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Representation and perturbations of a system of additive-quadratic functional equations

Abasalt Bodaghi[a]

[a] Department of Mathematics, WT.C., Islamic Azad University, Tehran, Iran

Abstract

In the present article, we introduce a novel multimixed additive-quadratic mapping, and then show that the system of several mixed additive-quadratic equations defining a multimixed additive-quadratic mapping can be described as one single equation. We also show that such mappings under some conditions are multi-additive, multi-quadratic and multi-additive-quadratic. Moreover, we prove the Hyers stability for the multi-additive, multi-quadratic and multi-additive-quadratic mappings in the content of Banach spaces by two famous classical manners. As a outcome, by using a decomposition, we show that every multimixed additive-quadratic mapping is Hyers stable.

Keywords: Banach space, Hyers-Ulam stability, Multi-additive mapping, Multi-quadratic mapping, Multi-additive-quadratic mapping

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1. Introduction and preliminaries

A mathematical functional equation has the stability property when an approximate solution occurs near to the correct solution of the corresponding equation. Hyers [15] was the first author who gave an affirmative reply to Ulam's inquiry [21] in the context of Banach spaces regarding to the stability of additive functional equations as follows: Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces. Assume that $\Gamma : \mathbb{X} \longrightarrow \mathbb{Y}$ with

$$\|\Gamma(x+y) - \Gamma(x) - \Gamma(y)\| \leqslant \delta,$$

for all $x, y \in \mathbb{X}$ and for some $\delta \geqslant 0$. Then, there exists a unique additive mapping $\mathcal{A} : \mathbb{X} \longrightarrow \mathbb{Y}$ such that $\|\Gamma(x) - \mathcal{A}(x)\| \leqslant \delta$ for all $x \in \mathbb{X}$.

Aoki [1] presented an extension of the Hyers theorem and next the Hyers result for linear transformations was broadened by Rassias [19]. After that, Forti [13] and Găvruţa [14] expanded upon the earlier findings when they applied a general operater as a controller of the norms. Considering different upper bounds, there are many interesting and motivating stability results on various FEs pertaining to fascinating results which are available in literatures.

Throughout this paper, $\mathbb{N}$=positive integers, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\mathbb{Z}$=integers, $\mathbb{Q}$=rational numbers, $\mathbb{R}$=real numbers.

Email address: abasalt.bodaghi@gmail.com, abodaghi@iau.ir (Abasalt Bodaghi)

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Let $(G,+)$ be a commutative group, Y be a linear space over $\mathbb{Q}$. A mapping $\Gamma : G^n \cong \overbrace{G \times \cdots \times G}^{n-\text{times}} \longrightarrow Y$ is called

(i) *multi-additive* if it satisfies the Cauchy's equation in each of its n variables, that is, for each $i \in \{1,\dots,n\}$:

$$\begin{aligned}&\Gamma(\eta_1,\dots,\eta_{i-1},\eta_i+\eta_i',\eta_{i+1},\dots,\eta_n)\\&=\Gamma(\eta_1,\dots,\eta_{i-1},\eta_i,\eta_{i+1},\dots,\eta_n)+\Gamma(\eta_1,\dots,\eta_{i-1},\eta_i',\eta_{i+1},\dots,\eta_n);\end{aligned}$$

(ii) *multi-quadratic* if it fulfills the quadratic equation in each of its n components, that is, for each $i \in \{1,\dots,n\}$:

$$\begin{aligned}&\Gamma(\eta_1,\dots,\eta_{i-1},\eta_i+\eta_i',\eta_{i+1},\dots,\eta_n)+\Gamma(\eta_1,\dots,\eta_{i-1},\eta_i-\eta_i',\eta_{i+1},\dots,\eta_n)\\&=2\Gamma(\eta_1,\dots,\eta_{i-1},\eta_i,\eta_{i+1},\dots,\eta_n)+2\Gamma(\eta_1,\dots,\eta_{i-1},\eta_i',\eta_{i+1},\dots,\eta_n);\end{aligned}$$

(iii) k-additive and $n-k$-quadratic (briefly, *multi-additive-quadratic*) if Γ is additive in each of some k slots and is quadratic in each of the other slots.

All mappings defined in the above were represented as the single equation in [2], [10] and[25]. In other words,

(i) Γ is multi-additive if and only if it satisfies

$$\Gamma\left(\eta_1^{[n]}+\eta_2^{[n]}\right)=\sum_{j_1,\dots,j_n\in\{1,2\}}\Gamma(\eta_{j_1 1},\dots,\eta_{j_n n}), \tag{1.1}$$

where $\eta_i^{[n]}=(\eta_{i1},\dots,\eta_{in})\in G^n$ for $i\in\{1,2\}$ [10];

(ii) Γ is multi-quadratic if and only if it fulfills

$$\sum_{q\in\{-1,1\}^n}\Gamma\left(\eta_1^{[n]}+q\eta_2^{[n]}\right)=2^n\sum_{j_1,\dots,j_n\in\{1,2\}}\Gamma(\eta_{j_1 1},\dots,\eta_{j_n n}), \tag{1.2}$$

where $\eta_i^{[n]}=(\eta_{i1},\dots,\eta_{in})\in G^n$ for $j\in\{1,2\}$ [25];

