Math. & Comput. Sci., 7(3) (2026), 78–87

Online: ISSN 2717-2708

Mathematics and Computational Sciences

Journal Homepage: http://mcs.qut.ac.ir/mcs

A refined approach to Ulam-Hyers stability in trigonometric AQ functional equations

P. Agilan[a,*], V. Vallinayagam[a], S. Annadurai[a], Vaddi Seshagiri Rao[b]

[a]Department of Mathematics, St.Joseph's College of Engineering, OMR, Chennai, 600 119, Tamilnadu, India.
[b]Department of Mechanical Engineering, St.Joseph's College of Engineering, OMR, Chennai, 600 119, Tamilnadu, India.

Abstract

The primary objective of this paper is to formulate an extended version of a new mixed-type additive-quadratic functional equation and to analyze its Ulam-Hyers stability in three distinct cases: odd, even, and odd-even. Specifically, the odd case corresponds to the stability of additive functional equations, the even case corresponds to the stability of quadratic functional equations, and the odd-even case characterizes the stability behavior of mixed additive-quadratic (AQ) functional equations.

Keywords: Additive - Quadratic Functional Equations, Generalized Hyers - Ulam - Stability.

2020 MSC: 39B52, 39B82.

1. Introduction

The stability theory of functional equations originated from a question posed by Ulam in 1940 concerning the stability of group homomorphisms [25]. A partial affirmative answer was provided by Hyers [14] for Banach spaces, which later became known as Hyers–Ulam stability. This foundational result was further generalized by Aoki [9] and Rassias [21, 22], leading to the celebrated Hyers–Ulam–Rassias stability concept. Subsequent refinements and extensions were developed by Găvruta [12, 13], introducing control functions and broadening applicability. Over the decades, stability analysis has evolved into a rich and independent research field with deep connections to nonlinear analysis, operator theory, and Banach space geometry.

In recent years, significant progress has been made in extending stability concepts to differential and difference equations [1, 10, 16, 23, 24], fractional models [11, 17], nonlinear dynamical systems [2], and various generalized normed structures including fuzzy, random, and neutrosophic normed spaces [3, 4, 5, 6, 7, 8, 20]. Moreover, modern approaches frequently employ direct estimation techniques and fixed point methods, which provide unified and powerful frameworks for proving stability results. Investigations into cubic and higher-order functional equations, Mittag-Leffler-type stability, and related

*Corresponding author
Email addresses: agilram@gmail.com (P. Agilan), vngam19@gmail.com (V. Vallinayagam), drsannadurai@gmail.com (S. Annadurai), raosvaddi@gmail.com (Vaddi Seshagiri Rao)

doi: 10.30511/mcs.2026.2079529.1648

Received: 01 December 2025 Accepted: 29 May 2026

analytical transforms [15, 18, 19] demonstrate that the scope of Ulam-type stability has expanded considerably beyond classical additive mappings.

$$\begin{aligned}
&\mathscr{Q}\left(\mathscr{X}\operatorname{cosec}^{\mathscr{L}}\left(\frac{\pi}{4}\right)+2\mathscr{Y}\sec^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)+\mathscr{Q}\left(\mathscr{Y}\sec^{\mathscr{L}}\left(\frac{\pi}{4}\right)-\mathscr{X}\operatorname{cosec}^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)\\
&+\mathscr{Q}\left(\mathscr{X}\operatorname{cosec}^{\mathscr{L}}\left(\frac{\pi}{4}\right)-\mathscr{Y}\sec^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)=\left(\frac{\operatorname{cosec}^{\mathscr{L}}\left(\frac{\pi}{4}\right)+3\operatorname{cosec}^{2\mathscr{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathscr{Q}(\mathscr{X})\\
&\qquad+\left(\frac{3\operatorname{cosec}^{2\mathscr{L}}\left(\frac{\pi}{4}\right)-\operatorname{cosec}^{\mathscr{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathscr{Q}(-\mathscr{X})\\
&+\left(\sec^{\mathscr{L}}\left(\frac{\pi}{4}\right)+3\sec^{2\mathscr{L}}\left(\frac{\pi}{4}\right)\right)\mathscr{Q}(\mathscr{Y})+\left(3\sec^{2\mathscr{L}}\left(\frac{\pi}{4}\right)-\sec^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)\mathscr{Q}(-\mathscr{Y})
\end{aligned}$$

