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One end node domination of Cartesian product of simple path semigraphs with 1,2 intermediate nodes

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Abstract

Domination concepts in graph theory, have various practical applications in real-world scenarios. In the context of networks, systems, and optimization, domination theory can offer several advantages. In communication or computer networks, domination can be used to optimize the placement of resources such as routers, servers, or communication hubs. A dominating set can represent a minimal number of resources needed to monitor or control the entire network. In this paper, 1e-domination number of the Cartesian product of elementary semigraphs with one and two intermediate nodes are discussed.

Keywords: Semigraph, Path Semigraph, Cartesian Product, 1e-Dominating set, 1e-Domination number.

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1. Introduction

The Cartesian product of simple path semigraphs has several real-world applications, particularly in fields that deal with network structures, optimization, and spatial planning. To understand the utility of these products, it's eminent to grasp that the Cartesian product of graphs combines two graphs in a way that creates a new graph, where the vertices represent pairs of vertices from the original graphs, and edges are drawn between vertices that are connected in either of the original graphs. In telecommunication systems, the Cartesian product of path semigraphs can be used to model complex communication networks. For example, multiple data transmission paths can be represented as a Cartesian product of simple paths, optimizing the network for both connectivity and bandwidth distribution. Each "path" represents a communication channel, and the Cartesian product helps in understanding how multiple channels can be interconnected efficiently.

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2. Definition

A semigraph A is a pair (N, J) where N is a nonempty set whose elements are observed components of A and J is a set of ordered n-tuples $n \geqslant 2$ of prescribed components called joins of A satisfying the following conditions : two joins $J_1 = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $J_2 = (\beta_1, \beta_2, \ldots, \beta_n)$ are said to be identical iff $m = n$ and either $\alpha_i = \beta_i$ or $\alpha_i = \beta_{n-i+1}$ for $1 \leqslant i \leqslant n$.

The components in a semigraph are splitted into three types namely end nodes, mid nodes and mid-end nodes, based on their positions in the join. The end nodes are described by thick dots, mid nodes are described by small circles, a small tangent is drawn at the small circles to describe mid-end nodes. It is simple to see that every simple connected graph is a semigraph, but not the converse.

A $1e$-dominating set is the subset U of V in which for all node $\beta \in V - U$, there exist a node $\alpha \in U$, such that α and β belong to the same join and atleast one of node is an end node of that join. The minimum cardinality of such a set U is said as $1e$-domination number of the semigraph A and it is notated as $\gamma_{1e}(A)$.

Domination and independent domination digits of a Graph was discussed by R.B. Allan and R. C. Laskar [1] on 1978. E. S. S. Kamath and R. S. Bhat [6] explored domination in semigraphs in 2003. S. T. Hedetniemi et al. [2] introduced fundamental definitions of domination parameters in 1990. Subsequently, N. Murugesan and D. Narmatha investigated the domination digits of the Cartesian product of path semigraphs in [[5], [7], [**?**]]. Hossein et al. discussed detecting paths in graphs [3].

The domination number often repeats in modular patterns because many graphs (especially paths, cycles, grids, etc.) have repeating structural blocks. When a graph grows by adding similar units, the minimum dominating set also grows in a periodic way.Domination number repeats in modular patterns because:Each dominating vertex covers fixed-size neighborhood,Many graph families have repeating structure and Optimal dominating sets are placed periodically.For example think like tiling a floor:If one tile covers 3 squares,then the number of tiles depends on how many complete groups of 3 you have,the leftover (n mod 3) determines if you need one more tile.So modular arithmetic naturally enters.

3. Cartesian product

Let N_1 and N_2 be the node sets and J_1 and J_2 be the join sets of path semigraphs A_1 and A_2 respectively . The Cartesian product of A_1 and A_2 is defined as
$A_1 \square A_2 = (N_1 \times N_2, J_1 \times J_2)$ such that $N_1 \times N_2 = \{(\alpha_p, \beta_q)/\alpha_p \in N_1, \beta_q \in N_2\}$ and

(i) Any node $\alpha \in N_1$ and any join $J = (\beta_1, \beta_2, \ldots, \beta_p)$ in J_2, $((\alpha, \beta_1), (\alpha, \beta_2), \ldots, (\alpha, \beta_i))$ is an element of $J_1 \times J_2$

(ii) Any join $J = (\alpha_1, \alpha_2, \ldots, \alpha_j)$ in J_1 and for any node $\beta \in N_2$ $((\alpha_1, \beta), (\alpha_2, \beta), \ldots, (\alpha_j, \beta))$ is an element of $J_1 \times J_2$.

