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Bounded approximate solutions of additive functional equations in Banach spaces

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Abstract

The objectives of this article is to demonstrate the Stanislaw Marcin Ulam stability in Banach spaces and to offer a new type of additive functional equation. The findings were achieved using a traditional direct and fixed point technique with several general control mapping.

Keywords: Additive Functional Equations, Generalized Hyers - Ulam - Stability, Banach spaces

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1. Introduction

S.M. Ulam [18] was the first to highlight the issue of the stability of functional equation. D. H. Hyers [9] presented a positive response to the issue of Ulam for Banach spaces in 1941. T. Aoki [7] explore this subject for additive maps in 1950. Th.M. Rassias [16] was successful in extending Hyers' Theorem's result by weakening the condition for the Cauchy difference. Taking into account the significant effect of Ulam, Hyers, and Rassias on the development of stability issues of functional equation, the stability phenomena demonstrated by Th.M. Rassias is known as Hyers-Ulam-Rassias stability cited by [11, 12, 13, 14, 15, 17]. In the spirit of Rassias' method, P. Gavruta [8] in 1994 by substituting the unbounded Cauchy difference with a generic control function. Different types of additive functional equation and their Ulam stability are addressed. In recent years, the stability theory of additive functional equations has develo ped in a systematic and progressive manner. Several important contributions have been made toward extending classical UlamHyers stability results to more generalized and complex functional settings.Agilan et.al., [1] studied the generalized UlamHyers stability of complex additive functional equations, thereby extending the classical additive model to complex domains.

$$\begin{aligned}(\lambda^n+1)\,\mathcal{G}\left(e^{\frac{in\pi}{3}}\vartheta+e^{\frac{-in\pi}{3}}\omega\right)+(\lambda^n-1)\,\mathcal{G}\left(e^{\frac{-in\pi}{3}}\vartheta+e^{\frac{in\pi}{3}}\omega\right)&\\ +e^{\frac{-in\pi}{3}}\mathcal{G}(\vartheta-\omega)+e^{\frac{in\pi}{3}}\mathcal{G}\,(\omega-\vartheta)&=2\lambda^n\cos\left(\frac{n\pi}{3}\right)[\mathcal{G}(\vartheta)+\mathcal{G}(\omega)]\end{aligned}$$

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Later, Agilan et.al., [6] examined the Hyers stability of additive functional equations and strengthened the analytical framework of stability theory.

$$\begin{aligned}&\left(e^{2\left(\frac{n\pi}{3}\right)}+6e^{\left(\frac{n\pi}{3}\right)}\right)\mathcal{G}\left(x\sin\left(\frac{n\pi}{3}\right)+y\cos\left(\frac{n\pi}{3}\right)\right)+\left(9-6e^{\left(\frac{n\pi}{3}\right)}\right)\mathcal{G}\left(y\sin\left(\frac{n\pi}{3}\right)+z\cos\left(\frac{n\pi}{3}\right)\right)\\&+6e^{\left(\frac{n\pi}{3}\right)}\mathcal{G}\left(z\sin\left(\frac{n\pi}{3}\right)+x\cos\left(\frac{n\pi}{3}\right)\right)-6e^{\left(\frac{n\pi}{3}\right)}\sin\left(\frac{n\pi}{3}\right)\mathcal{G}(x-y)-6e^{\left(\frac{n\pi}{3}\right)}\cos\left(\frac{n\pi}{3}\right)\mathcal{G}(y-z)\\&=\left(e^{2\left(\frac{n\pi}{3}\right)}\sin\left(\frac{n\pi}{3}\right)+6e^{\left(\frac{n\pi}{3}\right)}\cos\left(\frac{n\pi}{3}\right)\right)\mathcal{G}(x)+\left(9\sin\left(\frac{n\pi}{3}\right)+e^{2\left(\frac{n\pi}{3}\right)}\cos\left(\frac{n\pi}{3}\right)\right)\mathcal{G}(y)\\&+\left(6e^{\left(\frac{n\pi}{3}\right)}\sin\left(\frac{n\pi}{3}\right)+9\cos\left(\frac{n\pi}{3}\right)\right)\mathcal{G}(z)\end{aligned}$$

Further developments were made by Agilan and his collaborators in different normed structures. In [4], both direct and fixed point methods were applied to investigate the stability and instability of additive functional equations in Banach and quasi-beta normed spaces. This work showed that fixed point techniques provide a powerful alternative to classical methods.