$$\begin{aligned}
&\mathscr{D}\left(\mathscr{X}\sin^{\mathscr{L}}\left(\frac{\pi}{4}\right)+2\mathscr{Y}\cos^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)+\mathscr{D}\left(\mathscr{Y}\cos^{\mathscr{L}}\left(\frac{\pi}{4}\right)-\mathscr{X}\sin^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)\\
&+\mathscr{D}\left(\mathscr{X}\sin^{\mathscr{L}}\left(\frac{\pi}{4}\right)-\mathscr{Y}\cos^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)=\left(\frac{\sin^{\mathscr{L}}\left(\frac{\pi}{4}\right)+3\sin^{2\mathscr{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathscr{D}(\mathscr{X})\\
&\qquad+\left(\frac{3\sin^{2\mathscr{L}}\left(\frac{\pi}{4}\right)-\sin^{\mathscr{L}}\left(\frac{\pi}{4}\right)}{2}\right)\mathscr{D}(-\mathscr{X})\\
&+\left(\cos^{\mathscr{L}}\left(\frac{\pi}{4}\right)+3\cos^{2\mathscr{L}}\left(\frac{\pi}{4}\right)\right)\mathscr{D}(\mathscr{Y})+\left(3\cos^{2\mathscr{L}}\left(\frac{\pi}{4}\right)-\cos^{\mathscr{L}}\left(\frac{\pi}{4}\right)\right)\mathscr{D}(-\mathscr{Y})
\end{aligned}$$

Motivated by these developments, the present study aims to further investigate stability phenomena for a new class of functional equations within generalized normed frameworks. Despite the extensive literature, several open questions remain regarding the interplay between higher-order structures, nonlinear perturbations, and generalized metric environments. In particular, there is a need to unify direct and fixed point techniques while improving stability bounds under weaker assumptions. The objective of this work is therefore to contribute to the ongoing advancement of Ulam-type stability theory by establishing new stability results, refining existing methodologies, and extending applicability to broader analytical settings.

$$\begin{aligned}
&\mathscr{H}\left(\mathcal{U}\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)+2\mathcal{V}\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)+\mathscr{H}\left(\mathcal{V}\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)-\mathcal{U}\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\\
&+\mathscr{H}\left(\mathcal{U}\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)-\mathcal{V}\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)=\left(\frac{\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)+3\tan^{2\mathcal{R}}\left(\frac{\pi}{3}\right)}{2}\right)\mathscr{H}(\mathcal{U})\\
&\qquad+\left(\frac{3\tan^{2\mathcal{R}}\left(\frac{\pi}{3}\right)-\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)}{2}\right)\mathscr{H}(-\mathcal{U})\\
&+\left(\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)+3\cot^{2\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathscr{H}(\mathcal{V})+\left(3\cot^{2\mathcal{R}}\left(\frac{\pi}{3}\right)-\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathscr{H}(-\mathcal{V}) \qquad (1.1)
\end{aligned}$$

Let

$$\begin{aligned}
\mathscr{H}(\mathcal{U},\mathcal{V})=\;&\mathscr{H}\left(\mathcal{U}\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)+2\mathcal{V}\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)+\mathscr{H}\left(\mathcal{V}\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)-\mathcal{U}\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\\
&+\mathscr{H}\left(\mathcal{U}\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)-\mathcal{V}\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\\
&-\left(\frac{\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)+3\tan^{2\mathcal{R}}\left(\frac{\pi}{3}\right)}{2}\right)\mathscr{H}(\mathcal{U})-\left(\frac{3\tan^{2\mathcal{R}}\left(\frac{\pi}{3}\right)-\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)}{2}\right)\mathscr{H}(-\mathcal{U})\\
&-\left(\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)+3\cot^{2\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathscr{H}(\mathcal{V})-\left(3\cot^{2\mathcal{R}}\left(\frac{\pi}{3}\right)-\cot^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathscr{H}(-\mathcal{V})
\end{aligned}$$

2. ADDITIVE MAPPING

Theorem 2.1. Assume that $\mathcal{E}$ be a additive function and

$$\sum_{\mathcal{N}=0}^{\infty} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{V}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}} \quad \text{converges in } \mathcal{R} \quad \text{also}$$

$$\lim_{\mathcal{N}\to\infty} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{V}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}} = 0. \tag{2.1}$$