(iii) Γ is multi-additive-quadratic if and only if it satisfies

$$\sum_{t\in\{-1,1\}^{n-k}}\Gamma\left(\eta_1^{[k]}+\eta_2^{[k]},\eta_1^{[n-k]}+t\eta_2^{[n-k]}\right)=2^{n-k}\sum_{j_1,\dots,j_n\in\{1,2\}}\Gamma(\eta_{j_1 1},\cdots,\eta_{j_n n}) \tag{1.3}$$

where $\eta_i^{[k]}=(\eta_{i1},\dots,\eta_{ik})\in G^k,\eta_1^{[n-k]}=(\eta_{i,k+1},\dots,\eta_{in})\in G^{n-k}$, $i\in\{1,2\}$ [2].

More information and details about the structure of multi-additive and multi-quadratic mappings and also the Hyers-Ulam stability (in certain types of Banach spaces) for such mappings are available as follows: the generalized multi-quadratic functional equations and their stability by using a fixed point technique [4], the Hyers-Ulam stability of multi-quadratic mappings in non-Archimedean spaces [6], Some remarks on the Hyers-Ulam stability of vaious systems of functional equations [8]; we also refer to a comprehensive book on the varied functional equations [16].

To make better decisions on approximations, we may need to increase reliable and useful information on different aspects of an approximation. In other words, for obtaining the best quality and certainty of a proximity to the exact solution of a functional equation, we need to measure both the certainty and quality of the approximation and errors as well. It can be an application of decision theory on the approximations of functional equations. Recently, it was done for the multi-additive mappings via the concept of generalized Z-numbers [11].

The first mixed type of additive and quadratic functional equation has been introduced by Najati and Moghimi in [17] as the form

$$\mathbf{F}(2u+v)+\mathbf{F}(2u-v)=\mathbf{F}(u+v)+\mathbf{F}(u-v)+2\mathbf{F}(2u)-2\mathbf{F}(u).$$

It is known that there are many models of mixed type of additive and quadratic functional equations which are available in articles and books and so one can define a multimixed additive-quadratic mapping corresponding to each of them; see for instance [22]. In the last five years, miscellaneous multimixed additive-quadratic mappings were defined such that these mappings satisfy in the following list of mixed type of additive and quadratic functional equations in each of variable.

(i) $\mathbf{F}(u+2v)+\mathbf{F}(u-2y)+8\mathbf{F}(v)=2\mathbf{F}(u)+4\mathbf{F}(2v)$ [24, 7];
(ii) $\mathbf{F}(u+2v)+\mathbf{F}(u-2v)-3\mathbf{F}(2v)=\mathbf{F}(u+v)+\mathbf{F}(u-v)-6\mathbf{F}(v)$ [3, 5];
(iii) $\mathbf{F}(u+sv)+\mathbf{F}(u-sv)=2\mathbf{F}(u)+s^2\mathbf{F}(2v)-2s^2\mathbf{F}(v)$, where s is an integer number with $s\neq 0,\pm 1$ [5, 12];
(iv) $2\mathbf{F}\left(\frac{u+v}{2}\right)+\mathbf{F}\left(\frac{u-v}{2}\right)+\mathbf{F}\left(\frac{v-u}{2}\right)=\mathbf{F}(u)+\mathbf{F}(v)$ [18, 20].

It is easily seen that the function $F:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $\mathbf{F}(x)=ax^2+bx$ gives as a common solution of the above mixed additive-quadratic functional equations.

In this paper, motivated by equation

$$\Phi(2u+v)+\Phi(2u-v)=\Phi(u+v)+\Phi(u-v)+4\Phi(u)+2\Phi(-u), \tag{1.4}$$

which was presented in [23], we define a multimixed additive-quadratic mapping as the general system of the mixed type of additive and quadratic functional equations, and the represent such systems as an unified equation. We also show that such mappings under some conditions can be multi-additive, multi-quadratic and multi-additive-quadratic. Furthermore, we prove the Hyers stability of the multi-additive-quadratic mappings in the setting of Banach spaces by two known classical methods, namely, the direct and fixed point.

2. Representation of multimixed additive-quadratic mappings

Throughout this section, it is assumed that X and Y are real vector spaces. For any $l\in\mathbb{N}_0, n\in\mathbb{N}$, $t=(t_1,\ldots,t_n)\in\{-1,1\}^n$ and $\eta=(\eta_1,\ldots,\eta_n)\in X^n$ we write $l\eta:=(l\eta_1,\ldots,l\eta_n)$ and $t\eta:=(t_1\eta_1,\ldots,t_n\eta_n)$. Applying (1.4), we commence this section with the definition of multimixed additive-quadratic mappings.