$$\begin{aligned}&(\lambda^n+1)\,\mathcal{G}\left(e^{\frac{in\pi}{3}}\vartheta+e^{\frac{-in\pi}{3}}\omega\right)+(\lambda^n-1)\,\mathcal{G}\left(e^{\frac{-in\pi}{3}}\vartheta+e^{\frac{in\pi}{3}}\omega\right)\\&+e^{\frac{-in\pi}{3}}\mathcal{G}(\vartheta-\omega)+e^{\frac{in\pi}{3}}\mathcal{G}\left(\omega-\vartheta\right)=2\lambda^n\cos\left(\frac{n\pi}{3}\right)\left[\mathcal{G}(\vartheta)+\mathcal{G}(\omega)\right]\end{aligned}$$

The research direction was extended to uncertainty frameworks in [5], where intuitionistic fuzzy stability of an EulerLagrange symmetry additive functional equation was established.

$$\zeta\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\zeta\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\zeta(\gamma-\kappa)+\eta^{\hbar}\zeta(\kappa-\mu)=2\left(\eta^{\ell+\wp}\zeta(\gamma)+\eta^{\hbar}\zeta(\kappa)\right)$$

Similarly, in [2], the stability of bilateral symmetric additive functional equations was analyzed in fuzzy and random normed spaces using classical and fixed point approaches.

$$\begin{aligned}&(\gamma^a+1)\,\mathcal{G}\left(\mathcal{U}_1+\gamma^{b+c}\mathcal{U}_2\right)+(1-\gamma^a)\,\mathcal{G}\left(\gamma^{b+c}\mathcal{U}_2+\mathcal{U}_3\right)+\gamma^{b+c}\mathcal{G}(\mathcal{U}_1-\mathcal{U}_2)+\gamma^a\mathcal{G}(\mathcal{U}_2-\mathcal{U}_3)\\&=\left(\gamma^a+\gamma^{b+c}+1\right)\mathcal{G}(\mathcal{U}_1)+\left(\gamma^a+\gamma^{b+c}\right)\mathcal{G}(\mathcal{U}_2)+\left(1-2\gamma^a\right)\mathcal{G}(\mathcal{U}_3)\end{aligned}$$

More recently, in [3], a new class of series-type additive functional equations was introduced, and its stability was studied in Banach spaces through direct and fixed point techniques.

$$¥_1\left(\left(\sum_{\vartheta=1}^{n}(-1)^{\vartheta-1}\vartheta^2\right)\mathfrak{r}\right)=(-1)^{n-1}\frac{n(n+1)}{2}¥_1(\mathfrak{r})$$

$$¥_2\left(\left(\sum_{\vartheta=1}^{n}n(n-1)+2\vartheta-1\right)\mathfrak{r}\right)=n^3¥_2(\mathfrak{r})$$

These works clearly indicate a gradual expansion of stability theory from classical additive equations to more generalized forms involving symmetry, series structures, fuzzy settings, and advanced normed spaces. They also highlight the effectiveness of combining classical Hyers-type methods with modern fixed point techniques to obtain broader and stronger stability results.

Motivated by these developments, the present work aims to further contribute to this growing area by studying a new form of additive-type functional equation and analyzing its stability behavior. Since many generalized functional equations ultimately relate to classical additivity, it is both natural and important to examine how structural modifications influence stability properties. Therefore, investigating the stability of newly formulated additive equations not only enriches the theory of functional equations but also strengthens the ongoing development of UlamHyers stability theory in contemporary analysis.

The goal of this present study is to propose a new type of additive FE and determine its solution, as well as its Ulam stability of this equation

$$f\left(u+\frac{1}{n^2}v\right)+f\left(v+\frac{1}{n^2}w\right)-f\left(w+\frac{1}{n^2}u\right) = \left(1-\frac{1}{n^2}\right)f(u)+\left(1+\frac{1}{n^2}\right)f(v)+\left(\frac{1}{n^2}-1\right)f(w) \tag{1.1}$$

with $n \neq 0, \pm 1$ in Banach spaces.

2. General Solution

In this section, we derived the general solution of the functional equation (1.1).

Lemma 2.1. *Let the odd mapping* $f : X \to Y$ *satisfies the FE*

$$f(u+v) = f(u)+f(v) \tag{2.1}$$

if only if the function $f : X \to Y$ *satisfies (1.1)* $\forall u, v, w \in X$.