Cartesian product affects domination by:

1. Increasing dimension of coverage:Coverage becomes multi-directional.
2. Creating layered interaction:Each layer influences adjacent layers.
3. Producing 2D tiling problems:Domination becomes a covering/tiling optimization problem.
4. Changing density:Dominating sets become more sparse in higher dimensions.

3.1. Lemma

$\gamma_{1e}\left[A_{1m(1)} \square A_{1m(1)}\right] = 3$
$\gamma_{1e}\left[A_{1m(2)} \square A_{1m(1)}\right] = 3$

$$\gamma_{1e}\left[A_{1m(s)}\square A_{1m(1)}\right]=\begin{cases}\frac{4(s)}{3}+1 & \text{if } s=3t\\ \frac{4(s-1)}{3}+2 & \text{if } s=3t+1\\ \frac{4(s-2)}{3}+3 & \text{if } s=3t+2,\ t=1,2,3,\dots\end{cases}\quad \text{where } s,t \text{ denotes the number of nodes.}$$

Proof:
Let $A_{1m(1)}$ be single mid node path semigraph.
Fig. 1 represents Two copies of $A_{1m(1)}$

Figure 1: Two copies of $A_{1m(1)}$

The Cartesian product $A_{1m(1)}\square A_{1m(1)}$ is a graph with node set

$$N = \{(\alpha_1,\beta_1),(\alpha_2,\beta_1),(\alpha_3,\beta_1),(\alpha_1,\beta_2),(\alpha_2,\beta_2),(\alpha_3,\beta_2),(\alpha_1,\beta_3),(\alpha_2,\beta_3),(\alpha_3,\beta_3)\}$$

and join set

$$E=\{[(\alpha_1,\beta_1),(\alpha_2,\beta_1),(\alpha_3,\beta_1)],[(\alpha_1,\beta_2),(\alpha_2,\beta_2),(\alpha_3,\beta_2)],\\ [(\alpha_1,\beta_3),(\alpha_2,\beta_3),(\alpha_3,\beta_3)],[(\alpha_1,\beta_1),(\alpha_1,\beta_2),(\alpha_1,\beta_3)],\\ [(\alpha_2,\beta_1),(\alpha_2,\beta_2),(\alpha_2,\beta_3)],[(\alpha_3,\beta_1),(\alpha_3,\beta_2),(\alpha_3,\beta_3)]\}.$$

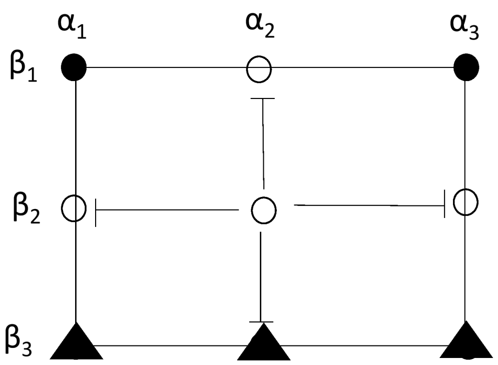

Figure 2: $A_{1m(1)}\square A_{1m(1)}$

Note that from Fig. 2 (α_2,β_2) is the only mid node, and $(\alpha_2,\beta_1),(\alpha_1,\beta_2),(\alpha_3,\beta_2),(\alpha_2,\beta_3)$ are mid-end nodes, and $(\alpha_1,\beta_1),(\alpha_3,\beta_1),(\alpha_1,\beta_3),(\alpha_3,\beta_3)$ are terminate nodes.

Here select all the three end nodes,which dominates all the other nodes in that join. Dominating nodes are represented by triangle shapes. Similarly if we select three mid-end nodes in a row (or column) is also form a minimal dominating set. Hence any three end nodes or mid nodes from a row or column which form a minimal $1e-$ dominating set. Therefore $\gamma_{1e}\left[A_{1m(1)}\square A_{1m(1)}\right]$ is three.