Definition 2.1. A mapping $\Gamma:X^n\longrightarrow Y$ is called n-*mixed additive-quadratic* or *multimixed additive-quadratic* if Γ satisfies the system of additive-quadratic functional equations

$$\begin{aligned}&\Gamma(\vartheta_1,\ldots,\vartheta_{j-1},2\vartheta_j+\vartheta_j',\vartheta_{j+1},\ldots,\vartheta_n)+\Gamma(\vartheta_1,\ldots,\vartheta_{j-1},2\vartheta_j-\vartheta_j',\vartheta_{j+1},\ldots,\vartheta_n)\\&=\Gamma(\vartheta_1,\ldots,\vartheta_{j-1},\vartheta_j+\vartheta_j',\vartheta_{j+1},\ldots,\vartheta_n)+\Gamma(\vartheta_1,\ldots,\vartheta_{j-1},\vartheta_j-\vartheta_j',\vartheta_{j+1},\ldots,\vartheta_n)\\&\quad+4\Gamma(\vartheta_1,\ldots,\vartheta_{j-1},\vartheta_j,\vartheta_{j+1},\ldots,\vartheta_n)+2\Gamma(\vartheta_1,\ldots,\vartheta_{j-1},-\vartheta_j,\vartheta_{j+1},\ldots,\vartheta_n),\end{aligned}$$

for all $j\in\{1,\ldots,n\}$.

Note that each multimixed additive-quadratic mapping satisfies (1.4) in each of its n slots.

Definition 2.2. Given a mapping $\Gamma:X^n\longrightarrow Y$.

(i) We say Γ is *odd* in the j^{th} variable if

$$\Gamma(x_1,\ldots,x_{j-1},-x_j,x_{j+1},\ldots,x_n)=-\Gamma(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_n);$$

(ii) We say Γ is *even* in the j^{th} component if

$$\Gamma(x_1,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_n)=\Gamma(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_n);$$

(iii) Let $s\in\mathbb{Z}\backslash\{0\}$. We say Γ has the *s-power property* in the j^{th} slot if

$$\Gamma(x_1,\dots,x_{j-1},2x_j,x_{j+1},\dots,x_n)=2^s\Gamma(x_1,\cdots,x_{j-1},x_j,x_{j+1},\dots,x_n),$$

for all $x_1,\dots,x_n\in X^n$. In particular, 1-power property and 2-power property is called the *additive property* and the *quadratic property*, respectively.

For a mapping $\Gamma:X^n\longrightarrow Y$, we consider the following hypotheses:

(H1): Γ is odd in all variables;
(H2): Γ is even in each variable;
(H3): Γ is odd in some k components $(k<n)$ and is even in the other components.
(H4): Γ has the additive property in all slots;
(H5): Γ has the quadratic property in all variables.
(H6): Γ has the additive property in some k components $(k<n)$ and has the quadratic property in the other components.

Suppose that Γ is an n-mixed additive-quadratic mapping. If Γ satisfies (H1) [resp. (H2)], then it is a multi-additive mapping [resp. multi-quadratic mapping]; see [10] and [25]. In the case that Γ fulfills (H3), then it is a multi-additive-quadratic mapping [2]. Define the function $\Phi:\mathbb{R}^n\longrightarrow\mathbb{R}$ through $\Phi(x_1,\dots,x_n)=\prod_{i=1}^n(a_ix_i^2+b_ix_i)$, where $a_i,b_i\in\mathbb{R}$, $i\in\{1,\dots,n\}$. A routine verification shows Γ is a multimixed additive-quadratic mapping and indeed is a solution of the system of functional equations given in Definition 2.1. If moreover Γ fulfills hypothesis (H1), then its solution has the form $\Phi(x_1,\dots,x_n)=b\prod_{i=1}^n x_i$, where $b\in\mathbb{R}$. Furthermore, if ψ satisfies (H2), then one of its solutions can be given as $\Phi(x_1,\dots,x_n)=a\prod_{i=1}^n x_i^2$, where $a\in\mathbb{R}$. Lastly, If (H3) is valid for Γ, then its solution has the representation as $\Phi(x_1,\dots,x_n)=c\prod_{i=1}^n x_i^2x_i$, where $c\in\mathbb{R}$.

In the rest of this section, we represent a multimixed additive-quadratic mapping which is defined as a system of n mixed type additive-quadratic functional equations, as an unified equation.

Let $n\in\mathbb{N}$ with $n\geqslant 2$ and $x_i^{[n]}=(x_{i1},x_{i2},\dots,x_{in})\in X^n$, where $i\in\{1,2\}$. For $x_1^{[n]},x_2^{[n]}\in X^n$,

$$\mathfrak{X}^n=\left\{\mathfrak{x}_n=(\mathbb{X}_1,\dots,\mathbb{X}_n)|\ \mathbb{X}_j\in\{x_{1j}\pm x_{2j},x_{1j},-x_{1j}\}\right\},$$

where $j\in\{1,\dots,n\}$. Let $r,s\in\mathbb{N}_0$ with $0\leqslant r,s\leqslant n$. The subset $\mathfrak{X}_{r,s}^n$ of $\mathfrak{X}^n$ is given as follows:

$$\mathfrak{X}_{r,s}^n:=\left\{\mathfrak{x}_n\in\mathfrak{X}^n|\ \mathrm{Card}\{\mathbb{X}_j:\mathbb{X}_j=-x_{1j}\}=r,\ \mathrm{Card}\{\mathbb{X}_j=x_{1j}\}=s\right\},$$

in which $\mathrm{Card}\Omega$ is the cardinality of a set Ω.