Proof. The mapping $f : X \to Y$ satisfy the FE (2.1). Take $u = v = 0$ in (2.1), we reach $f(0) = 0$. Let u by $-v$ in (2.1), we have $f(-v) = -f(v)$ $\forall v \in X$. Considering v by u in (2.1), we get

$$f(2u) = 2f(u) \tag{2.2}$$

for all $u \in X$. In general

$$f(nu) = n\, f(u) \tag{2.3}$$

Taking $u = u, v = \frac{1}{n^2}v$ in (2.1) and using (2.3), we get

$$f\left(u+\frac{1}{n^2}v\right) = f(u)+\frac{1}{n^2}f(v) \tag{2.4}$$

Taking $u = v, v = \frac{1}{n^2}w$ in (2.1) and using (2.3), we get

$$f\left(v+\frac{1}{n^2}w\right) = f(v)+\frac{1}{n^2}f(w) \tag{2.5}$$

$\forall v, w \in X$. Taking $u = w, v = \frac{1}{n^2}u$ in (2.1) and using (2.3), we get

$$f\left(w+\frac{1}{n^2}u\right) = f(w)+\frac{1}{n^2}f(u) \tag{2.6}$$

$\forall u, w \in X$. Adding (2.4), (2.5), and subtracting (2.6) , we reach (1.1). Coversely, Put $v = 0$ in the original equation (1.1) to obtain the scaling property

$$f(u)+f\left(\frac{1}{n^2}w\right)-f\left(w+\frac{1}{n^2}u\right) = \left(1-\frac{1}{n^2}\right)f(u)+\left(\frac{1}{n^2}-1\right)f(w).$$

$\forall u, w \in X$. Simplifying coefficients, we have

$$f\left(w+\frac{1}{n^2}u\right) = \frac{1}{n^2}f(u)+f(w).$$

$\forall u, w \in X$. Interchanging w by u and u by n^2v, Hence we reach

$$f(u+v) = f(u)+f(v)$$

$\forall u, v \in X$. $\square$

Remarks

The classical Cauchy additive functional equation is

$$f(x+y) = f(x) + f(y).$$

This equation is very simple in nature. It involves only two variables and directly describes the idea of additivity in a clear and elegant manner. There are no extra parameters or complicated coefficients in the expression. The structure is also symmetric in x and y, since interchanging these variables does not change the form of the equation. Because of its simplicity and clarity, it is considered the basic and most fundamental additive functional equation.In contrast, the proposed equation (1.1) is structurally more involved. It contains three variables u, v, and w, and also includes the parameter $\frac{1}{n^2}$. The equation is formed by combining three translated function arguments with specific coefficients depending on this parameter. Due to the presence of these coefficients and additional variables, the equation appears more complicated than the classical Cauchy equation. Moreover, it does not possess an obvious symmetry among u, v, and w. Even though it looks more complex, it can still be viewed as an extension of the additive structure, since it ultimately reflects a generalized form of additivity.

3. Stability Results: Direct Method

In this section, we obtained the Ulam-Hyers stability of the functional equation (1.1).

Assume $\mathbb{L}^*-$ normed linear space, $\mathbb{M}^*-$ Banach space. Let

$$\begin{aligned}\mathfrak{B}f(u,v,w) = f\left(u + \frac{1}{n^2}v\right) + f\left(v + \frac{1}{n^2}w\right) - f\left(w + \frac{1}{n^2}u\right)\\ - \left(1 - \frac{1}{n^2}\right)f(u) - \left(1 + \frac{1}{n^2}\right)f(v) - \left(\frac{1}{n^2} - 1\right)f(w)\end{aligned}$$

where $n \neq 0, \pm 1$ $\forall u, v, w \in X$.

Theorem 3.1. *Suppose that the mapping* $f : \mathbb{L}^* \to \mathbb{M}^*$ *satisfying*

$$\|\mathfrak{B}f\,(u,v,w)\| \leqslant \mathfrak{Q}\,(u,v,w) \tag{3.1}$$

$\forall u, v, w \in \mathbb{L}^*$ *and Let* $\mathfrak{Q} : \mathbb{L}^* \times \mathbb{L}^* \times \mathbb{L}^* \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{\mathfrak{Q}\,(\mathcal{A}^n u, \mathcal{A}^n v, \mathcal{A}^n w)}{\mathcal{A}^n} = 0 \tag{3.2}$$

$\forall u, v, w \in \mathbb{L}^*$. *Then, there occurs a distinct additive function* $A : \mathbb{L}^* \to \mathbb{M}^*$ *and satisfying the FE (1.1) such that*

$$\|A(u) - f(u)\| \leqslant \sum_{q=1}^{\infty} \frac{\mathfrak{Q}(\mathcal{A}^{(q-1)}u, \mathcal{A}^{(q-1)}u, \mathcal{A}^{(q-1)}u)}{\mathcal{A}^s} \tag{3.3}$$

$\forall u \in \mathbb{L}^*$. *If* $A(u)$ *is the additive mapping defined as*

$$A(u) = \lim_{n\to\infty} \frac{f(\mathcal{A}^n u)}{\mathcal{A}^n} \tag{3.4}$$

$\forall u \in \mathbb{L}^*$ *where* $\mathcal{A} = \left(\dfrac{n^2+1}{n^2}\right)$.