Let the mapping $\mathscr{H}_a : \mathscr{H} \to \mathscr{I}$ be an odd function satisfying the inequality

$$\|\mathscr{H}_a(\mathcal{U}, \mathcal{V})\| \leqslant \mathcal{E}\left(\mathcal{U}, \mathcal{V}\right) \tag{2.2}$$

and the defined condition $A(\mathcal{U})$ is

$$A(\mathcal{U}) = \lim_{\mathcal{U}\to\infty} \frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}}. \tag{2.3}$$

There exists $A : \mathscr{H} \to \mathscr{I}$ is an unique odd mapping, then

$$\|\mathscr{H}_a(\mathcal{U}) - A(\mathcal{U})\| \leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U}, 0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}}. \tag{2.4}$$

Proof. Take $\mathcal{V}$ as 0 in (2.2) and applying the odd condition, we reach

$$\left\|\mathscr{H}_a(\mathcal{U}) - \frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)}\right\| \leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)}\mathcal{E}\left(\mathcal{U}, 0\right) \tag{2.5}$$

for all$\mathcal{U} \in \mathscr{H}$. Now replacing $\mathcal{U}$ by $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\mathcal{U}\right)$ and dividing by $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)$ in (2.5), we obtain

$$\left\|\frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} - \frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\right\| \leqslant \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U}, 0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2} \tag{2.6}$$

for all$\mathcal{U} \in \mathscr{H}$. It follows from (2.5) and (2.6) that

$$\left\|\mathscr{H}_a(\mathcal{U}) - \frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\right\| \leqslant \left\|\mathscr{H}_a(\mathcal{U}) - \frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)}\right\| \tag{2.7}$$

$$+ \left\|\frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} - \frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\right\|$$

$$\leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)}\left[\mathcal{E}\left(\mathcal{U}, 0\right) + \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U}, 0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)}\right] \tag{2.8}$$

for all$\mathcal{U} \in \mathscr{H}$. In general for any positive integer N , we get

$$\left\|\mathscr{H}_a(\mathcal{U}) - \frac{\mathscr{H}_a(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}}\right\| \leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\mathcal{N}-1} \frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U}, 0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}}$$

$$\leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U}, 0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}} \tag{2.9}$$

for all$\mathcal{U} \in \mathscr{H}$. We have to show the sequence of convergence

$$\left\{ \frac{\mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \right\},$$

take $\mathcal{U}$ as $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M x$ also divide the following term $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M$ in (2.9), and $M, N > 0$

$$\begin{aligned}
&\left\| \frac{\mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M} - \frac{\mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N+M} \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{(N+M)}} \right\| \\
&= \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M} \left\| \mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M \mathcal{U}\right) - \frac{\mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \cdot \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \right\| \\
&\leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\mathcal{N}-1} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+M} \mathcal{U}, 0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+M}} \\
&\leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+M} \mathcal{U}, 0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+M}} \\
&\to 0 \quad \text{as} \quad M \to \infty
\end{aligned}$$

for all $\mathcal{U} \in \mathscr{H}$. Hence the sequence $\left\{ \frac{\mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \right\}$ is a Cauchy.

Since $\mathscr{I}$ is complete, there exists a mapping $A : \mathscr{H} \to \mathscr{I}$ such that

$$A(\mathcal{U}) = \lim_{\mathcal{U} \to \infty} \frac{\mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \quad \forall \ \mathcal{U} \in \mathscr{H}.$$

To prove A satisfies (1.1). By taking suitable substitions, we obtain

$$\begin{aligned}
&\frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \left\| \mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{V}\right) \right\| \\
&\qquad \leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{V}\right).
\end{aligned}$$

