmatrix}$$

D_{P_q} matrix is obtained as follows,

$$D_{Pq} = \begin{bmatrix} k-a_1 & k-a_2 & k-a_3 & k-a_4 & \cdots & k-a_n \\ k-a_2 & k-a_1 & k-a_2 & k-a_3 & \ddots & \vdots \\ k-a_3 & k-a_2 & k-a_1 & k-a_2 & \ddots & \vdots \\ k-a_4 & k-a_3 & k-a_2 & \ddots & \ddots & k-a_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & k-a_2 \\ k-a_n & \cdots & k-a_4 & k-a_3 & k-a_2 & k-a_1 \end{bmatrix}$$

where $a_1 > a_2 > \cdots > a_n$ and $k - a_1 < k - a_2 < \cdots < k - a_n$. D_{Pq} matrix has the diagonal elements as $k - a_1$.

$$R_{Pq} = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Every row of R_{P_q} has exactly one element with a value 1. By Theorem 2.12 the system has a unique solution. To find that solution, collect minimum element in every column of D_{p_q}. Finally column minimum elements of D_{p_q} are listed as maximal solution of the system $P \otimes x = q$ which is $x^* = [k - a_1]$. □

Proposition 4.2. *Let* $P \in M_{n\times n}(\tilde{Z})$ *be a symmetric toeplitz ,* $n > 2$*,* $a_2 > a_1 > \cdots a_n$*. If* $q = [k]$ *then the tropical symmetric toeplitz system* $P \otimes x = q$ *has infinitely many solution* $x^* = [k - a_2]$.

Proof. Assume that $P \otimes x = q$ be a tropical symmetric toeplitz system with $q = [k]$. Given $P \otimes x = q$ be a tropical symmetric toeplitz system with $a_2 > a_1 > \cdots > a_n$. The discrepancy matrix of given system is obtained as follows,

$$D_{Pq} = \begin{bmatrix} k-a_1 & k-a_2 & k-a_3 & k-a_4 & \cdots & k-a_n \\ k-a_2 & k-a_1 & k-a_2 & k-a_3 & \ddots & \vdots \\ k-a_3 & k-a_2 & k-a_1 & k-a_2 & \ddots & \vdots \\ k-a_4 & k-a_3 & k-a_2 & \ddots & \ddots & k-a_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & k-a_2 \\ k-a_n & \cdots & k-a_4 & k-a_3 & k-a_2 & k-a_1 \end{bmatrix}$$

where $a_2 > a_1 > \cdots > a_n$ and $k - a_2 < k - a_1 < \cdots < k - a_n$. The reducible discrepancy matrix is obtained as follows,

$$R_{Pq} = \begin{bmatrix} 0 & \boxed{1} & 0 & 0 & \cdots & 0 \\ \boxed{1} & 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & \boxed{1} & 0 & \boxed{1} & \ddots & \vdots \\ 0 & 0 & \boxed{1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \boxed{1} \\ 0 & \cdots & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

Except first and last row, every rows of reducible discrepancy matrix has more than one element with a value 1. By Theorem 2.11 system has a infinitely many solutions. Collect every minimum element in matrix D_{p_q} columnwise. Finally column minimum elements of D_{p_q} are listed as maximal solution of the system $P \otimes x = q$ and which is obtained as $x^* = [k - a_2]$. □

Proposition 4.3. *Let* $P \in M_{n\times n}(\tilde{Z})$ *be a tropical tridiagonal toeplitz matrix* P_α *with* $a > b > c > \alpha$*. If* $q = [k]$ *then the tropical tridiagonal toeplitz system* $P \otimes x = q$ *has a unique solution* $x^* = [k - a]$.

Proof. Assume that $P \otimes x = q$ be a tropical tridiagonal toeplitz system with $q = [k]$

$$\begin{bmatrix} a & b & \alpha & \alpha & \cdots & \alpha \\ c & a & b & \alpha & \ddots & \vdots \\ \alpha & c & a & b & \ddots & \vdots \\ \alpha & \alpha & c & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & \ddots & b \\ \alpha & \cdots & \alpha & \alpha & c & a \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \\ k \\ \vdots \\ k \end{bmatrix}$$

The discrepancy matrix of the system $P \otimes x = q$ is obtained as follows,

$$D_{Pq} = \begin{bmatrix} k-a & k-b & k-\alpha & k-\alpha & \cdots & k-\alpha \\ k-c & k-a & k-b & k-\alpha & \ddots & \vdots \\ k-\alpha & k-c & k-a & k-b & \ddots & \vdots \\ k-\alpha & k-\alpha & k-c & \ddots & \ddots & k-\alpha \\ \vdots & \ddots & \ddots & \ddots & \ddots & k-b \\ k-\alpha & \cdots & k-\alpha & k-\alpha & k-c & k-a \end{bmatrix}$$

where $a > b > c > \alpha$ and which implies $k-a < k-b < k-c < k-\alpha$. The reducible discrepancy matrix is obtained as follows

$$R_{Pq} = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Every row of R_{Pq} has exactly one element with a value 1. By Theorem 2.12 the system has a unique solution. The column minimum elements of D_{p_q} are listed as maximal solution of the system $P \otimes x = q$ and $x^* = [k-a]$.

