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Hull numbers of Bicyclic and Tricyclic graphs

Kobra Mirzaei[a], Zahra Yarahmadi [a,*]

[a] Department of Mathematics, Khorramabad Branch, Islamic Azad University, Khorramabad, Iran.

Abstract

For two vertices u and v of a connected graph G, the set $I[u,v]$ consists of all vertices lying on some u-v geodesic in G. Let S be a subset of the vertex set $V(G)$, and denote by $I[S]$ the union of all sets $I[u,v]$ with $u,v \in S$. The set S is called *convex* if $I[S] = S$. The *convex hull* $[S]$ of S is defined as the smallest convex set in G containing S. The *hull number* $h(G)$ of a graph G is the cardinality of a smallest subset $S \subseteq V(G)$ such that $[S] = V(G)$. In this paper, we characterize the hull numbers of bicyclic and tricyclic graphs and provide a classification of these values.

Keywords: Hull number, Bicyclic graphs, Tricyclic graphs, Extreme vertex.

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1. Introduction

Assume $G(V,E)$ is a finite simple connected graph with no loops or multiple edges. The length of the shortest path between two vertices u and v in G is called the distance between them in G, denoted by $d(u,v)$. Clearly, the distance on the vertex set $V(G)$ is a metric space. For two vertices u and v of a connected graph G, a path between u and v with length $d(u,v)$ is called a u-v geodetic. The set $I[u,v]$ includes all vertices lying on some u-v geodesic in G. Suppose S is a subset of the vertices of G. Then the union of all sets $I[u,v]$, for $u,v \in S$, is denoted by $I[S]$, in fact;

$$I[S] = \bigcup_{u,v \in S} I[u,v]. \tag{1.1}$$

The set S is convex if $I[S] = S$. The convex hull of S is the smallest convex set that contains S. The convex hull $[S]$ can be represented as a sequence $\{I^k[S]\}$ for $k \geqslant 0$ such that

$$I^0[S] = S, \quad I^1[S] = I[S], \quad \ldots, \quad I^k[S] = I[I^{k-1}[S]] \quad \text{for } k \geqslant 2. \tag{1.2}$$

This sequence eventually becomes constant. Let p be the smallest natural number such that

$$I^p[S] = I^{p+1}[S]. \tag{1.3}$$

*Corresponding author
Email addresses: kobra.mirzaei@iau.ir (Kobra Mirzaei), zahra.yarahmadi@iau.ac.ir, z.yarahmadi@gmail.com (Zahra Yarahmadi)

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Then $I^p[S]$ is the convex hull of S. A set S of vertices in G is a convex hull of G if $[S] = V(G)$. We denote the set of all convex hulls of graph G by $H(G)$, hence

$$H(G) = \{S \subseteq V(G) \mid [S] = V(G)\}. \tag{1.4}$$

The smallest set in $H(G)$, called the minimum convex hull of G, has cardinality equal to the hull number of the graph G, denoted by $h(G)$.
A vertex x is an *extreme vertex* of G if the induced subgraph of its neighborhood is complete, or equivalently, if $V(G) - \{x\}$ is convex in G.

It is clear that vertices of degree one are extreme vertices. Vertices with degree one are called *end* or *terminal* vertices. A *unicyclic graph* is a graph with exactly one cycle. A simple connected graph G is termed *bicyclic* if the number of edges is at least one more than the number of vertices of G. An *induced cycle* is a cycle that does not contain another cycle as a subgraph. A *bicyclic graph* and a *tricyclic graph* refer to graphs that contain exactly two or three induced cycles, respectively. The hull number is an important parameter in graph theory, introduced by Evert and Seidman [8] and has been extensively studied in [3, 4, 5]. This concept has many applications in facility location theory and convexity theory, as well as in road design and communication network design problems. For basic graph theoretic terminology, we refer to [6]. We also refer to [1] for results on distance in graphs. In this paper, we investigate the hull number of bicyclic and tricyclic graphs and classify these graphs based on their hull numbers. If S is a hull set of the connected graph G, and $u, v \in S$, then any vertex lying on the u-v geodesic belongs to I[S]. It means that, if $u, v \in S$, then $I[u, v] \subseteq I[S]$.

Lemma 1.1 ([3, 7]). *If S is a hull set of the connected graph G ($S \in H(G)$), and if $u, v \in S$, then any vertex $w \notin \{u, v\}$ lying on the u-v geodesic of G satisfies $w \notin S$.*

Lemma 1.2 ([3, 7]). *Every end vertex of a connected graph G belongs to some hull set of G. In fact, if S is the set of all end vertices of a hull set of the graph G, then S is the unique hull set of G.*

In [2] the hull number of certain families of graphs are obtained, see table 1. Assume $n, m \geqslant 2$ for the complete bipartite graph $K_{n,m}$.

Table 1: The hull number of certain families of graphs G

G	P_n	C_p	C_q	T_n	K_n	$K_{n,m}$
$h(G)$	2	2	3	k	n	2

Lemma 1.3 ([1, 8]). *If G is a connected graph of order n with k extreme vertices and diameter d, then the hull number h satisfies in the following inequality:*

$$k \leqslant h \leqslant n - d + 1. \tag{1.5}$$

2. Hull number of unicyclic and bicyclic graphs

In this section, we investigate the hull numbers of unicyclic and bicyclic graphs by applying the concepts of convexity and geodesics introduced earlier. By analyzing the structural properties of these graphs, we extend the preliminary results presented in the introduction and obtain a deeper understanding of how their structure influences their hull numbers. This study leads to new classifications and provides further insight into these classes of graphs.