As N approaches to ∞ in the above and applying additive condition, we arrive

$$\mathscr{H}_a(\mathcal{U}, \mathcal{V}) = 0.$$

Next, A satisfies the uniqueness

$$\begin{aligned}
\|A(\mathcal{U}) - B(\mathcal{U})\| &= \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \left\| A\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right) - B\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right) \right\| \\
&\leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N} \left\{ \left\| A\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right) - \mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right) \right\| \right. \\
&\qquad \left. + \left\| \mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right) - B\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right) \right\| \right\} \\
&\leqslant \frac{2}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+N} \mathcal{U}, 0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{(\mathcal{H}+N)}} \\
&\to 0 \quad \text{as} \quad N \to \infty
\end{aligned}$$

for all $\mathcal{U} \in \mathscr{H}$. Hence A is unique. □

Corollary 2.1. *Assume that $\mathcal{E}$ be a additive function and P are real constant, then*

$$\|\mathscr{H}_a(\mathcal{U},\mathcal{V})\| \leqslant \begin{cases} \mathcal{E}, & \\ \mathcal{E}\left\{\|\mathcal{U}\|^P + \|\mathcal{V}\|^P\right\}, & P \neq 1; \\ \mathcal{E}\left\{\|\mathcal{U}\|^P\|\mathcal{V}\|^P + \left\{\|\mathcal{U}\|^{2P} + \|\mathcal{V}\|^{2P}\right\}\right\}, & P \neq \frac{1}{2}; \end{cases} \tag{2.10}$$

and we have the following results

$$\|\mathscr{H}_o(\mathcal{U}) - A(\mathcal{U})\| \leqslant \begin{cases} \dfrac{\mathcal{E}}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) - 1}, \\ \dfrac{\mathcal{E}\|\mathcal{U}\|^P}{\left|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^P\right|}, \\ \dfrac{\mathcal{E}\|\mathcal{U}\|^{2P}}{\left|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2P}\right|}, \end{cases} \tag{2.11}$$

for all $\mathcal{U} \in \mathscr{H}$

3. QUADRATIC MAPPING

Theorem 3.1. *Assume that $\mathcal{E}$ be a quadratic function and*

$$\sum_{\mathcal{N}=0}^{\infty} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{V}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}} \quad \textit{converges} \quad \textit{in} \quad \mathcal{R} \quad \textit{also}$$

$$\lim_{\mathcal{N}\to\infty} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{V}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}} = 0. \tag{3.1}$$

Let the mapping $\mathscr{H}_q : \mathscr{H} \to \mathscr{I}$ be an even function satisfying the inequality

$$\|\mathscr{H}_q(\mathcal{U},\mathcal{V})\| \leqslant \mathcal{E}\left(\mathcal{U},\mathcal{V}\right) \tag{3.2}$$

and the defined condition $Q(\mathcal{U})$ is

$$Q(\mathcal{U}) = \lim_{\mathcal{U}\to\infty} \frac{\mathscr{H}_q(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{N}}\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}}. \tag{3.3}$$

There exists $Q : \mathscr{H} \to \mathscr{I}$ is an unique even mapping, then

$$\|\mathscr{H}_q(\mathcal{U}) - Q(\mathcal{U})\| \leqslant \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} \tag{3.4}$$

Proof. Consider $\mathcal{V}$ by 0 in (3.2) and applying the even condition, we reach

$$\left\|\mathscr{H}_q(\mathcal{U}) - \frac{\mathscr{H}_q(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\right\| \leqslant \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\mathcal{E}\left(\mathcal{U},0\right) \tag{3.5}$$

for all$\mathcal{U} \in \mathscr{H}$. Now replacing $\mathcal{U}$ by $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U}$ and dividing by $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2$ in (3.5), we obtain

$$\left\|\frac{\mathscr{H}_q(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2} - \frac{\mathscr{H}_q(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2\mathcal{U})}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^4}\right\| \leqslant \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U},0\right)}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^4} \tag{3.6}$$

for all$\mathcal{U} \in \mathscr{H}$. It follows from (3.5) and (3.6) that

$$\begin{aligned}\left\|\mathscr{H}_q(\mathcal{U}) - \frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2 \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^4}\right\| &\leqslant \left\|\mathscr{H}_q(\mathcal{U}) - \frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\right\| \\ &\quad + \left\|\frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2} - \frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2 \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^4}\right\| \\ &\leqslant \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\left[\mathcal{E}(\mathcal{U},0) + \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\right] \qquad (3.7)\end{aligned}$$