Here and subsequently, for a mapping $\Gamma:X^n\longrightarrow Y$, we denote

$$\Gamma\left(\mathfrak{X}_{r,s}^n\right):=\sum_{\mathfrak{x}_n\in\mathfrak{X}_{r,s}^n}\Gamma(\mathfrak{x}_n),\qquad \Gamma\left(\mathfrak{X}_{r,s}^n,x\right):=\sum_{\mathfrak{x}_n\in\mathfrak{X}_{r,s}^n}\Gamma(\mathfrak{x}_n,x),\quad (x\in X).\tag{2.1}$$

In the following proposition, we represent a multimixed additive-quadratic mappings as an unified equation.

Proposition 2.3. *Every multimixed additive-quadratic mapping $\Gamma:X^n\longrightarrow Y$ satisfies the equation*

$$\sum_{t\in\{-1,1\}^n}\Gamma\left(2x_1^{[n]}+tx_2^{[n]}\right)=\sum_{r=0}^{n}\sum_{s=0}^{n-r}2^r\times 4^s\Gamma\left(\mathfrak{X}_{r,s}^n\right),\tag{2.2}$$

where $\Gamma\left(\mathfrak{X}_{r,s}^n\right)$ is defined in (2.1).

Proof. . The proof is according to the mathematical induction on m so that Γ satisfies (2.2) when $1 \leqslant m \leqslant n$. For $m = 1$, it is obvious that Γ satisfies (1.4). Assume that Γ fulfills (2.2) for an arbitrary and fixed $m-1 \in \mathbb{N}_0$ with $1 \leqslant m \leqslant n$. Using (2.1), we have

$$\begin{aligned}
\sum_{t\in\{-1,1\}^m} \Gamma\left(2x_1^{[n]} + tx_2^{[m]}\right) &= \sum_{t\in\{-1,1\}^{m-1}} \sum_{q\in\{-1,1\}} \Gamma\left(2x_1^{[m-1]} + tx_2^{[m-1]}, 2x_{1m} + qx_{2m}\right) \\
&= \sum_{r=0}^{m-1} \sum_{s=0}^{m-1-r} \sum_{q\in\{-1,1\}} 2^r \times 4^s \Gamma\left(\mathcal{X}_{r,s}^{m-1}, 2x_{1m} + qx_{2m}\right) \\
&= \sum_{r=0}^{m-1} \sum_{s=0}^{m-1-r} \sum_{q\in\{-1,1\}} 2^r \times 4^s \Gamma\left(\mathcal{X}_{r,s}^{m-1}, x_{1m} + qx_{2m}\right) \\
&\quad + 2\sum_{r=0}^{m-1} \sum_{s=0}^{m-1-r} 2^r \times 4^s \Gamma\left(\mathcal{X}_{r,s}^{m-1}, -x_{1m}\right) \\
&\quad + 4\sum_{r=0}^{m-1} \sum_{s=0}^{m-1-r} 2^r \times 4^s \Gamma\left(\mathcal{X}_{r,s}^{m-1}, x_{1m}\right) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{n-r} 2^r \times 4^s \Gamma\left(\mathcal{X}_{r,s}^{n}\right).
\end{aligned}$$

The proof is now finished. □

In the upcoming result, we give a converse of Proposition 2.3.

Proposition 2.4. *Suppose that $\Gamma : X^n \longrightarrow Y$ is a mapping satisfies equation* (2.2)*.*

(i) *If Γ satisfies* (H1) *and* (H4)*, then it is multi-additive and so it satisfies* (1.1)*;*
(ii) *If* (H2) *and* (H5) *hold for Γ, then it is multi-quadratic and therefore Γ fulfills* (1.2)*;*
(iii) *If Γ fulfills* (H3) *and* (H6)*, then it is multi-additive-quadratic and hence equation* (1.3) *is valid for Γ.*

Proof. (i) Assume that Γ satisfies (2.2). We show that $\mathcal{M}$ is mixed type additive and quadratic in the j^{th} variable, where $j \in \{1, \dots, n\}$ is arbitrary and fixed. Set

$$\Gamma^\bullet(2x_{1j}, x_{2j}) := \sum_{q\in\{-1,1\}} \Gamma\left(x_{11}, \dots, x_{1,j-1}, 2x_{1j} + qx_{2j}, x_{1,j+1}, \dots, x_{1n}\right),$$

$$\Gamma^\bullet(x_{1j}, x_{2j}) := \sum_{q\in\{-1,1\}} \Gamma\left(x_{11}, \dots, x_{1,j-1}, x_{1j} + qx_{2j}, x_{1,j+1}, \dots, x_{1n}\right),$$

$$\Gamma^\bullet(x_{1j}) := \Gamma\left(x_{11}, \dots, x_{1,j-1}, x_{1j}, x_{1,j+1}, \dots, x_{1n}\right),$$

and

$$\Gamma^\bullet(-x_{1j}) := \Gamma\left(x_{11}, \dots, x_{1,j-1}, -x_{1j}, x_{1,j+1}, \dots, x_{1n}\right),$$