Theorem 2.1. *Let G be a unicyclic graph with k end vertices. Then $h(G) \in \{k, k+1, k+2\}$.*

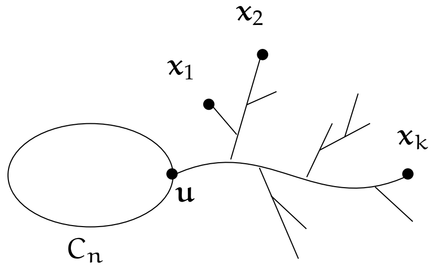

Figure 1: Unicyclic graph formed by joining a cycle and a tree.

Proof. Let G be a unicyclic graph with k end vertices, obtained by attaching a cycle C_n and t trees ($1 \leqslant t \leqslant n$) at the connection vertices $u_1, u_2, \dots, u_t$. Firstly, assume that $t = 1$ and G is a unicyclic graph formed by connecting a tree and a cycle C_n, where the tree contains k end vertices $x_1, x_2, \dots, x_k$, and u is common vertex joining of the tree and the cycle (see Figure 1). We consider the following two cases:

Case 1: If n is even, then there exists a vertex $v \in V(C_n)$ such that $S_1 = \{u, v\}$ is a minimum hull set of C_n. By Lemma 1.1, the set

$$S_2 = \{v, x_1, x_2, \dots, x_k\}$$

is a minimum hull set of G. Therefore, $h(G) = k + 1$.

Case 2: If n is odd, then there exist vertices $v, w \in V(C_n)$ such that $S_1 = \{u, v, w\}$ is a minimum hull set of C_n. By Lemma 1.1, the set

$$S_2 = \{v, w, x_1, x_2, \dots, x_k\}$$

is a minimum hull set of G. Therefore, $h(G) = k + 2$.

Now assume that $t > 1$. Let G be a unicyclic graph containing k end vertices $x_1, x_2, \dots, x_k$, where t trees are attached to the cycle C_n at the connection vertices $S_1 = \{u_1, u_2, \dots, u_t\}$. We consider two cases:

Case 1: If S_1 is a hull set of C_n, then by definition, $I[S_1] = V(C_n)$. Without loss of generality, suppose that for $1 \leqslant i \neq j \leqslant k$, the distinct end vertices x_i and x_j belong to two different trees attached to the cycle C_n at the vertices $u_{i'}$ and $u_{j'}$, respectively. It follows that $u_{i'}$ and $u_{j'}$ lie on every x_i–x_j geodesic. By Lemma 1.1, the set

$$S_2 = \{x_1, x_2, \dots, x_k\}$$

is a minimum hull set of G, and therefore $h(G) = k$.

Case 2: If S_1 is not a hull set of C_n, then there exists a vertex $u \in V(C_n)$ such that $S_1 \cup \{u\}$ forms a hull set of C_n. By a similar argument, the set $S_2 = \{x_1, x_2, \dots, x_k, u\}$ is a minimum hull set of G. Thus, $h(G) = k + 1$. □

Next, we investigate the hull numbers of bicyclic graphs with no end vertices. We begin with bicyclic graphs containing disjoint cycles, obtained by connecting two disjoint cycles C_m and C_n by a path P_t with joining vertices u and v. Such a graph is denoted by $C_m \sim_t C_n(u, v)$.

In the special case $t = 1$, the two cycles share a common vertex, say u, and we denote the graph by $C_m \sim_1 C_n(u)$. For simplicity, when the choice of the common vertex is unimportant, we write this graph simply as $C_m \sim_1 C_n$.

Similarly, when the two cycles are connected by a single edge ($t = 2$), we denote the graph by $C_m \sim_2 C_n$. More generally, if the two cycles are connected by a path P_t, we denote the graph by $C_m \sim_t C_n$.

Theorem 2.2. *Let* $G = C_m \sim_t C_n$. *Then the hull number of* G *satisfies;* $h(G) \in \{2, 3, 4\}$.

Proof. We only consider the case of $t = 1$, while considering different cases for whether m and n are even or odd. The case of $t > 1$ can be analyzed in a similar way. Suppose $t = 1$ and two cycles C_m and C_n are connected through a single common vertex u and denoted by $C_m \sim_1 C_n(u)$ or $C_m \sim_1 C_n$. According to the parity of m and n, three cases arise:

Case 1: If both m and n are even, then there exist vertices $v \in V(C_m)$ and $w \in V(C_n)$ such that $\{u, v\}$ and

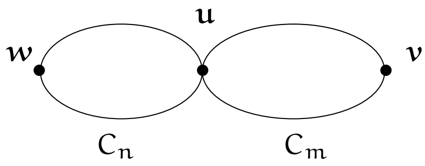

Figure 2: A bicyclic graph with two disjoint even cycles connected by a single vertex u.

$\{u, w\}$ are minimum hull sets of C_m and C_n, respectively (see Figure 2). By Lemma 1.1, it follows that $\{v, w\} \in H(G)$ is the smallest hull set of G and hence, $h(G) = 2$.