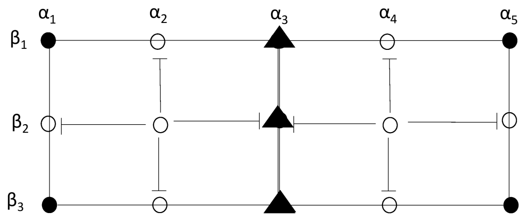

Figure 3: $A_{1m(2)}\square A_{1m(1)}$

Similarly, from Fig. 3 the 3 nodes $\{(\alpha_3, \beta_1), (\alpha_3, \beta_2), (\alpha_3, \beta_3)\}$ forms a minimal 1e− dominating set in $A_{1m(2)}\square A_{1m(1)}$.

Let $n = 3k$, $k = 1, 2, 3, \ldots$. Here the cardinality of $A_{1m(3)}\square A_{1m(1)}$ is $18k + 3$ and joins is $15k + 1$.

Here the node α_3, 1e−dominates α_1, α_2, α_4, α_5 in $A_{1m(2)}$. Since $A_{1m(2)}\square A_{1m(1)}$ is a lattice with 15 nodes with 3 imitations of $A_{1m(2)}$, nodes (α_3, β_1),(α_3, β_2),(α_3, β_3) design a minimal 1e−dominating set in $A_{1m(2)}\square A_{1m(1)}$. Also, $A_{1m(s)}\square A_{1m(1)}$, $s = 3t$, $t = 1, 2, 3, \ldots$ contains t copies of $A_{1m(2)}\square A_{1m(1)}$.

Hence the join set B such that

$$B = \{(\alpha_p, \beta_q)/p = 3, 9, .., (6t - 3), t = 1, 2, 3, ..q = 1, 2, 3\}$$

along 3t nodes is a minimal 1e−dominating set, dominating all nodes in $A_{1m(s)}\square A_{1m(1)}$ other than, (α_{6i}, β_1),(α_{6i}, β_2),(α_{6i}, β_3), where $i = 1, 2, 3, \ldots, t$ and the nodes (α_{6j+1}, β_1),(α_{6j+1}, β_2),(α_{6j+1}, β_3). Note that for all $i = 1, 2, 3, \ldots t$, the nodes (α_{6j+1}, β_1),(α_{6j+1}, β_2),(α_{6j+1}, β_3) form an join B_{6i} (say) in $A_{1m(s)}\square A_{1m(1)}$ with (α_{6i}, β_1) and (α_{6i}, β_3) as mid-end nodes, and (α_{6i}, β_2) as a mid node in which either of them 1e−dominating the remaining 2 nodes. Thus it should be t nodes to cover the nodes (α_{6i}, β_1), (α_{6i}, β_2),(α_{6i}, β_3), $i = 1, 2, 3, \ldots, t$. Finally, if we include any one node from the join $B_{6t+1} = ((\alpha_{6t+1}, \beta_1), (\alpha_{6t+1}, \beta_2), (\alpha_{6t+1}, \beta_3))$, the similar set carrying $4t + 1$ nodes, where $s = 3t$ is a minimal 1e−dominating set in $A_{1m(s)}\square A_{1m(1)}$. Hence $\gamma_{1e}[A_{1m(s)}\square A_{1m(1)}] = \frac{4s}{3} + 1$, if $s = 3t$.

Secondly, let $s = 3t + 1$,$t = 1, 2, 3, \ldots$. The product graph $A_{1m(s)}\square A_{1m(1)}$, when $s = 3t + 1$ contains all the nodes of $A_{1m(s)}\square A_{1m(1)}$ when $s = 3t$, and also the nodes (α_{6j+2}, β_1),(α_{6j+2}, β_2),(α_{6j+2}, β_3) and (α_{6j+3}, β_1),(α_{6j+3}, β_2),(α_{6j+3}, β_3).Hence, instead of a node chosen in the join $B_{6t} = ((\alpha_{6t}, \beta_1), (\alpha_{6t}, \beta_2), (\alpha_{6t}, \beta_3))$ when $s = 3t$, if we choose all the nodes in the join $B_{6t+1} = ((\alpha_{6t+1}, \beta_1), (\alpha_{6t+2}, \beta_2), (\alpha_{6t+3}, \beta_3))$, the similar nodes forms a minimal 1e−dominating set.

Hence,$\gamma_{1e}[A_{1m(s)}\square A_{1m(1)}]$, when $s = 3t + 1$ is equal to $\gamma_{1e}[A_{1m(s)}\square A_{1m(1)}] + 1$.