for all$\mathcal{U} \in \mathscr{H}$. For N , we arrive

$$\begin{aligned}\left\|\mathscr{H}_q(\mathcal{U}) - \frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}}\right\| &\leqslant \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\sum_{\mathcal{H}=0}^{\mathcal{N}-1}\frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} \\ &\leqslant \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\sum_{\mathcal{H}=0}^{\infty}\frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} \qquad (3.8)\end{aligned}$$

for all$\mathcal{U} \in \mathscr{H}$. We have to show the sequence of convergence

$$\left\{\frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}}\right\},$$

take $\mathcal{U}$ by $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2M}\mathcal{U}$ and taking the division of $\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2M}$to (3.8) and $M, N > 0$, we deduce

$$\begin{aligned}&\left\|\frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M x\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2M}} - \frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N+M} \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2(N+M)}}\right\| \\ &= \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2M}}\left\|\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M \mathcal{U}\right) - \frac{\mathscr{H}_a\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N\cdot\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^M \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}}\right\| \\ &\leqslant \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\sum_{\mathcal{H}=0}^{\mathcal{N}-1}\frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+M}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2(\mathcal{H}+M)}} \\ &\leqslant \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2}\sum_{\mathcal{H}=0}^{\infty}\frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+M}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2(\mathcal{H}+M)}} \\ &\to 0 \quad \text{as} \quad M \to \infty\end{aligned}$$

The sequence $\left\{\frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}}\right\}$ is a Cauchy . Since $\mathscr{I}$ is complete, there exists a mapping $Q : \mathscr{H} \to \mathscr{I}$ such that

$$Q(\mathcal{U}) = \lim_{\mathcal{U}\to\infty}\frac{\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}} \quad \forall\ \mathcal{U} \in \mathscr{H}.$$

To prove Q satisfies (1.1). By taking suitable substitions, we obtain

$$\begin{aligned}&\frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}}\left\|\mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{V}\right)\right\| \\ &\leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}}\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{U}, \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^N \mathcal{V}\right)\end{aligned}$$

As N approaches to ∞ in the above and applying quadratic condition, we arrive

$$Q(\mathcal{U},\mathcal{V}) = 0.$$

Next, Q satisfies the uniqueness

$$\begin{aligned}
\|Q(\mathcal{U}) - B(\mathcal{U})\| &= \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}} \left\| A\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}\mathcal{U}\right) - B\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}\mathcal{U}\right)\right\| \\
&\leqslant \frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2N}} \left\{ \left\| A\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}\mathcal{U}\right) - \mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}\mathcal{U}\right)\right\| \right. \\
&\qquad \left. + \left\| \mathscr{H}_q\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}\mathcal{U}\right) - B\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{N}\mathcal{U}\right)\right\| \right\} \\
&\leqslant \frac{2}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2}} \sum_{\mathcal{H}=0}^{\infty} \frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}+N}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2(\mathcal{H}+N)}} \\
&\to 0 \quad \text{as} \quad N \to \infty
\end{aligned}$$

for all $\mathcal{U} \in \mathscr{H}$. □

Corollary 3.2. *Assume that $\mathcal{E}$ be a quadratic function and P are real constant, then*

$$\|\mathscr{H}_q(\mathcal{U},\mathcal{V})\| \quad \leqslant \begin{cases} \mathcal{E}, & \\ \mathcal{E}\left\{\|\mathcal{U}\|^P + \|\mathcal{V}\|^P\right\}, & P \neq 2; \\ \mathcal{E}\left\{\|\mathcal{U}\|^P\|\mathcal{V}\|^P + \left\{\|\mathcal{U}\|^{2P} + \|\mathcal{V}\|^{2P}\right\}\right\}, & P \neq 1; \end{cases} \tag{3.9}$$

and we have the following results

$$\|\mathscr{H}_q(\mathcal{U}) - A(\mathcal{U})\| \leqslant \begin{cases} \dfrac{\mathcal{E}}{3\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2} - 1\right)}, \\ \dfrac{\mathcal{E}\|\mathcal{U}\|^P}{3\left|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2} - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{P}\right|}, \\ \dfrac{\mathcal{E}\|\mathcal{U}\|^{2P}}{3\left|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2} - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2P}\right|}, \end{cases} \tag{3.10}$$

4. ADDITIVE -QUADRATIC MAPPING

Theorem 4.1. *Assume that $\mathcal{E}$ be a additive-quadratic function and satisfying*

$$\|\mathscr{H}(\mathcal{U},\mathcal{V})\| \leqslant \mathcal{E}(\mathcal{U},\mathcal{V}). \tag{4.1}$$