Case 2: Let m and n be odd, there exist vertices $x, y \in V(C_m)$ and $v, w \in V(C_n)$ such that $\{u, x, y\}$ and $\{u, v, w\}$ are minimum hull sets of C_m and C_n, respectively (see Figure 3). In a similar way one can see that $\{v, w, x, y\} \in H(G)$ is the minimum hull set of G. Therefore $h(G) = 4$.

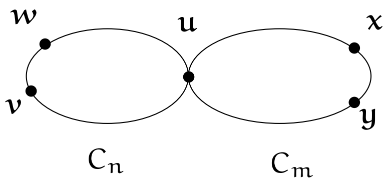

Figure 3: A bicyclic graph with two disjoint odd cycles connected by a single vertex u.

Case 3: Suppose that one of the cycles is even and the other one is odd (or vice versa). Without loss of generality, suppose m is odd and n is even. Then there exist vertices $x, y \in V(C_m)$ and $v \in V(C_n)$ such that $\{u, x, y\}$ and $\{u, v\}$ are minimum hull sets of C_m and C_n, respectively. By Lemma 1.1, $\{v, x, y\} \in H(G)$ forms a minimum hull set of the bicyclic graph $C_m \sim_1 C_n$, so $h(G) = 3$. □

In analyzing hull numbers, we find important relationships between different types of graphs. Next, we now consider bicyclic graphs consisting of two cycles C_m and C_n with k end vertices. The following theorem shows that the hull number of such graphs depends on the structure and parity of the cycles as well as the arrangement of the end vertices. n particular, although the hull number remains within a bounded range, its exact value is determined by the specific way in which the cycles are connected and how the end vertices are attached.

Theorem 2.3. *Let* G *be a bicyclic graph with disjoint cycles* $C_m \sim_t C_n$ *and* k *end vertices, then:*

$$h(G) = k,\ k+1,\ k+2,\ k+3 \text{ or } k+4. \tag{2.1}$$

Proof. Suppose that G is a bicyclic graph contained disjoint cycles C_m and C_n which constructed by connecting a path P_t with joining vertices u and v, in other words G contained $C_m \sim_t C_n(u, v)$. For the positions of the end vertices, there are three cases.

Case 1: Assume that $P_t : u = a_1, a_2, \dots, a_t = v$. If k end vertices are connected to the vertices of the path P_t by k' trees $(k' \leqslant k, t)$. it is evident that, according to Lemma 1.1, Theorem 2.2 and the fact that the end vertices belong to the hull set, we conclude that the hull number will be one of the values $k+2$, $k+3$ or $k+4$.

Case 2: If a tree with k end vertices $z_1, z_2, \dots, z_k$ is connected to a vertex $w \neq u, v$ of the cycles C_m or C_n. Without loss of generality, assume that this tree is connected to vertex w as a vertex of C_m. If C_m and C_n are even cycles, then there exists $w' \in V(C_m)$ (may be $w' = u$) and $v' \in V(C_n)$ such that $\{w, w'\}$ and $\{v, v'\}$ are minimum hull sets of C_m or C_n, respectively. By Lemma 1.1 and Theorem 2.2, one can see that, if $w' \neq u$ then $\{v', w'z_1, z_2, \dots, z_k\} \in H(G)$ is minimum hull set of G and then $h(G) = k+2$ and for $w' = u$, we have $\{v', z_1, z_2, \dots, z_k\} \in H(G)$ is minimum hull set of G and then $h(G) = k+1$, see Figure 4. By similar argument, when one of the cycles are odd and another one are even, one can see that $h(G) = k+2$ and when C_m and C_n are odd cycles $h(G) = k+3$.

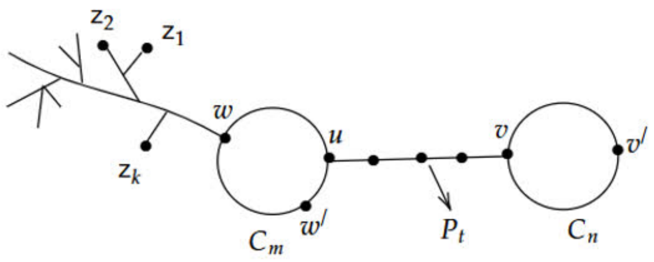

Figure 4: A bicyclic graph $C_m \sim_t C_n$ with a tree attached to one of its cycles, excluding the connecting path.

Case 3: If G is a bicyclic graph with disjoint cycles $C_m \sim_t C_n$, and k end vertices which are located in t trees and these trees are connected to some vertices of the cycles C_m and C_n except at the vertices u and v. Assume that t_1 trees are connected to the vertices $x_1, x_2, x_3, \dots, x_{t_1} \in V(C_m)$ and t_2 trees are connected to the vertices $y_1, y_2, y_3, \dots, y_{t_2} \in V(C_n)$, $t_1 + t_2 = t$ (see Figure 5). Now we consider four different situations:

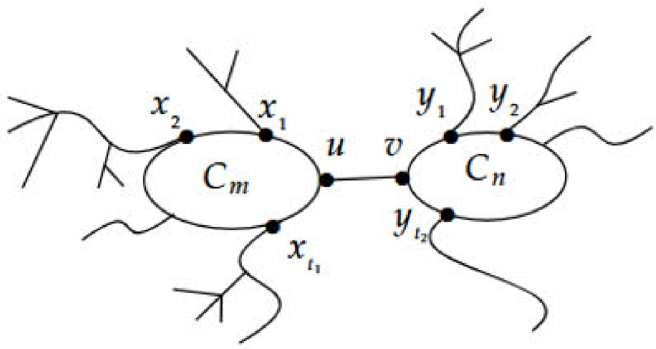

Figure 5: A bicyclic graph with disjoint cycles and multiple trees attached to each cycle, excluding the connecting path.