Proof. Considering (u, v, w) by $(0, 0, 0)$ in (3.1), then we have $\left\|\frac{1}{n^2}f(u)\right\| = 0$ or $f(u) = 0$. Switching $u = v = w = u$ in (3.1), we get

$$\|f(\mathcal{A}u) - \mathcal{A}f(u)\| \leqslant \mathfrak{Q}(u, u, u) \tag{3.5}$$

$\forall u \in \mathbb{L}^*$, where $\mathcal{A} = \left(\frac{n^2+1}{n^2}\right)$. We replace u by $\mathcal{A}^{q-1}u$ (for $q \in \mathbb{N}$ and $q \geqslant 1$) in (3.5) , and we obtain

$$\left\|f(\mathcal{A}^q u) - \mathcal{A}f(\mathcal{A}^{q-1}u)\right\| \leqslant \mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)$$

$\forall u \in \mathbb{L}^*$. Multiplying both sides of the above inequality by $\frac{1}{\mathcal{A}^q}$ and we have the result by adding n inequalities then

$$\sum_{q=1}^{n} \frac{1}{\mathcal{A}^s}\left\|f(\mathcal{A}^q u) - \mathcal{A}f(\mathcal{A}^{q-1}u)\right\| \leqslant \sum_{q=1}^{n} \frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)}{\mathcal{A}^q}$$

Utilizing the triangle inequality

$$|\alpha + \beta| \leqslant |\alpha| + |\beta|$$

and we reach left side of the inequality after simplying

$$\left\|\frac{1}{\mathcal{A}^n}f(\mathcal{A}^n u) - f(u)\right\| \leqslant \sum_{q=1}^{n} \frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)}{\mathcal{A}^q} \tag{3.6}$$

Since

$$\sum_{q=1}^{n} \frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)}{\mathcal{A}^q} \leqslant \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)}{\mathcal{A}^q}$$

the inequality (3.6) yields

$$\left\|\frac{1}{\mathcal{A}^n}f(\mathcal{A}^n u) - f(u)\right\| \leqslant \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)}{\mathcal{A}^q}$$

$\forall u \in \mathbb{L}^*$. By induction it will be demonstrated that (3.6) is exist $\forall$ $\mathbb{N}$. If $m > n > 0$, then $m - n \in \mathbb{N}$ and supplanting n by $m - n$ in (3.6), we have

$$\left\|\frac{1}{\mathcal{A}^{m-n}}f\left(\mathcal{A}^{m-n}u\right) - f(u)\right\| \leqslant \sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)}{\mathcal{A}^q} \tag{3.7}$$

which is

$$\left\|\frac{1}{\mathcal{A}^m}f\left(\mathcal{A}^{m-n}u\right) - \frac{1}{\mathcal{A}^n}f(u)\right\| \leqslant \frac{1}{\mathcal{A}^n}\sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u, \mathcal{A}^{q-1}u\right)}{\mathcal{A}^q} \tag{3.8}$$

$\forall u \in \mathbb{L}^*$. Interchanging u by $\mathcal{A}^n u$ in (3.8), we obtain

$$\left\|\frac{1}{\mathcal{A}^m}f(\mathcal{A}^m u) - \frac{1}{\mathcal{A}^n}f(\mathcal{A}^n u)\right\| \leqslant \frac{1}{\mathcal{A}^n}\sum_{q=1}^{\infty} \frac{\mathfrak{Q}\left(\mathcal{A}^{q+n-1}u, \mathcal{A}^{q+n-1}u, \mathcal{A}^{q+n-1}u\right)}{\mathcal{A}^q} \tag{3.9}$$

Since

$$\lim_{n\to\infty} \frac{1}{\mathcal{A}^n} = 0$$

and hence from (3.9), we obtain

$$\lim_{n\to\infty}\left\|\frac{1}{\mathcal{A}^m}f\left(\mathcal{A}^m x\right)-\frac{1}{\mathcal{A}^n}f(\mathcal{A}^n u)\right\|=0$$