There exists a AQ mapping $A, Q : \mathscr{H} \to \mathscr{I}$ and, we have

$$\begin{aligned}
\|\mathscr{H}(\mathcal{U}) - A(\mathcal{U}) - Q(\mathcal{U})\| \leqslant \frac{1}{2}\Bigg[&\frac{1}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \left(\frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0\right.}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}} + \frac{\mathcal{E}\left(-\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}} \right) \\
&+ \frac{1}{3\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2}} \sum_{\mathcal{H}=0}^{\infty} \left(\frac{\mathcal{E}\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} + \frac{\mathcal{E}\left(-\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0\right)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} \right) \Bigg]
\end{aligned} \tag{4.2}$$

for all $\mathcal{U} \in \mathscr{H}$.

Proof. By the priciple of oddness, we have $\mathscr{H}_o(\mathcal{U}) = \dfrac{\mathscr{H}_a(\mathcal{U}) - \mathscr{H}_a(-\mathcal{U})}{2}$ and by applying this condition

$$\|\mathscr{H}_o(\mathcal{U},\mathcal{V})\| \leqslant \frac{\mathcal{E}(\mathcal{U},\mathcal{V})}{2} + \frac{\mathcal{E}(-\mathcal{U},-\mathcal{V})}{2} \tag{4.3}$$

For 2.1

$$\|\mathscr{H}_o(\mathcal{U}) - \mathrm{A}(\mathcal{U})\| \leqslant \frac{1}{2\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \left(\frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}} + \frac{\mathcal{E}(-\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}} \right) \tag{4.4}$$

By the priciple of evenness, we have $\mathscr{H}_e(\mathcal{U}) = \dfrac{\mathscr{H}_q(\mathcal{U}) + \mathscr{H}_q(-\mathcal{U})}{2}$ and by applying this condition

$$\|\mathscr{H}_e(\mathcal{U},\mathcal{V})\| \leqslant \frac{\mathcal{E}(\mathcal{U},\mathcal{V})}{2} + \frac{\mathcal{E}(-\mathcal{U},-\mathcal{V})}{2} \tag{4.5}$$

For 3.1

$$\|\mathscr{H}_e(\mathcal{U}) - \mathrm{Q}(\mathcal{U})\| \leqslant \frac{1}{6\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2} \sum_{\mathcal{H}=0}^{\infty} \left(\frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} + \frac{\mathcal{E}(-\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} \right) \tag{4.6}$$

By Combining the above two conditions, we reach

$$\mathscr{H}(\mathcal{U}) = \mathscr{H}_e(\mathcal{U}) + \mathscr{H}_o(\mathcal{U}) \tag{4.7}$$

and substitute the results, then we obtain the following

$$\begin{aligned}
\|\mathscr{H}(\mathcal{U}) - \mathrm{A}(\mathcal{U}) - \mathrm{Q}(\mathcal{U})\| &= \|\mathscr{H}_e(\mathcal{U}) + \mathscr{H}_o(\mathcal{U}) - \mathrm{A}(\mathcal{U}) - \mathrm{Q}(\mathcal{U})\| \\
&\leqslant \|\mathscr{H}_o(\mathcal{U}) - \mathrm{A}(\mathcal{U})\| + \|\mathscr{H}_e(\mathcal{U}) - \mathrm{Q}(\mathcal{U})\| \\
&\leqslant \frac{1}{2\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)} \sum_{\mathcal{H}=0}^{\infty} \left(\frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}} + \frac{\mathcal{E}(-\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}} \right) \\
&\quad + \frac{1}{6\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2} \sum_{\mathcal{H}=0}^{\infty} \left(\frac{\mathcal{E}(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} + \frac{\mathcal{E}(-\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathcal{H}}\mathcal{U},0)}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathcal{H}}} \right)
\end{aligned}$$

for all $\mathcal{U} \in \mathscr{H}$ □

Corollary 4.2. *Assume that* $\mathcal{E}$ *be a additive-quadratic function and* P *is real constant, then*