- Suppose that $\{x_1, x_2, x_3, \dots, x_{t_1}, u\} \in H(C_m)$ and $\{y_1, y_2, y_3, \dots, y_{t_2}, v\} \in H(C_n)$. In this case, according to Lemma 1.1, the set $S = \{z_1, z_2, z_3, \dots, z_k\} \in H(G)$, and since all vertices are extreme, it follows that the hull set of G is unique, hence $h(G) = k$.

- Let $\{x_1, x_2, x_3, \dots, x_{t_1}, u\} \notin H(C_m)$ and $\{y_1, y_2, y_3, \dots, y_{t_2}, v\} \notin H(C_n)$. Then there exist vertices a, b such that $\{x_1, x_2, x_3, \dots, x_t, u, a\} \in H(C_m)$ and $\{y_1, y_2, y_3, \dots, y_{t_2}, v, b\} \in H(C_n)$. In this case, according to lemma 1.1 and Theorem 2.2, the set $S = \{z_1, z_2, z_3, \dots, z_k, a, b\} \in H(G)$ will be the unique hull set of G, hence $h(G) = k + 2$.

- Suppose that $\{x_1, x_2, x_3, \dots, x_{t_1}, u\} \in H(C_m)$ and $\{y_1, y_2, y_3, \dots, y_{t_2}, v\} \notin H(C_n)$. Then there exists a vertex a such that $\{y_1, y_2, y_3, \dots, y_{t_2}, v, a\} \in H(C_n)$. In this case, according to lemma 1.1 and Theorem 2.2, G has just one hull set $S = \{z_1, z_2, z_3, \dots, z_k, a\}$, hence $h(G) = k + 1$.

- If $\{x_1, x_2, x_3, \dots, x_{t_1}, u\} \notin H(C_m)$ and $\{y_1, y_2, y_3, \dots, y_{t_2}, v\} \in H(C_n)$, by similar argument, its easy to see that $h(G) = k + 1$. Thus the proof is completed.

□

Next, we consider the hull numbers of *induced bicyclic graphs*, i.e., graphs in which two cycles share a common path.

Notation: Let G be a bicyclic graph with induced cycles C_m and C_n with a common path P_t, without end vertices. we denote this graph as $G(C_m, C_n; P_t)$,

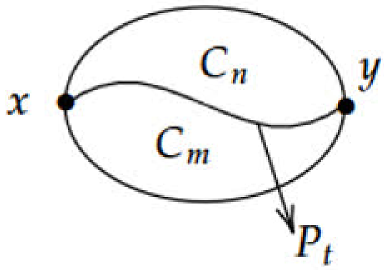

Figure 6: The graph $G(C_n, C_m; P_t)$.

Lemma 2.4. *Let* $G = G(C_m, C_n; P_t)$*, then* $h(G) = 2$ *or* 3.

Proof. According to the structure of G, there are three paths, such as P_{m-t+2}, P_{n-t+2} and P_t in $G = G(C_m, C_n; P_t)$. Let x and y be common vertices in these paths. Depending on which paths are x-y geodesic, we analyze two cases:

Case 1: If the x-y geodesic is P_t, then the cycle formed by the two remaining paths P_{m-t+2} and P_{n-t+2} has length $m+n-2t+2$.

- If $m+n-2t+2$ is even, there exist vertices $u, v \in V(C_{m+n-2t+2})$ such that $S = \{u, v\}$ is the minimum hull set of $G(C_m, C_n; P_t)$, because $I[S] = V(C_{m+n-2t+2})$ and $I^2[S] = V(G(C_m, C_n; P_t))$. Hence, $h(G) = 2$.

- If $m+n-2t+2$ is odd, there exist vertices $u, v, w \in V(C_{m+n-2t+2})$ such that $S = \{u, v, w\}$ is the minimum hull set of $G(C_m, C_n; P_t)$, which also covers the vertices of P_t, i.e., $I^2[S] = V(G(C_m, C_n; P_t))$ and then $h(G) = 3$.

Case 2: If the x-y geodesic is P_{m-t+2} or P_{n-t+2}. Without compromising the overall proof, suppose that the x-y geodesic is P_{m-t+2}. Hence the cycle formed by the other two paths P_{n-t+2} and P_t is C_n. The hull number depends on the parity of n:

$$h(G) = \begin{cases} 2, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases} \tag{2.2}$$

It can be observed that in any case, the hull number of $G(C_m, C_n; P_t)$ equals 2 or 3, and this concludes the proof

□

Theorem 2.5. *Let* G *be an induced bicyclic graph with* k *end vertices. Then* $h(G) = k, k+1$ *or* $k+2$.

Proof. Let G be an induced bicyclic graph contained $G(C_m, C_n; P_t)$ with end vertices $u_1, u_2, \ldots, u_k$ and l trees be attached to $G(C_m, C_n; P_t)$. Suppose that vertices x and y are common in paths P_{m-t+2}, P_{n-t+2} and P_t. Firstly, assume $l = 1$ and a tree T is attached to one of the vertices x or y, without loosing generality suppose that T is attached to x, see Figure 7. Depending on which paths between x and y are x-y geodesic, by Lemma 2.4, there are three cases. Let C_p denote the largest cycle, where p can be m, n, or $m+n-2t+2$.