Therefore

$$\left\{\frac{f(\mathcal{A}^n u)}{\mathcal{A}^n}\right\}_{n=1}^{\infty}$$

is Cauchy sequence. Then the sequence has a limit in $\mathbb{L}^*$. We Define

$$A(u)=\lim_{n\to\infty}\frac{f(\mathcal{A}^n u)}{\mathcal{A}^n}$$

$\forall u\in\mathbb{L}^*$. First we showthat $A:\mathbb{L}^*\to\mathbb{L}^*$ is additive.
Consider

$$\begin{aligned}
&\left\|A\left(u+\frac{1}{n^2}v\right)+A\left(v+\frac{1}{n^2}w\right)-A\left(w+\frac{1}{n^2}u\right)\right.\\
&\qquad\left.-\left(1-\frac{1}{n^2}\right)A(u)-\left(1+\frac{1}{n^2}\right)A(v)-\left(\frac{1}{n^2}-1\right)A(w)\right\|\\
=&\frac{1}{\mathcal{A}^n}\left\|A\left(\mathcal{A}u+\frac{\mathcal{A}}{n^2}v\right)+A\left(\mathcal{A}v+\frac{\mathcal{A}}{n^2}w\right)-A\left(\mathcal{A}w+\frac{\mathcal{A}}{n^2}u\right)\right.\\
&\qquad\left.-\left(1-\frac{1}{n^2}\right)A(\mathcal{A}u)-\left(1+\frac{1}{n^2}\right)A(\mathcal{A}v)-\left(\frac{1}{n^2}-1\right)A(\mathcal{A}w)\right\|\\
\leqslant&\lim_{n\to\infty}\frac{1}{\mathcal{A}^n}\mathfrak{Q}\left(\mathcal{A}^n u,\mathcal{A}^n v,\mathcal{A}^n w\right)=0
\end{aligned}$$

Hence

$$\begin{aligned}
&A\left(u+\frac{1}{n^2}v\right)+A\left(v+\frac{1}{n^2}w\right)-A\left(w+\frac{1}{n^2}u\right)\\
&\qquad=\left(1-\frac{1}{n^2}\right)A(u)+\left(1+\frac{1}{n^2}\right)A(v)+\left(\frac{1}{n^2}-1\right)A(w)
\end{aligned}$$

$\forall u\in\mathbb{L}^*$. Next, we consider

$$\begin{aligned}
||A(u)-f(u)||&=||\lim_{n\to\infty}\frac{f(\mathcal{A}^n u)}{\mathcal{A}^n}-f(u)||\\
&=\lim_{n\to\infty}||\frac{f(\mathcal{A}^n u)}{\mathcal{A}^n}-f(u)||\\
&\leqslant\lim_{n\to\infty}\sum_{q=1}^{\infty}\frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u\right)}{\mathcal{A}^q}
\end{aligned}$$

Hence, we get

$$||A(u)-f(u)||\leqslant\sum_{q=1}^{\infty}\frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u\right)}{\mathcal{A}^q}$$

for all $u\in\mathbb{L}^*$.
Next, we show A is unique. Then there occur another mapping $B:\mathbb{L}^*\to\mathbb{L}^*$ we have

$$||B(u)-f(u)||\leqslant\sum_{q=1}^{\infty}\frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u\right)}{\mathcal{A}^q}$$

Hence

$$\begin{aligned}\|B(u)-A(u)\| &\leqslant \|B(u)-f(u)\|+\|A(u)-f(u)\| \\ &\leqslant \sum_{q=1}^{n}\frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u\right)}{\mathcal{A}^{q}}+\sum_{q=1}^{\infty}\frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u\right)}{\mathcal{A}^{q}} \\ &= 2\sum_{q=1}^{\infty}\frac{\mathfrak{Q}\left(\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u,\mathcal{A}^{q-1}u\right)}{\mathcal{A}^{q}}\end{aligned}$$

Since the additive mappings are A and B then we see

$$\begin{aligned}\|A(u)-B(u)\| &= \frac{2}{\mathcal{A}^{n}}\|A(\mathcal{A}^{n}u)-B(\mathcal{A}^{n}u)\| \\ &\leqslant \frac{2}{\mathcal{A}^{n}}\sum_{q=1}^{\infty}\frac{\mathfrak{Q}\left(\mathcal{A}^{q+n-1}u,\mathcal{A}^{q+n-1}u,\mathcal{A}^{q+n-1}u\right)}{\mathcal{A}^{q}}\end{aligned} \tag{3.10}$$