$$\|\mathscr{H}(\mathcal{U},\mathcal{V})\| \leqslant \begin{cases} \mathcal{E}, & \\ \mathcal{E}\left\{\|\mathcal{U}\|^{\mathrm{P}} + \|\mathcal{V}\|^{\mathrm{P}}\right\}, & \mathrm{P} \neq 1,2; \\ \mathcal{E}\left\{\|\mathcal{U}\|^{\mathrm{P}}\|\mathcal{V}\|^{\mathrm{P}} + \left\{\|\mathcal{U}\|^{2\mathrm{P}} + \|\mathcal{V}\|^{2\mathrm{P}}\right\}\right\}, & \mathrm{P} \neq \frac{1}{2}, 1; \end{cases} \tag{4.8}$$

for all $\mathcal{U},\mathcal{V} \in \mathscr{H}$*. Then there exists a unique additive mapping* $\mathrm{A} : \mathscr{H} \to \mathscr{I}$ *and a quadratic mapping* $\mathrm{Q} : \mathscr{H} \to \mathscr{I}$ *such that*

$$\begin{aligned}
&\|\mathrm{A}(\mathcal{U}) - \mathrm{Q}(\mathcal{U}) - \mathscr{H}(\mathcal{U})\| \\
&\leqslant \begin{cases} \left(\dfrac{\mathcal{E}}{\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) - 1}\right) + \left(\dfrac{\mathcal{E}}{\left(\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2 - 1\right)}\right), \\ \left[\dfrac{1}{|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathrm{P}}|} + \dfrac{1}{3|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2 - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{\mathrm{P}}|}\right] \|\mathcal{U}\|^{\mathrm{P}}, \\ \left[\dfrac{1}{|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right) - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathrm{P}}|} + \dfrac{1}{3|\left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^2 - \left(\tan^{\mathcal{R}}\left(\frac{\pi}{3}\right)\right)^{2\mathrm{P}}|}\right] \|\mathcal{U}\|^{2\mathrm{P}} \end{cases}
\end{aligned} \tag{4.9}$$

for all $\mathcal{U} \in \mathscr{H}$

5. Conclusion

We formulated an extended version of a new mixed-type additive-quadratic functional equation and established its Ulam-Hyers stability under three distinct structural cases: odd, even, and odd-even. By decomposing the given functional equation according to symmetry properties, we demonstrated that the odd case reduces to the stability behavior of additive functional equations, while the even case corresponds to quadratic functional equations. The mixed odd-even case was shown to characterize the stability structure of additive-quadratic (AQ) functional equations, thereby unifying both classical behaviors within a single generalized framework. The obtained results not only generalize several existing stability results in the literature but also provide a systematic approach for analyzing mixed-type functional equations through symmetry decomposition techniques. The methods employed in this work offer a clear mechanism to control perturbations and to derive explicit stability bounds, thereby strengthening the connection between classical Ulam-Hyers stability theory and modern mixed functional equations. The framework developed in this study opens new directions for further research. In particular, the present approach can be extended to investigate generalized Ulam-Hyers-Rassias stability, stability in fuzzy, random, or neutrosophic normed spaces, and higher-order or fractional mixed-type functional equations. These potential extensions may contribute to deeper structural understanding and broader applications of stability theory in nonlinear analysis.

References

[1] Agarwal, R.P., Hristova, S. and O'Regan, D., (2024). Ulam stability for boundary value problems of differential equations-main misunderstandings and how to avoid them. Mathematics, 12(11), p.1626. 1

[2] Agarwal, G., Singh, M.M., Jan, R. and Purohit, S.D., (2025). Nonlinear dynamics and stability analysis of a pandemic model using homotopy perturbation. Critical Reviews in Biomedical Engineering, 53(3). 1

[3] Agilan, P., Julietraja, K., Mlaiki, N. and Mukheimer, A., (2022). Intuitionistic fuzzy stability of an Euler–Lagrange symmetry additive functional equation via direct and fixed point technique. Symmetry, 14, p.2454. 1

[4] Agilan, P., Julietraja, K., Kanimozhi, B. and Alsinai, A., (2024). Hyers stability of AQC functional equation. Dynamics of Continuous, Discrete and Impulsive Systems, Series B, 31, pp.63–75. 1

[5] Agilan, P., Almazah, M.A.A., Julietraja, K. and Alsinai, A., (2023). Classical and fixed point approach to the stability analysis of a bilateral symmetric additive functional equation in fuzzy and random normed spaces. Mathematics, 11, p.681. 1