$\square$

Proposition 4.4. *Let $P \in M_{n\times n}(\tilde{Z})$ be a block tropical tridiagonal toeplitz matrix with submatrix of order m. If $P \otimes x = q$ be a system with $q = [r]$ and $a_i^s = k$, $b, c = 0$ then the system has infinitely many solutions.*

Proof. Given $P \otimes x = q$ is a tropical linear system with block tropical tridiagonal toeplitz matrix P and $a_i^s = k$, $b, c = 0$. Note that every row of block tropical tridiagonal toeplitz matrix of order k exactly s times where $n = ms$ $1 \leqslant s \leqslant n$.

$$\begin{bmatrix}
\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} & \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} & \cdots & \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} \\
\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} \\
\vdots & \ddots & \ddots & \ddots \\
\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} & \cdots & \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} & \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix} & \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \alpha \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & k \end{bmatrix}
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} r \\ r \\ r \\ r \\ \vdots \\ r \end{bmatrix}$$

The discrepancy matrix of the system $P \otimes x = q$ is obtained as follows,

$$D_{pq} = \begin{bmatrix} \begin{bmatrix} r-k & r & r & \cdots & r \\ r & r-k & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & r-k \end{bmatrix} & \cdots & \begin{bmatrix} r-k & r & r & \cdots & r \\ r & r-k & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & r-k \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} r-k & r & r & \cdots & r \\ r & r-k & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & r-k \end{bmatrix} & \cdots & \begin{bmatrix} r-k & r & r & \cdots & r \\ r & r-k & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & r-k \end{bmatrix} \end{bmatrix}$$

$r > r-k, \forall\, r \neq k$.
Now the reducible discrepancy matrix of the system $P \otimes x = q$ is obtained as follows,

$$R_{pq} = \begin{bmatrix} \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} & \cdots & \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} & \cdots & \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} \end{bmatrix}$$

Every row of the R_{Pq} matrix has an element 1 more than once. By Theorem 2.11 the system has infinitely many solutions. The maximal solution of the given system is $x^* = [r-k]$. Suppose $r = k$ and D_{pq} obtained as folows,

$$D_{pq} = \begin{bmatrix} \begin{bmatrix} 0 & r & r & \cdots & r \\ r & 0 & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & r & r & \cdots & r \\ r & 0 & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & 0 \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} 0 & r & r & \cdots & r \\ r & 0 & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & r & r & \cdots & r \\ r & 0 & r & \ddots & \vdots \\ r & r & \ddots & \ddots & r \\ \vdots & \ddots & \ddots & \ddots & r \\ r & \cdots & r & r & 0 \end{bmatrix} \end{bmatrix}$$

Suppose $r > 0$ then 0 is a minimum entry among the elements of D_{Tq} and R_{Tq} is obtained as follows,

$$R_{pq} = \begin{bmatrix} \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} & \cdots & \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} & \cdots & \begin{bmatrix} \boxed{1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{1} & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \boxed{1} \end{bmatrix} \end{bmatrix}$$

The maximal solution is $x^* = [0]$.
Suppose $r < 0$ then the maximal solution is $x^* = [r]$. □

Proposition 4.5. *Let $P \in M_{n\times n}(\tilde{Z})$ be a block tropical symmetric toeplitz matrix with submatrix of order m and $a_1 > a_2 > \cdots > a_n > b_1 > \cdots > b_{n-1}$. If $P \otimes x = q$ is a system with $q = [k]$ then system has unique solution.*

Proof. In Definition 2.4, substitute the value of a_1 which is a maximum among the block tropical symmetric toeplitz matrix. Similarly the value $k - a_1$ is minimum and which is placed exactly one time in D_{Pq} matrix. Also in R_{Pq} matrix every row contains an element exactly once. By Theorem 2.12 system has a unique solution. □

Conclusion

In conclusion, this paper has focused on the analysis of determinant and different types of ranks of tropical toeplitz matrices. We have presented conditions for determining the non-singularity and singularity of these matrices, as well as investigated their properties of balance and unbalance. Furthermore, we have analysed the conditions under which tropical toeplitz linear systems have solutions. The motivation for this research is from the profound connection between the toeplitz and hankel matrices. The essential role played by their determinants in diverse mathematical disciplines and real-world applications. Lastly, we have successfully solved tropical toeplitz linear systems, offering cognizance into solving certain tropical cryptographic algorithms. Overall, this research contributes to a deeper understanding of tropical toeplitz matrices and their implications in various mathematical domains and practical scenarios. Our future research will be concentrated on solving cryptographical algorithms by using the obtained results. Also in future we will extend the applications of hankel matrices to the applications of toeplitz matrices using tropical semiring.

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