- If p is even, there exists a vertex w such that $\{x, w\} \in H(C_p)$. Then the set $S = \{x_1, x_2, \ldots, x_k, w\} \in H(G)$ is the minimum hull set of G, so $h(G) = k+1$.

- If p is odd, there exist vertices z, w such that $\{x, w, z\} \in H(C_l)$. Then the set $S = \{x_1, x_2, \ldots, x_k, w, z\} \in H(G)$ is the minimum hull set of G, so $h(G) = k+2$.

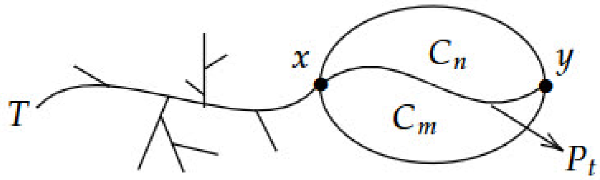

Figure 7: An induced bicyclic graph where a tree is attached to one end vertex of the common path.

If a tree is attached to $z \in V(G(C_m, C_n; P_t))$ and $z \neq x, y$, then, similar to the previous case, depending on the largest cycle created, the hull number $h(G)$ takes the values $k+1$ or $k+2$, as shown in Figure 8. The only special case occurs when the x-y geodesic is P_{m-t+2}, in which case C_n is the largest cycle. Then a vertex z is closer to one of the end vertices, say x. In this situation, vertices a and b can be identified such that one of the sets $\{x, a\}$ or $\{x, a, b\}$ forms a minimum hull set of C_n. In either case, the hull number is again $k+1$ or $k+2$.

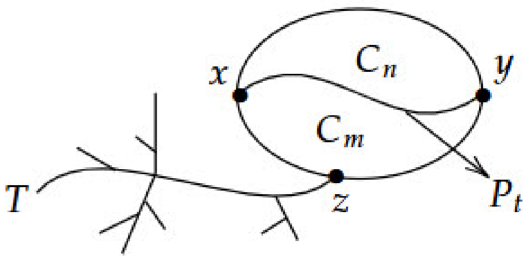

Figure 8: An induced bicyclic graph with a tree attached to a vertex outside the end vertices of the common path.

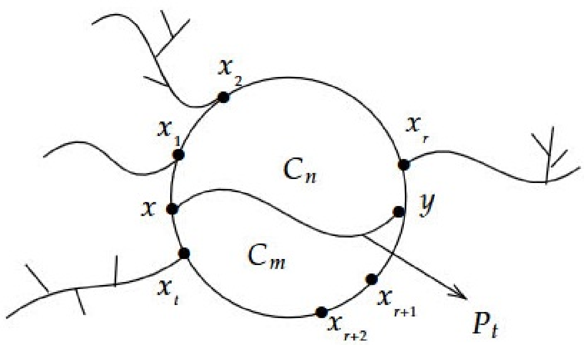

Figure 9: An induced bicyclic graph with t trees attached to vertices outside the common vertices x and y.

Now $l > 1$ and l trees be attached to $G(C_m, C_n; P_t)$ vertices $x_1, x_2, \dots, x_l$ such that vertices $x_1, \dots, x_r$ lie on the path P_{n-t+2} and vertices $x_{r+1}, \dots, x_t$ lie on the path P_{m-t+2}, as shown in Figure 9. Suppose

x, y are common vertices of the three paths P_{n-t+2}, P_{m-t+2}, and P_t. Then three cases arise depending on the x-y geodesic.

Case 1: If the x-y geodesic is P_t, then $C_{m+n-2t+2}$ is the largest cycle.

- If $\{x_1, x_2, \ldots, x_l\} \in H(C_{m+n-2t+2})$, then $S = \{x_1, \ldots, x_l\}$ satisfies $[S] = V(C_{m+n-2t+2})$ and by Lemma 1.1 and this fact that all end vertices contain hull convex set of graph, so $\{u_1, u_2, \ldots, u_k\}$ is minimum hull set of G, then $h(G) = k$.

- If $\{x_1, x_2, \ldots, x_l\} \notin H(C_{m+n-2t+2})$ and $m + n - 2t + 2$ is even, for each x_i, $1 \leqslant i \leqslant l$, there exists a vertex $y_j \in V(C_{m+n-2t+2})$ such that $\{x_i, y_j\} \in H(C_{m+n-2t+2})$. Then for each $1 \leqslant j \leqslant l$, the set $\{u_1, u_2, \ldots, u_k, y_j\}$ form the minimum hull set, so $h(G) = k + 1$. Now if $m + n - 2t + 2$ is odd, for each pair x_i, x_j there exists y_j such that $\{x_i, x_j, y_j\} \in H(C_{m+n-2t+2})$. Again, the end vertices with y_j form the minimum hull set, so $h(G) = k + 1$.

Case 2: If $P_{n-t+2}(P_{m-t+2})$ is x-y geodesic, then $C_m(C_n)$ is the largest cycle of G.

- If $\{x_{r+1}, x_{r+2}, \ldots, x_l\} \in H(C_m)$, then $h(G) = k$.