(i) Putting $x_{2k} = 0$ for all $k \in \{1, \dots, n\}\backslash\{j\}$ in (2.2), using the assumptions, we get

$$
\begin{aligned}
4^{n-1}\Gamma^{\bullet}(2x_{1j},x_{2j}) &= \sum_{r=0}^{n-1}\sum_{s=0}^{n-1}\left[\binom{n-1}{r}\binom{n-1-r}{s}(-2)^r\times 4^s\times 2^{n-1-r-s}\right]\Gamma^{\bullet}(x_{1j},x_{2j})\\
&\quad +2\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r}\left[\binom{n-1}{r}\binom{n-1-r}{s}(-2)^r\times 4^s\times 2^{n-1-r-s}\right]\Gamma^{\bullet}(-x_{1j})\\
&\quad +\sum_{r=0}^{n-1}\sum_{s=1}^{n-r}\left[\binom{n-1}{r}\binom{n-1-r}{s-1}(-2)^r\times 4^s\times 2^{n-r-s}\right]\Gamma^{\bullet}(x_{1j}),
\end{aligned}
\tag{2.3}
$$

where $\binom{m}{r}$ is the binomial coefficient defined for all $m,r\in\mathbb{N}_0$ with $m\geqslant r$ by $m!/(r!(m-r)!)$. We compute one of coefficients. We have

$$
\begin{aligned}
&\sum_{r=0}^{n-1}\sum_{s=1}^{n-r}\binom{n-1}{r}\binom{n-1-r}{s-1}(-2)^r\times 4^s\times 2^{n-r-s}\\
&\qquad =4\sum_{r=0}^{n-1}\sum_{s=0}^{n-1-r}\binom{n-1}{r}\binom{n-1-r}{s}(-2)^r\times 4^s\times 2^{n-1-r-s}\\
&\qquad =4\sum_{r=0}^{n-1}(-2)^r\times 6^{n-1-r}=4(-2+6)^{n-1}=4\times 4^{n-1}.
\end{aligned}
\tag{2.4}
$$

Similarly, we have

$$
\sum_{r=0}^{n-1}\sum_{s=0}^{n-1}\binom{n-1}{r}\binom{n-1-r}{s}(-2)^r\times 4^s\times 2^{n-1-r-s}=4^{n-1}.
\tag{2.5}
$$

It follows from (2.3)-(2.5) that

$$
\Gamma^{\bullet}(2x_{1j},x_{2j})=\Gamma^{\bullet}(x_{1j},x_{2j})+4\Gamma^{\bullet}(x_{1j})+2\Gamma^{\bullet}(-x_{1j}).
$$

Now, [23, Lemma 2.2] implies that Γ is additive in each of variable. The last part was shown in [10, Theorem 2].

(ii) The result can be obtained similar to the first part and applying [23, Lemma 2.1].

(iii) This part is a consequence of (i), (ii) and Theorem 1 from [2, Theorem 1]. □

3. Stability results for (2.2)

In this section, we prove the Hyers stability of equation (2.2) by two classical methods in the setting of Banach spaces.

For a mapping $\Gamma : X^n \longrightarrow Y$, we use the difference operation

$$
\Delta\Gamma\left(x_1^{[n]},x_2^{[n]}\right):=\sum_{t\in\{-1,1\}^n}\Gamma\left(2x_1^{[n]}+tx_2^{[n]}\right)-\sum_{r=0}^{n}\sum_{s=0}^{n-r}2^r\times 4^s\,\Gamma\left(\mathcal{X}_{r,s}^n\right),
$$

where $\Gamma\left(\mathcal{X}_{r,s}^n\right)$ is defined in (2.1).

3.1. Direct technique

In the next theorem, we bring Hyers stability result for equation (2.2) by the direct way which is our main result in this subsection.

Theorem 3.1. *Let X be a linear space over $\mathbb{Q}$ and Y be a Banach space. Suppose that $\Gamma : X^n \longrightarrow Y$ is a mapping satisfies* (H3) *such that*

$$\left\|\Delta\Gamma\left(x_1^{[n]}, x_2^{[n]}\right)\right\| \leqslant \delta, \tag{3.1}$$

for all $x_1^{[n]}, x_2^{[n]} \in X^n$ for some $\delta \in [0, \infty)$. Then, there exists a solution $\mathcal{A}_q : X^n \longrightarrow Y$ of (2.2) *such that*

$$\left\|\Gamma\left(x^{[n]}\right) - \mathcal{A}_q\left(x^{[n]}\right)\right\| \leqslant \frac{\delta}{2^n(2^{2n-k} - 1)}, \tag{3.2}$$

for all $x^{[n]} \in X^n$. Moreover, if $\mathcal{A}_q$ fulfills (H6)*, then it is a unique generalized multi-additive-quadratic mapping satisfying* (3.2)*. In particular,*