[6] Agilan, P., Julietraja, K., Almazah, M.A.A. and Alsinai, A., (2023). Stability analysis of a new class of series type additive functional equation in Banach spaces: direct and fixed point techniques. Mathematics, 11, p.887. 1

[7] Agilan, P., Julietraja, K., Aljohani, S. and Mlaiki, N., (2024). Generalised Ulam-Hyers stability analysis for system of additive functional equation in fuzzy and random normed spaces: direct and fixed point approach. International Journal of Analysis and Applications, 22, p.201. 1

[8] Agilan, P., Vijayan, V., Sophia, M. and Ganapathy, G., (2025). Exploring advanced stability of higher-order functional equations in neutrosophic normed spaces via Hyers-Ulam methodologies. Communications on Applied Nonlinear Analysis, 32(7s), pp.806–822. 1

[9] Aoki, T., (1950). On the stability of the linear transformation in Banach spaces. Journal of the Mathematical Society of Japan, 2, pp.64–66. 1

[10] Baias, A.R. and Popa, D., (2020). On Ulam stability of a linear difference equation in Banach spaces. Bulletin of the Malaysian Mathematical Sciences Society, 43(2), pp.1357–1371. 1

[11] Bhatter, S., Bhatia, B., Kumawat, S. and Purohit, S.D., (2025). Modeling and simulation of COVID-19 disease dynamics via Caputo-Fabrizio fractional derivative. Computational Methods for Differential Equations, 13(2). 1

[12] Găvruta, P., (1999). An answer to a question of J.M. Rassias concerning the stability of Cauchy functional equation. Advances in Equations and Inequalities, pp.67–71. 1

[13] Găvruta, P., (2001). On a problem of G. Isac and Th.M. Rassias concerning the stability of mappings. Journal of Mathematical Analysis and Applications, 261, pp.543–553. 1

[14] Hyers, D.H., (1941). On the stability of the linear functional equation. Proceedings of the National Academy of Sciences of the USA, 27, pp.222–224. 1

[15] Kalvandi, V. and Samei, M.E., (2021). Mittag-Leffler-Hyers-Ulam-Rassias stability of cubic functional equation. Mathematics and Computational Sciences, 2(3), pp.14–21. 1

[16] Kumam, P., Ali, A., Shah, K. and Khan, R.A., (2017). Existence results and Hyers–Ulam stability to a class of nonlinear arbitrary order differential equations. Journal of Nonlinear Sciences and Applications, 10(6), pp.2986–2997. 1

[17] Kumawat, S., Bhatter, S., Bhatia, B., Purohit, S.D., Baskonus, H.M. and Suthar, D.L., (2025). Novel application of q-HAGTM to analyze Hilfer fractional differential equations in diabetic dynamics. Mathematics and Computers in Simulation, 238, pp.136–149. 1

[18] Movahednia, E., (2024). Introduce gradual Banach lattices and Hyers stability. Mathematics and Computational Sciences, 5(4), pp.85–97. 1

[19] Panda, S., Agarwal, P. and Sinha, A.K., (2025). On q-double modified Laplace transform. Mathematics and Computational Sciences, 6(2), pp.92–101. 1

[20] Pasupathi, A., Konsalraj, J., Fatima, N., Velusamy, V., Mlaiki, N. and Souayah, N., (2022). Direct and fixed-point stability–instability of additive functional equation in Banach and quasi-beta normed spaces. Symmetry, 14, p.1700. 1

[21] Rassias, J.M., (1982). On approximately linear mappings by linear mappings. Journal of Functional Analysis, 46, pp.126–130. 1

[22] Rassias, J.M., 1984. On approximately linear mappings by linear mappings. Bulletin des Sciences Mathématiques, 108, pp.445–446. 1

[23] Rus, I.A., (2009). Ulam stability of ordinary differential equation. Studia Universitatis Babe-Bolyai Mathematica, 4. 1

[24] Tripathy, A.K., (2021). Hyers-Ulam Stability of Ordinary Differential Equations. Boca Raton: Chapman and Hall/CRC. 1

[25] Ulam, S.M., (1964). Problems in Modern Mathematics. New York: Wiley. 1