- If $\{x_{r+1}, x_{r+2}, \ldots, x_l\} \notin H(C_m)$ and m is even, for each $x_{r+i}, 1 \leqslant i \leqslant l - r$ there exists $y_j \in V(C_m)$ such that $\{x_{r+i}, y_j\} \in H(C_m)$. If m is odd, for each pair x_{r+i}, x_{r+j} there exists $y_j \in V(C_m)$ such that $\{x_{r+i}, x_{r+j}, y_j\} \in H(C_m)$. In all cases, the leaves together with the y_j vertices form the minimum hull set, so $h(G) = k + 1$.

□

3. Hull Numbers of Some Classes of Tricyclic Graphs

In this section, we study the hull numbers of several classes of tricyclic graphs containing no end vertices. Let G be a graph consisting of three disjoint cycles C_m, C_n, and C_l, with no end vertices. According to [9], all possible relative arrangements of these three cycles fall into seven configurations, as shown in Figures 10 and 11. By examining the hull number for each configuration, we find that these disjoint tricyclic graphs can be classified into two categories based on their hull numbers, as stated in the following theorem.

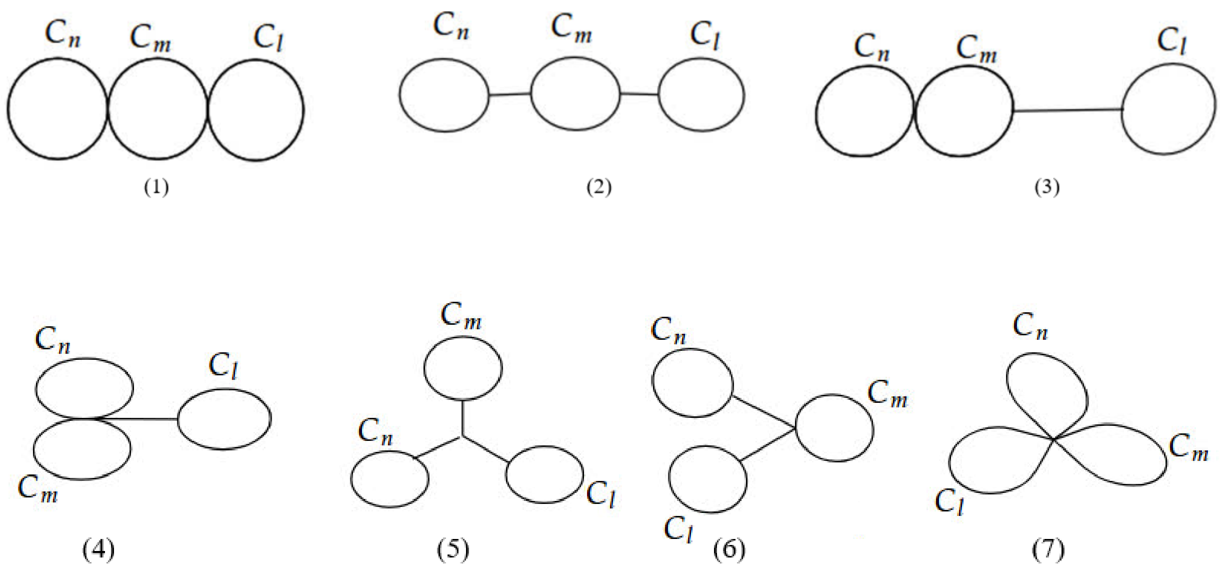

Figure 10: All possible arrangements of three disjoint cycles in a tricyclic graph.

Theorem 3.1. *Let* G *be a graph with three disjoint cycles and no end vertices. Then* $2 \leqslant h(G) \leqslant 6$

Proof. First, we classify G as a graph containing three disjoint cycles with no end vertices into two classes and consider the proof for each class separately. Let disjoint cycles C_m, C_n, and C_l be arranged as shown in Figure 10 (cases 1, 2, and 3). Depending on the parity of m, n, and l, four different cases arise.

- **All cycles are even:** We begin by investigating case 1 of Figure 10. Let $u \in V(C_n) \cap V(C_m)$ and $v \in V(C_m) \cap V(C_l)$, then there exists $a \in V(C_n)$ and $b \in V(C_l)$ such that $\{u, a\} \in H(C_n)$ and $\{v, b\} \in H(C_l)$ are the minimum convex hull sets of C_n and C_l respectively. If $\{u, v\} \in H(C_m)$, then by Lemma 1.1, $\{a, b\} \in H(G)$ forms a minimum hull set of the disjoint tricyclic graph G, and thus $h(G) = 2$. Now suppose $\{u, v\} \notin H(C_m)$, then there exists avertex $c \in V(C_m)$ such that $\{u, c\} \in H(C_m)$ or $\{v, c\} \in H(C_m)$. Again, by Lemma 1.1, $h(G) = 3$, see Figure 11.