Hence, $\gamma_{1e}[A_{1m(s)}\square A_{1m(1)}] = 4k + 2 = \frac{4(s-1)}{3} + 2$.

Finally, let $s = 3t + 2$. Here 1e−dominations in $A_{1m(s)}\square A_{1m(1)}$, is same for $s = 3t, 3t + 1, 3t + 2$ from the join $B_1 = ((\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_1, \beta_3))$ to
$B_{6t-3} = ((\alpha_{6t-3}, \beta_1), (\alpha_{6t-3}, \beta_2), (\alpha_{6t-3}, \beta_3))$.

Hence γ_{1e} changes for various nodes of s according to the remaining joins. Hence for 3s nodes, $B_{6t-i}, i = -2, ..., 1$ joins the Minimal γ_{1e} dominating nodes are $(\alpha_{6t}, \beta_1), (\alpha_{6t+1}, \beta_2)$. For $3s + 1$ nodes, $B_{6t-i}, i = -2, ..., 3$ joins the Minimal γ_{1e} dominating nodes are $(\alpha_{6t+1}, \beta_1), (\alpha_{6t+1}, \beta_2), (\alpha_{6t+1}, \beta_3)$.For $3s + 2$ nodes, $B_{6t-i}, i = -2, ..., 5$ joins the Minimal γ_{1e} dominating nodes are $(\alpha_{6t}, \beta_2), (\alpha_{6t+3}, \beta_1), (\alpha_{6t+3}, \beta_2)$, (α_{6t+3}, β_3).

Hence $\gamma_a[P_{s(n)}\square P_{s(1)}]$, when $n = 3k+2$ is equal to $\gamma_{1e}[A_{1m(s)}\square A_{1m(1)}] + 1 = 4k+3 = \frac{4(n-2)}{3} + 3$. Hence the lemma.

3.2. Lemma

$\gamma_{1e}\left[A_{2m(1)}\square A_{2m(1)}\right] = 4$
$\gamma_{1e}\left[A_{2m(2)}\square A_{2m(1)}\right] = 4$

$$\gamma_{1e}\left[A_{2m(s)}\square A_{2m(1)}\right] = \begin{cases} 2s+1 & \text{if } \ s\text{=3t} \\ 2(s-1)+2 & \text{if } \ s\text{=3t+1} \\ 3(s-2)+4 & \text{if } \ s\text{=3t+2}, \ t = 1,2,3,\ldots \end{cases} \quad \text{where } s,t \text{ denotes the number of nodes.}$$

Proof:

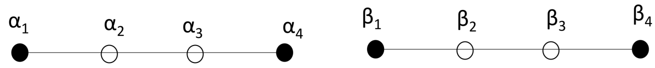

Figure 4: Two copies of $A_{2m(1)}$

From Fig. 4 $A_{2m(1)}$ two copies of two intermediate nodes semigraph with single join.

Here $A_{2m(1)}\square A_{2m(1)}$ having the node set

$$N = \{(\alpha_1,\beta_1),(\alpha_2,\beta_1),(\alpha_3,\beta_1),(\alpha_4,\beta_1),(\alpha_1,\beta_2),(\alpha_2,\beta_2),\\ (\alpha_3,\beta_2),(\alpha_4,\beta_2),(\alpha_1,\beta_3),(\alpha_2,\beta_3),(\alpha_3,\beta_3),(\alpha_4,\beta_3),\\ (\alpha_1,\beta_4),(\alpha_2,\beta_4),(\alpha_3,\beta_4),(\alpha_4,\beta_4)\}$$

and the join set

$$J = \{[(\alpha_1,\beta_1),(\alpha_2,\beta_1),(\alpha_3,\beta_1),(\alpha_4,\beta_1)],\\ [(\alpha_1,\beta_2),(\alpha_2,\beta_2),(\alpha_3,\beta_2),(\alpha_4,\beta_2)],\\ [(\alpha_1,\beta_3),(\alpha_2,\beta_3),(\alpha_3,\beta_3),(\alpha_4,\beta_3)],\\ [(\alpha_1,\beta_4),(\alpha_2,\beta_4),(\alpha_3,\beta_4),(\alpha_4,\beta_4)],\\ [(\alpha_1,\beta_1),(\alpha_1,\beta_2),(\alpha_1,\beta_3),(\alpha_1,\beta_4)],\\ [(\alpha_2,\beta_1),(\alpha_2,\beta_2),(\alpha_2,\beta_3),(\alpha_2,\beta_4)],\\ [(\alpha_3,\beta_1),(\alpha_3,\beta_2),(\alpha_3,\beta_3),(\alpha_3,\beta_4)],\\ [(\alpha_4,\beta_1),(\alpha_4,\beta_2),(\alpha_4,\beta_3),(\alpha_4,\beta_4)]\}.$$