(i) if Γ satisfies (H1)*, that is $k = n$, then there exists a unique multi-additive mapping $\mathcal{A} : X^n \longrightarrow Y$ such that*

$$\left\|\Gamma\left(x^{[n]}\right) - \mathcal{A}\left(x^{[n]}\right)\right\| \leqslant \frac{\delta}{2^n(2^n - 1)},$$

for all $x^{[n]} \in X^n$;

(ii) if Γ satisfies (H2)*, that is $k = 0$, then there exists a unique multi-quadratic mapping $\mathcal{Q} : X^n \longrightarrow Y$ such that*

$$\left\|\Gamma\left(x^{[n]}\right) - \mathcal{Q}\left(x^{[n]}\right)\right\| \leqslant \frac{\delta}{2^n(2^{2n} - 1)},$$

for all $x^{[n]} \in X^n$.

Proof. Putting $v_2^{[n]} = \left(\overbrace{0, \dots, 0}^{n-\text{times}}\right)$ in (3.1) and using (H3), we have

$$\left\|2^n\Gamma\left(2x_1^{[n]}\right) - ST\Gamma\left(x_1^{[n]}\right)\right\| \leqslant \delta, \tag{3.3}$$

for all $x_1^{[n]} \in V^n$, where

$$S = \sum_{r=0}^{k}\sum_{s=0}^{k-r}\binom{n}{r}\binom{n-r}{s}(-2)^r \times 4^s \times 2^{k-r-s}$$

and

$$T = \sum_{r=0}^{n-k}\sum_{s=0}^{n-k-r}\binom{n-k}{r}\binom{n-k-r}{s}2^r \times 4^s \times 2^{n-k-r-s}.$$

One can find that $S = 4^n$ and $T = 8^{n-k}$. For the rest of proof, we consider $x^{[n]}$ instead of $x_1^{[n]}$. It follows from relation (3.3) and last equalities that

$$\left\|2^n\Gamma\left(2x^{[n]}\right) - 2^{3n-k}\Gamma\left(x^{[n]}\right)\right\| \leqslant \delta,$$

for all $x^{[n]} \in X^n$, and so the equation above can be rewritten as

$$\left\| \frac{\Gamma\left(2x^{[n]}\right)}{2^{2n-k}} - \Gamma\left(x^{[n]}\right) \right\| \leqslant \frac{\delta}{2^{3n-k}}, \tag{3.4}$$

for all $x^{[n]} \in X^n$. Substituting $x^{[n]}$ into $2x^{[n]}$ in (3.4) and continuing this method, we find

$$\left\| \frac{\Gamma\left(2^m x^{[n]}\right)}{2^{(2n-k)m}} - \frac{\Gamma\left(2^l x^{[n]}\right)}{2^{(2n-k)l}} \right\| \leqslant \frac{\delta}{2^{3n-k}} \sum_{j=l}^{m-1} \frac{1}{2^{(2n-k)j}}, \tag{3.5}$$

for all $x^{[n]} \in X^n$ and $m > l \geqslant 0$. Thus, the sequence $\left\{ \frac{\Gamma\left(2^m x^{[n]}\right)}{2^{(2n-k)m}} \right\}$ is Cauchy by (3.5). Due to the completeness of W, there exists a mapping $\mathcal{A}_q : X^n \longrightarrow Y$ such that

$$\lim_{m\to\infty} \frac{\Gamma\left(2^m x^{[n]}\right)}{2^{(2n-k)m}} = \mathcal{A}_q\left(x^{[n]}\right), \qquad \left(x^{[n]} \in X^n\right). \tag{3.6}$$

Letting $l = 0$ in (3.5), letting the limit as $m \to \infty$ and applying (3.6), the validity of (3.2) is now shown. Switching $\left(x_1^{[n]}, x_2^{[n]}\right)$ into $\left(2^m x_1^{[n]}, 2^m x_2^{[n]}\right)$ in (3.1) and dividing the resultant to $2^{(2n-k)m}$, we obtain

$$\frac{1}{2^{(2n-k)m}} \left\| \Delta\Gamma\left(2^m x_1^{[n]}, 2^m x_2^{[n]}\right) \right\| \leqslant \frac{\delta}{2^{(2n-k)m}}.$$