  We now consider Case 2 of Figure 10. Based on the positions of the vertices u and v, choose

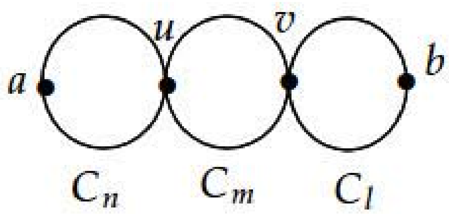

Figure 11: Chosen some vertices for convex hull set in Case 1.

  vertices $a \in V(C_n)$ and $b \in V(C_l)$ such that $\{u, a\}$ and $\{v, b\}$ are minimum hull sets of C_n and C_l, respectively. Consider the vertices x and y in $V(C_m)$. If $\{x, y\} \in H(C_m)$ then by Lemma 1.1, $\{a, b\} \in H(G)$ and then $h(G) = 2$. Otherwise $\{x, y\} \notin H(C_m)$ then there exists $c \in V(C_m)$ such that $\{x, c\} \in H(C_m)$ or $\{y, c\} \in H(C_m)$. Again, by Lemma 1.1, we obtain $h(G) = 3$ (see Figure 12).

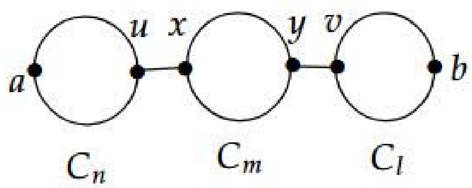

Figure 12: Chosen some vertices for convex hull set in Case 2.

  Finally, we consider Case 3 of Figure 10. Let $u \in V(C_n) \cap V(C_m)$ and choose $v \in v(C_l)$ as shown in Figure 10(3), such that $\{u, a\} \in H(C_n)$ and $\{v, b\} \in H(C_l)$ are minimum convex hull sets of C_n and C_l respectively. Let $y \in V(C_m)$ be the vertex shown in Figure 13. If $\{u, y\} \in H(C_m)$ then u, v and y lie on an $a - b$ geodesic of G and then $h(G) = 2$. Otherwise, if $\{u, y\} \notin H(C_m)$ by similar argument, there exists $y' \in V(C_m)$ such that $\{u, y'\} \in H(C_m)$ and $h(G) = 3$, (see Figure 13).

- **Two even cycles and one odd cycle:** First, we consider the case in which two consecutive cycles are even and the third cycle is odd (i.e., n and m are even and l is odd, or m and l are even and n is odd). By reasoning similar to Case 1, $h(G) = 3$ or $h(G) = 4$. Next, we examine the case in which the middle cycle is odd, while its adjacent cycles are even (n and l even, m odd).By a similar argument, one can see that $h(G) = 3$.

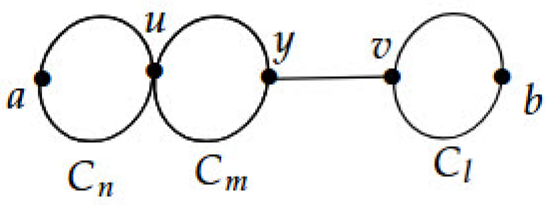

Figure 13: Chosen some vertices for convex hull set in Case 3.

- **Two odd cycles and one even cycle:** Let two consecutive cycles are odd (i.e., n and m are odd and l is even, or m and l are odd and n is even). It is straightforward to see that $h(G) = 4$. Next, we consider the case in which the middle cycle is even, while the adjacent cycles are odd (n and l are odd and m is even). By a similar argument, it follows that $h(G) = 4$ or $h(G) = 5$.

- **All cycles are odd:** In this case, all cycles in the graphs shown in Figure 10 are odd. Then $h(G) = 5$. t follows that, for all graph classes shown in Figure 10(1)–(3), we have

$$2 \leqslant h(G) \leqslant 5.$$

Now consider the disjoint cycles C_m, C_n, and C_l arranged as illustrated in Figure 10(4)–(7). Based on the parity of m, n, and l, we distinguish four cases. Since the proofs for the graph classes shown in Figure 10(4)–(7) are analogous, it suffices to present the proof for the graph shown in Figure 10(4). If all three cycles are even, then, by an argument similar to the one given above, we have $h(G) = 3$. Furthermore, if all three cycles are odd, then $h(G) = 6$. In the case where two cycles are even and one cycle is odd, we obtain $h(G) = 4$, whereas if two cycles are odd and one cycle is even, we obtain $h(G) = 5$. Therefore, for the graph classes depicted in Figure 10(4)–(7), we have

$$3 \leqslant h(G) \leqslant 6.$$

Consequently, for every graph G consisting of three disjoint cycles without end vertices, we conclude that

$$2 \leqslant h(G) \leqslant 6.$$

This completes the proof.

□

We now consider tricyclic graphs without end vertices, which contains an induced bicyclic subgraph $G(C_m, C_n : P_t)$ and a disjoint cycle connected by a path. According to [9], there are six graphs in this category, as shown in Figure 14.

Theorem 3.2. *Let G be a tricyclic graph containing an induced bicyclic graph $G(C_m, C_n, P_t)$ that is joined to the cycle C_l by a path. Then the hull number of G is equal to 2, 3 or 4.*

Proof. Without loss of generality, consider a graph shown in Figure 14. By Lemma 2.4, the hull number of the induced bicyclic subgraph $G_1(C_m, C_n; P_t)$ is either 2 or 3. Let u be a vertex in a minimum hull set of G_1. Similarly, the hull number of the cycle C_l is 2 or 3, depending on the parity of l; let v be a vertex in $H(C_l)$.