Hence taking the limit as $n\to\infty$ we get from (3.10)

$$\lim_{n\to\infty}\|A(u)-B(u)\| \leqslant 2\lim_{n\to\infty}\frac{1}{\mathcal{A}^{n}}\sum_{q=1}^{\infty}\frac{\mathfrak{Q}\left(\mathcal{A}^{q+n-1}u,\mathcal{A}^{q+n-1}u,\mathcal{A}^{q+n-1}u\right)}{\mathcal{A}^{q}}$$

Hence

$$\|A(u)-B(u)\| \leqslant 0$$

Therefore $A(u)=B(u)$ $\forall\, u\in\mathbb{L}^{*}$. Hence A is unique. □

Corollary 3.2. *Let* $\mathfrak{U}$ *and* p *be nonnegative real numbers. Let a function* $f:\mathbb{L}^{*}\to\mathbb{L}^{*}$ *fulfills the inequality*

$$\|\mathfrak{B}f(u,v,w)\| \leqslant \begin{cases}\mathfrak{U}, & \\ \mathfrak{U}\{\|u\|^{p}+\|v\|^{p}+\|w\|^{p}\}, & p\neq 1; \\ \mathfrak{U}\|u\|^{p}\|v\|^{p}\|w\|^{p}, & 3p\neq 1; \\ \mathfrak{U}\left\{\|u\|^{p}\|v\|^{p}\|w\|^{p}+\left\{\|u\|^{3p}+\|v\|^{3p}+\|w\|^{3p}\right\}\right\}, & 3p\neq 1;\end{cases} \tag{3.11}$$

$\forall\, u,v,w\in\mathbb{L}^{*}$. *Then the mapping* $A:\mathbb{L}^{*}\to\mathbb{L}^{*}$ *and we see*

$$\|f(u)-A(u)\| \leqslant \begin{cases}\dfrac{\mathfrak{U}}{|\mathcal{A}-1|}, \\ \dfrac{3\,\mathfrak{U}\|u\|^{p}}{|\mathcal{A}-\mathcal{A}^{p}|}, \\ \dfrac{\mathfrak{U}\|u\|^{3p}}{|\mathcal{A}-\mathcal{A}^{3p}|}, \\ \dfrac{4\mathfrak{U}\|u\|^{3p}}{|\mathcal{A}-\mathcal{A}^{3p}|}\end{cases} \tag{3.12}$$

$\forall\, u \in \mathbb{L}^{*}$.

4. Stability Results:Fixed Point Method

In this section, the authors have proved the generalized Ulam - Hyers stability of functional equation (1.1) in Banach spaces with the help of the fixed point method.

Theorem 4.1. *[10](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
(B_1) $$d(T^n x, T^{n+1} x) = \infty \quad \forall \ n \geqslant 0,$$
or
(B_2) *there exists a natural number n_0 such that:*
(i) $d(T^n x, T^{n+1} x) < \infty$ *for all $n \geqslant n_0$;*
(ii)*The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
(iii) *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0} x, y) < \infty\}$;*
(iv) $d(y^*, y) \leqslant \frac{1}{1-L}\ d(y, Ty)$ *for all $y \in Y$.*

Theorem 4.2. *Let $f : \mathbb{L}^* \to \mathbb{M}^*$ be a mapping for which there exists functions $\mathfrak{Q}, \mathfrak{Y} : \mathbb{L}^{*3} \to [0, \infty)$ with the condition*

$$\lim_{\xi \to \infty} \frac{\mathfrak{Q}(\chi_i^\xi u, \chi_i^\xi v,\ \chi_i^\xi w)}{\chi_i^\xi} = 0, \tag{4.1}$$

where

$$\chi_i = \begin{cases} \mathcal{A}, & i = 0, \\ \frac{1}{\mathcal{A}}, & i = 1 \end{cases}$$

satisfying the functional inequality

$$\|\mathfrak{B} f\,(u, v, w)\| \leqslant \mathfrak{Q}\,(u, v, w) \tag{4.2}$$

for all $u, v, w \in \mathbb{L}^*$. If there exists an $L = L(i) < 1$ such that the function

$$u \to \mathfrak{Y}(u) = \frac{1}{\mathcal{A}} \mathfrak{Q}\left(\frac{u}{\mathcal{A}}, \frac{u}{\mathcal{A}}, \frac{u}{\mathcal{A}}\right),$$

one has the property

$$\mathfrak{Y}(u) = L\, \chi_i\, \mathfrak{Y}\left(\frac{u}{\chi_i}\right) \tag{4.3}$$

for all $u \in \mathbb{L}^*$. Then there exists a unique additive function $\mathcal{A} : \mathbb{L}^* \to \mathbb{M}^*$ satisfying the functional equation (1.1) and

$$\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L} \mathfrak{Y}(u) \tag{4.4}$$

holds for all $u \in \mathbb{L}^*$.