Taking the limit as $m \to \infty$ in the last inequality and using (3.6), we arrive at $\Delta\mathcal{A}_q\left(x_1^{[n]}, x_2^{[n]}\right) = 0$ for all $x_1^{[n]}, x_2^{[n]} \in X^n$, and so $\mathcal{A}_q$ is a solution of (2.2). Assume now that $\mathcal{A}_q$ satisfies (H3), then it is a multi-additive-quadratic mapping by Proposition 2.4. Let now $\mathcal{A}'_q : X^n \longrightarrow Y$ be another multi-additive-quadratic mapping satisfying (3.2). Then, we have

$$\begin{aligned}
&\left\| \mathcal{A}_q\left(x^{[n]}\right) - \mathcal{A}'_q\left(x^{[n]}\right) \right\| \\
&= \frac{1}{2^{(2n-k)m}} \left\| \mathcal{A}_q\left(2^m x^{[n]}\right) - \mathcal{A}'_q\left(2^m x^{[n]}\right) \right\| \\
&\leqslant \frac{1}{2^{(2n-k)m}} \left(\left\| \mathcal{A}_q\left(2^m x^{[n]}\right) - \Gamma\left(2^m x^{[n]}\right) \right\| + \left\| \Gamma\left(2^m x^{[n]}\right) - \mathcal{A}'_q\left(2^m x^{[n]}\right) \right\| \right) \\
&\leqslant \frac{2}{2^{(2n-k)m}} \times \frac{\delta}{2^n(2^{2n-k}-1)},
\end{aligned}$$

for all $x^{[n]} \in X^n$. Letting $m \to \infty$ in the above inequality, we have $\mathcal{A}_q = \mathcal{A}'_q$. This completes the proof. □

The following corollary demonstrates that under some mild conditions a multimixed additive-quadratic can be Hyers stable.

Corollary 3.2. *Let $\delta_1, \delta_2 > 0$, X be a linear space over $\mathbb{Q}$ and Y be a Banach space. Suppose that $\Gamma : X^n \longrightarrow Y$ is a decomposable mapping as $\Gamma = \Gamma_o + \Gamma_e$ such that Γ_o and Γ_e satisfying* (H1) *and* (H2)*, respectively. If the inequalities*

$$\left\| \Delta\Gamma_o\left(x_1^{[n]}, x_2^{[n]}\right) \right\| \leqslant \delta_1 \ \text{ and } \ \left\| \Delta\Gamma_e\left(x_1^{[n]}, x_2^{[n]}\right) \right\| \leqslant \delta_2$$

hold $x_1^{[n]}, x_2^{[n]} \in X^n$, then there exist a unique multi-additive mapping $\mathcal{A} : X^n \longrightarrow Y$ and a unique multi-quadratic mapping $\mathcal{Q} : X^n \longrightarrow Y$ such that

$$\left\| \Gamma\left(x^{[n]}\right) - \mathcal{A}\left(x^{[n]}\right) - \mathcal{Q}\left(x^{[n]}\right) \right\| \leqslant \frac{\delta_1}{2^n(2^n-1)} + \frac{\delta_2}{2^n(2^{2n}-1)},$$

for all $x^{[n]} \in X^n$.

Proof. It follows from parts (i) and (ii) of Theorem 3.1 that there exists a unique multi-additive mapping $\mathcal{A} : X^n \longrightarrow Y$ and a unique multi-quadratic mapping $\mathcal{Q} : X^n \longrightarrow Y$ such that

$$\left\|\Gamma_o\left(x^{[n]}\right)) - \mathcal{A}\left(x^{[n]}\right)\right\| \leqslant \frac{\delta_1}{2^n(2^n-1)}, \tag{3.7}$$

and

$$\left\|\Gamma_o\left(x^{[n]}\right)) - \mathcal{Q}\left(x^{[n]}\right)\right\| \leqslant \frac{\delta_2}{2^n(2^{2n}-1)} \tag{3.8}$$

for all $x^{[n]} \in X^n$. The result now follows from the inequalities (3.7) and (3.8). □

3.2. *Fixed point approach*

In this subsection, we prove the Hyers stability of (2.2) by a fixed point result (Theorem 3.3) in Banach spaces. Here, we bring a well-known fixed point theorem which is a fundamental result to obtain our objective in this paper and was illustrated in [9, Theorem 1]. From now on, for two sets V and W, the set of all mappings from V to W is denoted by W^V.

Theorem 3.3. *Let X be a Banach space, $\mathcal{S}$ be a nonempty set, $j \in \mathbb{N}$, $h_1, \dots, h_j : \mathcal{S} \longrightarrow \mathcal{S}$ and $L_1, \dots, L_j : \mathcal{S} \longrightarrow \mathbb{R}_+$. Suppose that the following hypotheses hold:*

(A1) $\mathcal{T} : X^{\mathcal{S}} \longrightarrow X^{\mathcal{S}}$ *is an operator satisfying the inequality*

$$\|\mathcal{T}\lambda(s) - \mathcal{T}\mu(s)\| \leqslant \sum_{i=1}^{j} L_i(s)\,\|\lambda(h_i(s)) - \mu(h_i(s))\|, \quad \lambda, \mu \in X^{\mathcal{S}}, s \in \mathcal{S},$$

(A2) $\Lambda : \mathbb{R}_+^{\mathcal{S}} \longrightarrow \mathbb{R}_+^{\mathcal{S}}$ *is an operator defined through*

$$\Lambda\delta(s) := \sum_{i=1}^{j} L_i(s)\delta(h_i(s)) \qquad \delta \in \mathbb{R}_+^{\mathcal{S}}, s \in \mathcal{S}.$$

Moreover, a function $\theta : \mathcal{S} \longrightarrow \mathbb{R}_+$ and a mapping $\phi : \mathcal{S} \longrightarrow Y$ are fulfilling the following two conditions:

$$\|\mathcal{T}\phi(s) - \phi(s)\| \leqslant \theta(s), \quad \theta^*(s) := \sum_{l=0}^{\infty} \Lambda^l\theta(s) < \infty \qquad (s \in \mathcal{S}).$$

Then, there exists a unique fixed point ψ of $\mathcal{T}$ such that

$$\|\phi(s) - \psi(s)\| \leqslant \theta^*(s) \qquad (s \in \mathcal{S}).$$

Moreover, $\psi(s) = \lim_{l\to\infty} \mathcal{T}^l\phi(s)$ for all $s \in \mathcal{S}$.