By Lemma 1.1, in the combined graph G, the vertices u and v are no longer in a minimum hull set of G. The proof depends on the parity of the largest cycle in G_1 and the cycle C_l. Suppose that $C_{m+n-2t+2}$ is the largest cycle in G_1 and both $C_{m+n-2t+2}$ and C_l are even. Then there exist vertices $w \in V(C_{m+n-2t+2})$ and $x \in V(C_l)$ such that $\{u, w\} \in H(G_1)$ and $\{v, x\} \in H(C_l)$. By Lemma 1.1, $\{w, x\} \in H(G)$ is a minimum hull set of G, so $h(G) = 2$.

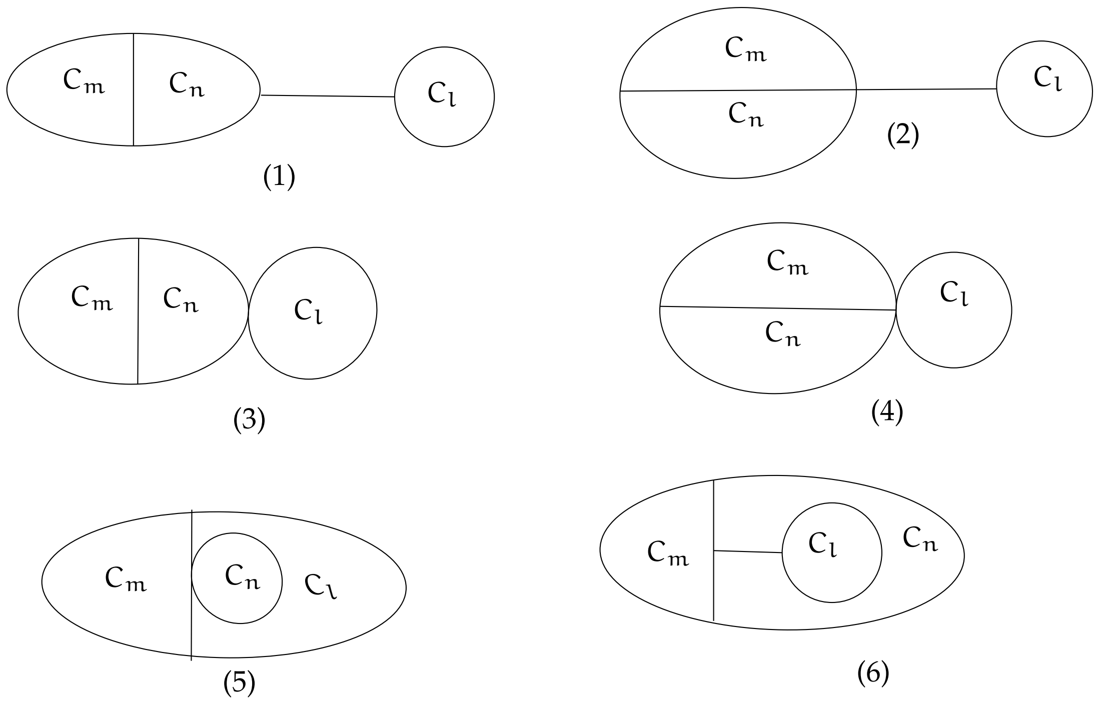

Figure 14: Tricyclic graphs containing an induced bicyclic graph C_m, C_n and a disjoint cycle C_l.

By a similar method, if one of the cycles $C_{m+n-2t+2}$ and C_l is even while the other is odd, then $h(G) = 3$, and if both cycles are odd, then $h(G) = 4$. Therefore, the hull number of G is in the set $\{2, 3, 4\}$.

By the same argument, the theorem also holds when C_m and C_n are the largest cycles in the induced bicyclic graph G_1. The proofs for graphs numbered (2), (3), and (4) in Figure 14 are analogous to that of graph (1), while graphs numbered (5) and (6) special cases of graphs (2) and (4), respectively. Therefore, the proof is complete. □

References

[1] Buckley, F., & Harary, F. (1990). Distance in graphs. New York, NY: Addison–Wesley Longman. 1, 1.3
[2] Cáceres, J., Hernando, C., Mora, M., I. M. Pelayo, & Puertas, M. L. (2010). On the geodetic and the hull numbers in strong product graphs. Computers & Mathematics with Applications, 60(11), 3020–3031. 1
[3] Chartrand, G., Harary, F., & Zhang, P. (2000). On the hull number of a graph. Ars Combinatoria, 57, 129–138. 1, 1.1, 1.2
[4] Chartrand, G., & Zhang, P. (2001). On graphs with a unique minimum hull set. Discussiones Mathematicae Graph Theory, 21(1), 31–42. 1
[5] Chartrand, G., & Zhang, P. (2001). The forcing hull number of a graph. Journal of Combinatorial Mathematics and Combinatorial Computing, 36, 81–94. 1
[6] Chartrand, G. (2006). Introduction to graph theory. Tata McGraw–Hill Education. 1
[7] Chandran, S. V. U., & Parthasarathy, G. J. (2016). The geodesic irredundant sets in graphs. International Journal of Mathematics and Combinatorics, 4, 135–143. 1.1, 1.2
[8] Everett, M. G., & Seidman, S. B. (1985). The hull number of a graph. Discrete Mathematics, 57(3), 217–223. 1, 1.3
[9] Yarahmadi, Z. (2020). A classification for bicyclic and tricyclic graphs with respect to Tutte polynomial. Hacettepe Journal of Mathematics and Statistics, 1–11. 3, 3