From the Fig. 5 observed that $(\alpha_1,\beta_1),(\alpha_4,\beta_1),(\alpha_1,\beta_4),(\alpha_4,\beta_4)$ are end nodes, $(\alpha_2,\beta_1),(\alpha_3,\beta_1),(\alpha_1,\beta_2),(\alpha_4,\beta_2),(\alpha_1,\beta_3),(\alpha_4,\beta_3),(\alpha_2,\beta_4),(\alpha_3,\beta_4)$ are mid-end nodes, and $(\alpha_2,\beta_2),(\alpha_3,\beta_2),(\alpha_2,\beta_3),(\alpha_3,\beta_3)$ are mid nodes. Hence $\gamma_{1e}\left[A_{2m(1)}\square A_{2m(1)}\right] = 4$.

From Fig. 5 $A_{2m(1)}\square A_{1m(1)}$ node set, $\{(\alpha_4,\beta_1),(\alpha_4,\beta_2),(\alpha_4,\beta_3),(\alpha_4,\beta_4)\}$ creates a minimal $1e-$ dominating set in $A_{2m(2)}\square A_{2m(1)}$.Hence $\gamma_{1e}\left[A_{2m(2)}\square A_{2m(1)}\right] = 4$.

To prove last case , let us take $s = 3t$, $k = t = 1,2,3,4,\ldots$ and it is observed that $A_{2m(2)}\square A_{2m(1)}$ is of cardinality $36t+4$ and $21t+1$ joins .

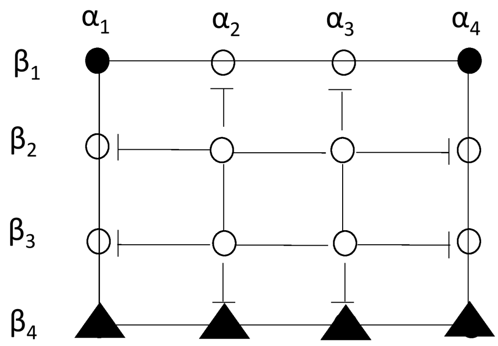

Figure 5: $A_{2m(1)} \square A_{2m(1)}$

It is noted that the mesh $A_{2m(2)} \square A_{2m(1)}$ containing fourty nodes with exactly 4 duplicates of $A_{2m(2)}$. Hence the end nodes (α_4, β_1),(α_4, β_2),(α_4, β_3), (α_4, β_4) dominates the other adjacent nodes in that edge. Also, $A_{2m(s)} \square A_{2m(1)}$, $s = 3t$, $t = 1, 2, 3, 4, \ldots$ containing p duplicates of $A_{2m(2)} \square A_{2m(1)}$.
Hence

$$B = \{(\alpha_p, \beta_q)/p = 4, 13, \ldots, (9t - 5), t = 1, \ldots q = 1, 2, 3, 4\}$$

with 4e nodes dominates all the other nodes in $A_{2m(s)} \square A_{2m(1)}$ apart from, (α_{9i1}, β_1),(α_{9i-1}, β_2),(α_{9i-1}, β_3), (α_{9i-1}, β_4) and (α_{9i}, β_1),(α_{9i}, β_2),(α_{9i}, β_3),(α_{9i}, β_4), $i = 1, 2, 3, \ldots, t$ nodes and (α_{9j+1}, β_1),(α_{9j+1}, β_2),(α_{9j+1}, β_3), (α_{9j+1}, β_4).