Remark 4.3. Using fixed point technique, we prove the stability results for Corollary (3.2) in the following way.

Proof. Let us set

$$\mathfrak{Q}(u, v, w) = \begin{cases} \mathfrak{U}, \\ \mathfrak{U}\{||u||^p + ||v||^p + ||w||^p\}, \\ \mathfrak{U}\ \ ||u||^p\ ||v||^p\ ||w||^p, \\ \mathfrak{U}\left\{||u||^p ||v||^p ||w||^p + (||u||^{3p} + ||v||^{3p} + ||w||^{3p})\right\} \end{cases}$$

for all $u, v, w \in \mathbb{L}^*$. Now

$$\frac{\mathfrak{Q}(\chi_i^{\xi}u, \chi_i^{\xi}v, \chi_i^{\xi}w)}{\chi_i^{\xi}} = \begin{cases} \frac{\mathfrak{U}}{\chi_i^{\xi}}, \\ \frac{\mathfrak{U}}{\chi_i^{\xi}}\left\{||\chi_i^{\xi}u||^p + ||\chi_i^{\xi}v||^p + ||\chi_i^{\xi}w||^p\right\}, \\ \frac{\mathfrak{U}}{\chi_i^{\xi}}||\chi_i^{\xi}u||^p\ ||\chi_i^{\xi}v||^p\ ||\chi_i^{\xi}w||^p, \\ \frac{\mathfrak{U}}{\chi_i^{\xi}}\left\{||\chi_i^{\xi}u||^p\ ||\chi_i^{\xi}v||^p\ ||\chi_i^{\xi}w||^p \left\{||\chi_i^{\xi}u||^{3p} + ||\chi_i^{\xi}v||^{3p} + ||\chi_i^{\xi}w||^{3p}\right\}\right\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } \xi \to \infty, \\ \to 0 \text{ as } \xi \to \infty, \\ \to 0 \text{ as } \xi \to \infty, \\ \to 0 \text{ as } \xi \to \infty. \end{cases}$$

i.e., (4.1) is holds. But, we have

$$\mathfrak{Y}(u) = \frac{1}{\mathcal{A}}\left[\mathfrak{Q}\left(\frac{u}{\mathcal{A}}, \frac{u}{\mathcal{A}}, \frac{u}{\mathcal{A}}\right)\right].$$

Hence

$$\mathfrak{Y}(u) = \frac{1}{\mathcal{A}}\left[\mathfrak{Q}\left(\frac{u}{\mathcal{A}}, \frac{u}{\mathcal{A}}, \frac{u}{\mathcal{A}}\right)\right] = \begin{cases} \frac{\mathfrak{U}}{\mathcal{A}}, \\ \frac{3\mathfrak{U}}{\mathcal{A}^p}||u||^p, \\ \frac{\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}, \\ \frac{4\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}. \end{cases}$$

Also,

$$\frac{1}{\chi_i}\mathfrak{Y}(\chi_i u) = \begin{cases} \frac{\mathfrak{U}}{\chi_i \cdot \mathcal{A}}, \\ \frac{3\mathfrak{U}}{\chi_i \cdot \mathcal{A}^p}||\chi_i u||^p, \\ \frac{\mathfrak{U}}{\chi_i \cdot \mathcal{A}^{3p}}||\chi_i u||^{3p}, \\ \frac{\mathfrak{U}}{\chi_i \cdot \mathcal{A}^{3p}}||\chi_i u||^{3p}. \end{cases}$$

$$= \begin{cases} \chi_i^{-1}\frac{\mathfrak{U}}{-}, \\ \chi_i^{p-1}\frac{3\mathfrak{U}}{\mathcal{A}^p}||u||^p, \\ \chi_i^{3p-1}\frac{\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}, \\ \chi_i^{3p-1}\frac{4\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}. \end{cases}$$

$$= \begin{cases} \chi_i^{-1}\mathfrak{Y}(u), \\ \chi_i^{p-1}\mathfrak{Y}(u), \\ \chi_i^{3p-1}\mathfrak{Y}(u), \\ \chi_i^{3p-1}\mathfrak{Y}(u). \end{cases}$$

we prove the following cases for conditions using (4.4)

Case:1 $L = \mathcal{A}^{-1}$ if $i = 0$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{\left(\mathcal{A}^{-1}\right)^{1-0}}{1-(\mathcal{A})^{-1}} \cdot \frac{\mathfrak{U}}{\mathcal{A}}\\ &= \frac{\mathfrak{U}}{(\mathcal{A}-1)}.\end{aligned}$$