Here, we present a different proof for Theorem 3.1.

A fixed point method for the proof of Theorem 3.1. Similar to the proof of Theorem 3.1, we have

$$\left\|2^n\Gamma\left(2x^{[n]}\right) - 2^{3n-k}\Gamma\left(x^{[n]}\right)\right\| \leqslant \delta, \tag{3.9}$$

for all $x^{[n]} \in X^n$. Then, relation (3.9) can be modified as

$$\left\|\Gamma\left(v^{[n]}\right) - \mathcal{T}\Gamma\left(x^{[n]}\right)\right\| \leqslant \zeta\left(x^{[n]}\right)) \qquad \left(x^{[n]} \in X^n\right),$$

where $\mathcal{T}\xi\left(x^{[n]}\right) := \frac{1}{2^{2n-k}}\xi\left(2x^{[n]}\right) \quad \left(\xi \in Y^{X^n}\right)$ and $\zeta\left(x^{[n]}\right) := \frac{\delta}{2^{3n-k}}$. Set $\Lambda\eta\left(x^{[n]}\right) := \frac{1}{2^{2n-k}}\eta\left(2x^{[n]}\right)$ for all $\eta \in \mathbb{R}_+^{X^n}, x^{[n]} \in X^n$. It is easily seen that Λ has the form described in (A2) with $\mathcal{S} = X^n$, $h_1\left(x^{[n]}\right) = 2x^{[n]}$ and $L_i\left(x^{[n]}\right) = \frac{1}{2^{2n-k}}$ for all $x^{[n]} \in X^n$ and all i. Moreover, for each $\lambda, \mu \in Y^{X^n}$ and $x^{[n]} \in X^n$, we obtain

$$\begin{aligned}\left\|\mathcal{T}\lambda\left(x^{[n]}\right) - \mathcal{T}\mu\left(x^{[n]}\right)\right\| &= \left\|\frac{1}{2^{2n-k}}\left[\lambda\left(2x^{[n]}\right) - \mu\left(2x^{[n]}\right)\right]\right\| \\ &\leqslant L_i\left(x^{[n]}\right)\left\|\lambda\left(h_1\left(x^{[n]}\right)\right) - \mu\left(h_1\left(x^{[n]}\right)\right)\right\|.\end{aligned}$$

Therefore the hypothesis (A1) is valid. By induction on l, it can be checked that for any $l \in \mathbb{N}_0$ and $x^{[n]} \in X^n$, we have

$$\Lambda^l\zeta\left(x^{[n]}\right) := \left(\frac{1}{2^{2n-k}}\right)^l \zeta\left(2^l x^{[n]}\right) = \frac{\delta}{2^{3n-k}}\left(\frac{1}{2^{2n-k}}\right)^l,$$

for all $x^{[n]} \in X^n$. We see that $\sum_{l=0}^{\infty}\Lambda^l\zeta\left(x^{[n]}\right)$ is convergent for all $x^{[n]} \in V^n$. Thus, all assumptions of Theorem 3.3 are satisfied and so there exists a mapping $\mathcal{A}_q : X^n \longrightarrow Y$ such that

$$\mathcal{A}_q\left(x^{[n]}\right) = \lim_{l\to\infty}(\mathcal{T}^l\Gamma)\left(x^{[n]}\right) = \frac{1}{2^{2n-k}}\mathcal{A}_q\left(2x^{[n]}\right) \qquad \left(x^{[n]} \in X^n\right),$$

and moreover (3.2) is true. Clearly,

$$\left\|\Delta\left(\mathcal{T}^l\Gamma)(x_1^{[n]}, x_2^{[n]}\right)\right\| \leqslant \delta, \tag{3.10}$$

for all $x_1^{[n]}, x_2^{[n]} \in X^n$ and $l \in \mathbb{N}_0$. Letting $l \to \infty$ in (3.10), we find $\Delta\mathcal{A}_q\left(x_1^{[n]}, x_2^{[n]}\right) = 0$ for all $x_1^{[n]}, x_2^{[n]} \in X^n$. This means that the mapping $\mathcal{A}_q$ satisfies (2.2). Moreover, if $\mathcal{A}_q$ satisfies (H6), then it is a multi-additive-quadratic mapping by Proposition 2.4. The proof of the uniqueness of $\mathcal{A}_q$ is routine.

Remark 3.4. It is seen that the results of this subsection and the previous one are the same while two approaches are different.

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