Observe that for all $i = 1, 2, 3, 4, \ldots t$, the nodes (α_{9i-1}, β_1),(α_{9i-1}, β_2),(α_{9i-1}, β_3) , (α_{9i-1}, β_4) create a join B_{9i-1} (say) and (α_{9i}, β_1),(α_{9i}, β_2),(α_{9i}, β_3) , (α_{9i}, β_4) create a join B_{9i} (say) with (α_{9i}, β_1), (α_{6i}, β_4) , (α_{9i-1}, β_1), (α_{9i-1}, β_4) mid-end nodes, (α_{9i}, β_2) , (α_{9i}, β_3), (α_{9i-1}, β_2), (α_{9i-1}, β_3) mid nodes in any one end node from the join B_{9i-1} and one end node from the join B_{9i} dominates others. Hence t end nodes (α_{9i-1}, β_1), (α_{9i-1}, β_2),(α_{9i-1}, β_3),(α_{9i-1}, β_4) must be chosen and (α_{9i}, β_1), (α_{9i}, β_2),(α_{9i}, β_3),(α_{9i}, β_4) $i = 1, 2, 3, 4, \ldots, t$.

At the end if we select only one end node from $B_{9j+1} = ((\alpha_{9j+1}, \beta_1), (\alpha_{9j+1}, \beta_2), (\alpha_{9j+1}, \beta_3), (\alpha_{9j+1}, \beta_4))$, the corresponding set containing $6t + 1$ nodes, where $s = 3t$. Therefore $\gamma_{1e}[A_{2m(s)} \square A_{2m(1)}] = 2s + 1$, if $s = 3t$.

Subsequenty, $s = 3t + 1$,$t = 1, 2, 3, 4, \ldots$. $A_{2m(s)} \square A_{2m(1)}$, when $s = 3t + 1$ contains all the nodes of $A_{2m(s)} \square A_{2m(1)}$ when $s = 3t$, and also the nodes (α_{9j+2}, β_1),(α_{9j+2}, β_2),(α_{9j+2}, β_3), (α_{9j+2}, β_4),(α_{9j+3}, β_1), (α_{9j+3}, β_2),(α_{9j+3}, β_3),(α_{9j+3}, β_4),(α_{9j+4}, β_1),(α_{9j+4}, β_2),(α_{9j+4}, β_3),(α_{9j+4}, β_4).

Hence, for selecting end nodes from the joins $B_{9t+1} = ((\alpha_{9t+1}, \beta_1), (\alpha_{9t+1}, \beta_2), (\alpha_{9t+1}, \beta_3), (\alpha_{9t+1}, \beta_4))$ makes a minimal $1e-$dominating set when $s = 3t + 1$, is $\gamma_{1e}[A_{2m(s)} \square A_{2m(1)}] + 1$.

Thus, $\gamma_{1e}[A_{2m(s)} \square A_{2m(1)}] = 6t + 2 = 2(t - 1) + 2$.

At last, let $s = 3t + 2$. It can be noted that $1e-$dominations in $A_{2m(s)} \square A_{2m(1)}$, is same for $s = 3t, 3t + 1, 3t + 2$ from the join $B_1 = ((\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_1, \beta_3), (\alpha_1, \beta_4))$ to $B_{9t-2} = ((\alpha_{9t-3}, \beta_1), (\alpha_{9t-3}, \beta_2), (\alpha_{9t-3}, \beta_3), (\alpha_{9t-3}, \beta_4))$.

For different s, γ_{1e} varies based on the other joins. Hence for 3s nodes, $B_{9t-i}, i = -4, ..., 1$ joins the Minimal γ_{1e} dominating nodes are $(\alpha_{9t-1}, \beta_4), (\alpha_{9t}, \beta_4), (\alpha_{9t+1}, \beta_4)$. For $3s + 1$ nodes, $B_{9t-i}, i = -4, ..., 4$ joins the Minimal γ_{1e} dominating nodes are $(\alpha_{9t+1}, \beta_1), (\alpha_{9t+1}, \beta_2), (\alpha_{9t+1}, \beta_3), (\alpha_{9t+1}, \beta_4)$. For $3s + 2$ nodes, $B_{9t-i}, i = -4, ..., 7$ joins the Minimal γ_{1e} dominating nodes are $(\alpha_{9t-1}, \beta_4), (\alpha_{9t}, \beta_4), (\alpha_{9t+4}, \beta_1), (\alpha_{9t+4}, \beta_2)$,

$(\alpha_{9t+4}, \beta_3), (\alpha_{9t+4}, \beta_4)$.