Case:2 $L = \mathcal{A}$ if $i = 1$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{(\mathcal{A})^{1-1}}{1-\mathcal{A}} \cdot \frac{\mathfrak{U}}{\mathcal{A}}\\ &= \frac{\mathfrak{U}}{(1-\mathcal{A})}.\end{aligned}$$

Case:1 $L = \mathcal{A}^{p-1}$ if $i = 0$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{\left(\mathcal{A}^{p-1}\right)^{1-0}}{1-\mathcal{A}^{p-1}} \frac{3\mathfrak{U}}{\mathcal{A}^{p}}||u||^{p}\\ &= \frac{\mathcal{A}^{r}}{\mathcal{A}-\mathcal{A}^{p}} \frac{3\mathfrak{U}}{\mathcal{A}^{p}}||u||^{p}\\ &= \frac{3\mathfrak{U}||u||^{p}}{(\mathcal{A}-\mathcal{A}^{p})}.\end{aligned}$$

Case:2 $L = \frac{1}{\mathcal{A}^{p-1}}$ if $i = 1$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{\left(\frac{1}{\mathcal{A}^{p-1}}\right)^{1-1}}{1-\frac{1}{\mathcal{A}^{p-1}}} \frac{3\mathfrak{U}}{\mathcal{A}^{p}}||u||^{p}\\ &= \frac{\mathcal{A}^{p}}{\mathcal{A}^{p}-\mathcal{A}} \frac{3\mathfrak{U}}{\mathcal{A}^{p}}||u||^{p}\\ &= \frac{3\mathfrak{U}||u||^{p}}{(\mathcal{A}^{p}-\mathcal{A})}.\end{aligned}$$

Case:1 $L = \mathcal{A}^{3p-1}$ if $i = 0$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{\left(\mathcal{A}^{3p-1}\right)^{1-0}}{1-\mathcal{A}^{3p-1}} \frac{\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{\mathcal{A}^{3p}}{\mathcal{A}-\mathcal{A}^{3p}} \frac{\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{\mathfrak{U}||u||^{3p}}{(\mathcal{A}-\mathcal{A}^{3p})}.\end{aligned}$$

Case:2 $L = \frac{1}{\mathcal{A}^{3p-1}}$ if $i = 1$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{\left(\frac{1}{\mathcal{A}^{3p-1}}\right)^{1-1}}{1-\frac{1}{\mathcal{A}^{3s-1}}} \frac{\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{\mathcal{A}^{3p}}{\mathcal{A}^{3p}-\mathcal{A}} \frac{\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{\mathfrak{U}||u||^{3p}}{(\mathcal{A}^{3p}-\mathcal{A})}.\end{aligned}$$

Case:1 $L = \mathcal{A}^{3p-1}$ if $i = 0$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{\left(\mathcal{A}^{3s-1}\right)^{1-0}}{1-\mathcal{A}^{3p-1}}\frac{4\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{\mathcal{A}^{3s}}{\mathcal{A}-\mathcal{A}^{3p}}\frac{4\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{4\mathfrak{U}||u||^{3p}}{(\mathcal{A}-\mathcal{A}^{3p})}.\end{aligned}$$

Case:2 $L = \frac{1}{\mathcal{A}^{3p-1}}$ if $i = 1$

$$\begin{aligned}\|f(u) - A(u)\| \leqslant \frac{L^{1-i}}{1-L}\mathfrak{Y}(u) &= \frac{\left(\frac{1}{\mathcal{A}^{3p-1}}\right)^{1-1}}{1-\frac{1}{\mathcal{A}^{3p-1}}}\frac{4\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{\mathcal{A}^{3p}}{\mathcal{A}^{3p}-\mathcal{A}}\frac{4\mathfrak{U}}{\mathcal{A}^{3p}}||u||^{3p}\\ &= \frac{4\mathfrak{U}||u||^{3p}}{(\mathcal{A}^{3p}-\mathcal{A})}.\end{aligned}$$

Hence the proof of the corollary. □

5. Conclusion

In conclusion, this article establishes the Stanisaw-Marcin Ulam stability of proposed additive functional equation in Banach spaces. By applying both the classical direct approach and fixed point methods, the study successfully proves the stability results under various general control mappings. The results not only extend existing work in the theory of functional equations but also provide a broader framework for analyzing stability problems in Banach spaces. These findings may serve as a foundation for further investigations into generalized functional equations and their applications in mathematical analysis.

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