Hence $\gamma_{1e}[A_{2m(s)} \square A_{2m(1)}]$, when $s = 3t + 2$ is $2(n - 2) + 6$.

Hence the lemma.

3.3. Lemma

$\gamma_{1e}\left[A_{1m(s)} \square A_{1m(2)}\right] = 2s + 1,\ s = 1, 2, 3, \ldots$

Proof:

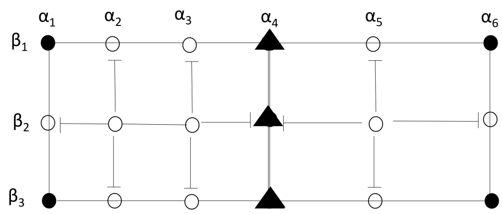

Figure 6: $A_{2m(1)} \square A_{1m(1)}$

From the Fig. 6 3 nodes are enough to dominate all other nodes.

Similarly the Cartesian product $A_{1m(1)} \square A_{1m(2)}$ is a graph with $5(2s + 1)$ nodes, and $9s + 2$ joins. Among $5(2s + 1)$ nodes, $3(s + 1)$ are end nodes, $2s$ are mid nodes and $5s + 2$ are mid-end nodes. It can be seen that, the nodes $(\alpha_{3e}, \beta_p), e = \begin{cases} 1, 3, 5, \ldots, \lceil \frac{s}{3} \rceil & \text{if } s \neq 3j + 1 \\ 1, 3, 5, \ldots, \lceil \frac{s}{4} \rceil & \text{if } s = 3j + 1 \end{cases}$

where $j = 1, 2, \ldots$ makes a minimal $1e-$dominating set, along with the nodes (α_{se}, β_3), (α_{se+1}, β_3), $e = 2$ if $s = 3, 6, 9, \ldots$

(α_{se-2}, β_3),(α_{se-1}, β_2),(α_{se}, β_3), (α_{se+1}, β_3), $s = 2$ if $s = 4, 7, 10, \ldots$

(α_{6e}, β_3), $e = 1, 2, 3, \ldots, \lfloor \frac{s}{4} \rfloor$ if $s = 4, 5, 6, 7, \ldots$

Thus the contain of such $1e-$dominating nodes of joins from B_1 to B_{3e+2} as defined above is

5	if	$s = 2, 3, 4$
$(2 \times 5) + 1$	if	$s = 5, 6, 7$
$(3 \times 5) + 2$	if	$s = 8, 9, 10$
$(4 \times 5) + 3$	if	$s = 11, 12, 13$ etc,...

Observe that the above nodes is the fullest set of minimal $1e-$dominating set for the product graphs $A_{s(2)} \square A_{1m(2)}$, $A_{1m(5)} \square A_{1m(2)}$, $A_{1m(8)} \square A_{1m(2)}, \ldots$ as these product graphs contain joins from B_1 to B_{3e+2}. But $A_{1m(3)} \square A_{1m(2)}$, $A_{1m(6)} \square A_{1m(2)}$, $A_{1m(9)} \square A_{1m(2)}, \ldots$ having two more additional joins and $A_{1m(4)} \square A_{1m(2)}$, $A_{1m(7)} \square A_{1m(2)}$, $A_{1m(10)} \square A_{1m(2)}, \ldots$ having 4 more joins except first set of the above product graphs. Hence for 2 nodes,the Minimal γ_{1e} dominating nodes are 5. For 3 nodes,the Minimal γ_{1e} dominating nodes are 7. For 4 nodes,the Minimal γ_{1e} dominating nodes are 9...,

Hence $\gamma_{1e}[A_{1m(s)} \square A_{1m(2)}] = 2s + 1,\ s = 1, 2, 3, \ldots$

4. Conclusion

The Cartesian product provides a powerful mathematical framework for modeling and optimizing complex systems involving multiple pathways or dimensions. Its applications span across various fields such as telecommunications, logistics, computer science, urban planning, biology, social networks, and robotics, helping solve practical problems involving connectivity, efficiency, and system optimization. 1e-domination in Cartesian products of path semigraphs not only advances theoretical semigraph theory but also contributes practical strategies for optimized network supervision and control.In this research work, $\gamma_{1e}\left[A_{1m(s)}\square A_{1m(1)}\right]$ and $\gamma_{1e}\left[A_{2m(s)}\square A_{2m(1)}\right]$ was discussed briefly.

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